Analytical Algorithms for the Inverse Source Problem in a Sphere

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Abstract

A homogeneous sphere is excited by a point source lying inside the sphere. Analytical inversion algorithms are established concerning the determination of the physical characteristics of the sphere as well as the location and strength of the source. The basic quantity utilized in these algorithms is the total field on the sphere which is assumed to be known. The investigation of the above described problem is motivated by various applications in medical imaging.

Introduction

A point source inside a homogeneous spherical conductor constitutes a simplified yet realistic model for investigating a variety of applications in brain imaging [1], [2]. Locating point sources using surface measurements is an example of an *inverse source problem* [3].

We consider the basic static problem consisting of Laplace's equation in a ball V_i with boundary ∂V. The goal is to identify a point source lying in V_i from Cauchy data on ∂V. There are fields both inside and outside the sphere, with appropriate interface conditions on the sphere. The inverse problem is to determine the location and strength of the source knowing the total field on the sphere. The internal conductivity is also to be found.

We obtain exact and complete results by developing analytical inversion algorithms utilizing the moments obtained by integrating the product of the total field on the spherical interface with spherical harmonic functions. All the information about the primary source and the ball's physical characteristics is encoded in these moments. The presented method is simple, explicit and exact (given exact data). Other analytic inversion algorithms for determining static point dipoles as well as acoustic point sources inside a homogeneous sphere are presented in [4].

1 Mathematical Formulation

Consider a homogeneous spherical object of radius a, surrounded by an infinite homogeneous medium. Denote the exterior by V_e and the interior by V_i. A point source lies inside the sphere at an unknown location $\mathbf{r}_1 \in V_\mathrm{i}$. We will determine the source, using information on the spherical interface.

Denote the field outside the sphere by u_e and the total field inside by u_i. Then, $u_\mathrm{i} = u^{\mathrm{pr}} + u^{\mathrm{sec}}$, where u^{pr} is the primary field due to the source (u^{pr} is singular at $\mathbf{r}_1$) and u^{sec} is the secondary (regular) field. The field u_e is regular and satisfies an appropriate far-field condition. The fields u_e and u_i are related by transmission conditions on the sphere.

For the primary field, we choose a point source,

$$u^{\mathrm{pr}}(\mathbf{r};\mathbf{r}_1) = \frac{A}{|\mathbf{r}-\mathbf{r}_1|}, \qquad \mathbf{r} \in \mathbb{R}^3 \backslash \{\mathbf{r}_1\}, \qquad (1)$$

where A is a real constant.

Introduce spherical polar coordinates (r,θ,ϕ) for the point at $\mathbf{r}$ so that the source is at (r_1,θ_1,ϕ_1) with $r_1 = |\mathbf{r}_1| < a$. Then, the transmission conditions are

$$u_\mathrm{e} = u_\mathrm{i} \quad \text{and} \quad \frac{1}{\rho_\mathrm{e}}\frac{\partial u_\mathrm{e}}{\partial r} = \frac{1}{\rho_\mathrm{i}}\frac{\partial u_\mathrm{i}}{\partial r} \quad \text{at } r = a, \qquad (2)$$

where ρ_e and ρ_i are constants.

Since we deal with a static problem, both u_e and u^{sec} are governed by Laplace's equation. The field u_e decays to zero at infinity. In the context of electrostatics, ρ_e and ρ_i are inverse conductivities.

2 Inverse Source Problem

A static point source lies at $\mathbf{r}_1$ and generates the field u^{pr}. Near the sphere ($r_1 < r < a$), separation of variables gives the expansion

$$u^{\mathrm{pr}}(\mathbf{r};\mathbf{r}_1) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m(\mathbf{r}_1)(a/r)^{n+1} Y_n^m(\hat{\mathbf{r}}),$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ and $Y_n^m(\hat{\mathbf{r}}) = Y_n^m(\theta,\phi)$ is a spherical harmonic (see [5, §3.2]), and the quantities f_n^m

characterizing the source, are given by

$$f_n^m(\mathbf{r}_1) = \frac{4\pi A}{a}\frac{(-1)^m}{2n+1}(r_1/a)^n Y_n^{-m}(\hat{\mathbf{r}}_1). \quad (3)$$

The secondary field inside the sphere is

$$u^{\mathrm{sec}}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \alpha_n f_n^m(\mathbf{r}_1)(r/a)^n Y_n^m(\hat{\mathbf{r}}),\ 0 \le r < a$$

whereas the field outside is given by

$$u_{\mathrm{e}}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \beta_n f_n^m(\mathbf{r}_1)(a/r)^{n+1} Y_n^m(\hat{\mathbf{r}}), \quad r > a$$

The transmission conditions at $r = a$, (2), give

$$\alpha_n = \frac{(1-\varrho)(n+1)}{n+\varrho(n+1)}, \quad \beta_n = \frac{2n+1}{n+\varrho(n+1)}, \quad (4)$$

where $\varrho = \rho_{\mathrm{i}}/\rho_{\mathrm{e}}$. Note that α_n and β_n do not depend on any characteristics of the source.

The field on the sphere is

$$u_{\mathrm{surf}}(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{2n+1}{n+\varrho(n+1)} f_n^m(\mathbf{r}_1) Y_n^m(\theta,\phi).$$

It is this quantity that we shall use to find the source.

The spherical harmonics are orthonormal,

$$\int_\Omega Y_n^m \overline{Y_\nu^\mu}\,\mathrm{d}\Omega = \int_0^\pi \int_{-\pi}^{\pi} Y_n^m(\theta,\phi)\overline{Y_\nu^\mu(\theta,\phi)}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \delta_{n\nu}\delta_{m\mu},$$

where Ω is the unit sphere and the overbar denotes complex conjugation. Hence, the moments

$$M_n^m \equiv \frac{1}{\sqrt{4\pi}}\int_\Omega u_{\mathrm{surf}}\overline{Y_n^m}\,\mathrm{d}\Omega = \frac{1}{\sqrt{4\pi}}\frac{2n+1}{n+\varrho(n+1)} f_n^m(\mathbf{r}_1), \quad (5)$$

are known, in principle, if u is known on $r = a$; the double integral over Ω could be approximated using a suitable quadrature rule and corresponding point evaluations of u_{surf}. The problem now is to determine properties of the source and the interior material (namely, $\rho_{\mathrm{i}} = \rho_{\mathrm{e}}\varrho$) from M_n^m.

For a point source, (3) and (5) give

$$M_n^m = (-1)^m \frac{\tilde{A}\tilde{r}_1^n\sqrt{4\pi}}{n+\varrho(n+1)} Y_n^{-m}(\theta_1,\phi_1), \quad (6)$$

with $\tilde{A} = \frac{A}{a}$ and $\tilde{r}_1 = \frac{r_1}{a}$. Thus, there are five unknowns, $\tilde{A}$, ϱ, $\tilde{r}_1$, θ_1 and ϕ_1.

As $Y_0^0 = (4\pi)^{-1/2}$, we obtain

$$M_0^0 = \tilde{A}/\varrho.$$

This ratio is all that can be recovered if the source is at the sphere's centre ($r_1 = \tilde{r}_1 = 0$). So, let us assume now that $\tilde{r}_1 \neq 0$.

For $n = 1$, we can use the expressions of Y_1^0, Y_1^1, and Y_1^{-1} (see e.g. [5, eqn (8.28)]) to obtain

$$M_1^0 = \tilde{A}\,\frac{\tilde{r}_1\sqrt{3}}{1+2\varrho}\cos\theta_1,$$

$$M_1^{\pm1} = \mp\tilde{A}\,\frac{\tilde{r}_1\sqrt{3/2}}{1+2\varrho}\,\mathrm{e}^{\mp\mathrm{i}\phi_1}\sin\theta_1.$$

If $M_1^{\pm1} = 0$, then $\theta_1 = 0$ or π (the source is on the z-axis and so ϕ_1 is irrelevant); to decide which, note that the sign of $M_0^0 M_1^0$ is the sign of $\cos\theta_1$. If $M_1^{\pm1} \neq 0$, ϕ_1 is determined by noting that the complex number $M_0^0 M_1^{-1}$ has argument ϕ_1.

If $M_1^0 = 0$, then $\theta_1 = \pi/2$. If $M_1^0 \neq 0$, $\sqrt{2}\,M_1^{-1}/M_1^0 = \mathrm{e}^{\mathrm{i}\phi_1}\tan\theta_1$ determines θ_1. Also

$$(1+2\varrho)^2\{(M_1^0)^2 - 2M_1^1 M_1^{-1}\} = 3(\tilde{A}\tilde{r}_1)^2. \quad (7)$$

To conclude, we take a measurement with $n = 2$

$$\varrho(2+3\varrho)M_0^0 M_2^m = (\tilde{A}\tilde{r}_1)^2(-1)^m\sqrt{4\pi}Y_2^{-m}(\theta_1,\phi_1). \quad (8)$$

Choosing m such that $Y_2^{-m}(\theta_1,\phi_1) \neq 0$ (namely take $m = 0$ unless $P_2(\cos\theta_1) = 0$), we eliminate $(\tilde{A}\tilde{r}_1)^2$ between (7) and (8) to give a quadratic equation for ϱ (which is real and positive). Then, $\tilde{A} = A/a = M_0^0\varrho$ and $\tilde{r}_1 = r_1/a$ follows from M_1^m or from (7).

References

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