Scattering by a Cage

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Abstract

Acoustic scattering by a ring comprising a large number of equally spaced small circles is considered, using a combination of Foldy-type approximations, circulant matrices and asymptotics.

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1 Introduction

Time-harmonic acoustic waves are scattered by N obstacles. Such multiple scattering problems can be solved exactly, in principle, by reducing them to integral equations or to infinite systems of linear algebraic equations [2].

In this paper, we are interested in the scattering of an incident wave by N identical parallel circular cylinders arranged in a particular way: in a cross-sectional plane, there are N circles (radius a) with their centres located on, and equally spaced around, a larger circle (radius b). We call this geometrical configuration a *ring* or a *cage*, the latter word being used because we can consider the configuration as giving a simple model of a Faraday cage.

Exact (numerical) methods have been applied to scattering by a cage. However, we are especially interested when N is large, so that we have many small circles around the ring with small gaps between them.

Intuitively we expect that, in the limit (when there are no gaps), we should approach the solution for scattering by a single large cylinder (with cross-section of radius b). Can this be shown, and, if so, how fast is the limit achieved?

In a recent paper [3], we gave an analysis of cage problems. The cylinders comprising the cage were assumed to be small, both geometrically ($a \ll b$) and acoustically ($ka \ll 1$, where $2\pi/k$ is the incident wavelength). For the scattering itself, we used *Foldy's method* [1], [2, §8.3]. This is an approximate theory, in which the scattering by each circular cylinder is represented by a single term (proportional to $H_0(kr)$, see below) instead of the usual infinite separation-of-variables series. However, all multiple scattering effects are taken into account. The result is an $N \times N$ linear algebraic system. This reduction works for N scatterers at more-or-less arbitrary locations. However, for a ring of equally-spaced identical scatterers, the matrix occurring has a special structure: it is a *circulant* matrix. This means that it can be inverted explicitly, using a discrete Fourier transform, and then the behaviour of the solution as N grows can be analysed. It turns out that the expected limit is achieved but the limit is approached slowly, at best as N^{-1}.

So far, we have not mentioned the boundary condition on each cylinder. Most exact methods can accommodate any choice, such as Dirichlet (sound-soft) or Neumann (sound-hard) conditions.

For the Foldy-type analysis described above, the underlying assumption is that each cylinder scatters *isotropically*: note the presence of $H_0(kr)$ with no dependence on the polar angle. This is entirely appropriate for Dirichlet problems: we know that small ($ka \ll 1$) sound-soft circles really do scatter like a monopole. However, sound-hard circles do not scatter isotropically: monopole and dipole contributions are equally important and both must be retained. The dipole gives a directional dependence to the waves scattered by one circle, and this must be incorporated into the calculation of the multiply scattered waves when there are N circles. Foldy's method can be extended to cover sound-hard scatterers, leading to a $3N \times 3N$ linear algebraic system. We can use this extension to study scattering by a sound-hard cage.

2 Foldy approach

Foldy's method, when applied to a cage of soft circles, leads to an $N \times N$ system,

$$\sum_{j=1}^{N} K_{n-j} A_j = f_n, \quad n = 1, 2, \ldots, N, \qquad (1)$$

where $f_n = -u_{\text{in}}(\mathbf{r}_n)$, u_{in} is the incident wave, $K_0 = -g^{-1}$, $g = -[J_0(ka)]/[H_0(ka)]$,

$$K_j = H_0(2kb\,|\sin(j\pi/N)|), \quad j \neq 0 \bmod N$$

and K_j is N-periodic: $K_{j+mN} = K_j$ for all integers m. The nth circle is centred at $\mathbf{r}_n$, $r = b$, $\theta = nh$, where $h = 2\pi/N$ is the angular spacing between adjacent circles.

The unknown coefficients A_n appear in our representation of the total field,

$$u(\mathbf{r}) = u_{\text{in}}(\mathbf{r}) + \sum_{j=1}^{N} A_j H_0(k|\mathbf{r} - \mathbf{r}_j|). \qquad (2)$$

The circulant structure means that we can solve (1) explicitly, using discrete Fourier transforms. Then all properties of the wavefield can be found. In particular, it is possible to extract asymptotic properties as N grows.

3 Extended Foldy approach

Sound-hard scatterers always generate a dipole field. Foldy's method can be generalized to cover these situations [2, §8.3.3]. Thus, we add

$$\sum_{j=1}^{N} \mathbf{q}_j \cdot \mathbf{g}(\mathbf{r} - \mathbf{r}_j)$$

to the right-hand side of (2), where $\mathbf{q}_j$ is an unknown vector (dipole strength and direction), $\mathbf{g}(\mathbf{r}) = \hat{\mathbf{r}}\, H_1(kr)$, $\hat{\mathbf{r}} = \mathbf{r}/r$ and $r = |\mathbf{r}|$.

For a cage, it is convenient to write $\mathbf{q}_j$ in terms of its radial and tangential components with respect to the cage. Let $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ be unit vectors in the x and y directions, respectively. Then $\hat{\mathbf{r}}_j = \mathbf{r}_j/b = \hat{\mathbf{\imath}} \cos\theta_j + \hat{\mathbf{\jmath}} \sin\theta_j$ with $\theta_j = jh$. Let $\hat{\boldsymbol{\theta}}_j = \hat{\mathbf{\jmath}} \cos\theta_j - \hat{\mathbf{\imath}} \sin\theta_j$ be a unit tangent vector, so that $\hat{\mathbf{r}}_j \cdot \hat{\boldsymbol{\theta}}_j = 0$. Write

$$\mathbf{q}_j = B_j \hat{\mathbf{r}}_j + C_j \hat{\boldsymbol{\theta}}_j,$$

so that the $3N$ unknowns are A_j, B_j and C_j, for $j = 1, 2, \ldots, N$. These satisfy the system

$$\sum_{j=1}^{N} \mathbf{K}_{n-j} \mathbf{x}_j = \mathbf{f}_n, \quad n = 1, 2, \ldots, N, \qquad (3)$$

with $\mathbf{f}_n = (-u_{\text{in}}(\mathbf{r}_n),\, -\hat{\mathbf{r}}_n \cdot \mathbf{v}_{\text{in}}(\mathbf{r}_n),\, \hat{\boldsymbol{\theta}}_n \cdot \mathbf{v}_{\text{in}}(\mathbf{r}_n))^T$, $\mathbf{x}_j = (A_j, B_j, C_j)^T$ and $\mathbf{v}_{\text{in}}(\mathbf{r}) = k^{-1} \text{grad}\, u_{\text{in}}$. $\mathbf{K}_j$ is a symmetric 3×3 matrix; in detail,

$$\mathbf{K}_0 = \mathbf{K}_N = \begin{pmatrix} Z_0^{-1} & 0 & 0 \\ 0 & (2Z_1)^{-1} & 0 \\ 0 & 0 & -(2Z_1)^{-1} \end{pmatrix},$$

$Z_n = J_n'(ka)/H_n'(ka)$, and, for $j \neq 0 \bmod N$,

$$\mathbf{K}_j = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{pmatrix},$$

with entries as follows:

$$K_{11} = H_0, \quad K_{12} = -(2b)^{-1} R_j H_1,$$
$$K_{13} = b R_j^{-1} H_1 \sin\theta_j,$$
$$K_{22} = \frac{H_1}{kR_j} \cos\theta_j + H_2 \frac{R_j^2}{4b^2},$$
$$K_{23} = \frac{H_1}{kR_j} \sin\theta_j - \frac{1}{2} H_2 \sin\theta_j,$$
$$K_{33} = -\frac{H_1}{kR_j} \cos\theta_j + H_2 \frac{b^2}{R_j^2} \sin^2\theta_j.$$

All Hankel functions have argument kR_j with $R_j = 2b|\sin(j\pi/N)|$. Clearly, $\mathbf{K}_j$ is N-periodic: $\mathbf{K}_{j+mN} = \mathbf{K}_j$, $m = \pm 1, \pm 2, \ldots$.

The system (3) gives $3N$ equations for $3N$ unknowns. Application of the discrete Fourier transform breaks the system into N 3×3 systems, one for each $\mathbf{x}_j$, thus permitting further analysis.

References

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