

## Acoustic Scattering by Spheres and Spheroids in the Time Domain

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### Abstract

The title problems are treated using Laplace transforms and separation of variables. This approach has been used for spheres since the 1950s. When applied to spheroids, we encounter new questions, such as how do spheroidal wavefunctions behave for complex parameters? We describe our recent work in this direction.

**Keywords:** acoustics, scattering, time domain

### 1 Introduction

We consider acoustic scattering of a sound pulse by a bounded three-dimensional obstacle with smooth boundary  $S$ . The scattered field  $u(\mathbf{r}, t)$  solves an initial-boundary value problem (IBVP) for the wave equation

$$\nabla^2 u = c^{-2} \partial^2 u / \partial t^2 \quad \text{in } B \text{ for } t > 0,$$

where  $B$  is the unbounded exterior of  $S$  and  $c$  is the constant speed of sound. In addition, there are zero initial conditions

$$u = 0 \quad \text{and} \quad \partial u / \partial t = 0 \quad \text{in } B \text{ at } t = 0 \quad (1)$$

and a boundary condition on  $S$  for  $t > 0$ .

The word ‘‘acoustic’’ in the title is important: we always assume that  $u$  is a velocity potential, so that  $\mathbf{v} = \text{grad } u$  is the fluid velocity and  $p = -\rho \partial u / \partial t$  is the (excess) pressure, where  $\rho$  is the constant background density.

Problems of physical interest often involve incident pulses, with moving wavefronts across which  $p$  or normal velocity  $v_n$  is discontinuous. However, in most cases, it can be arranged that  $u$  is continuous across wavefronts, even though  $p$  or  $v_n$  is not. Consequently, it is advantageous to solve for  $u$ , assumed to be continuous and piecewise-smooth; assuming too much smoothness may exclude interesting physical problems. Also, seeking weak solutions must be done with care: such solutions may not respect the proper jump conditions across wavefronts, conditions that stem from the underlying continuum mechanics. See [4] for details and references.

### 2 Use of Laplace transforms

The textbook method for solving IBVPs is to use Laplace transforms. Thus, define

$$U(\mathbf{r}, s) = \mathcal{L}\{u\} = \int_0^\infty u(\mathbf{r}, t) e^{-st} dt.$$

$U$  satisfies the modified Helmholtz equation,

$$\nabla^2 U - (s/c)^2 U = 0 \quad \text{in } B. \quad (2)$$

Here, we have used the continuity of  $u$  and the initial conditions (1).

We solve (2) using the Laplace transform of the boundary condition and a mild growth condition as  $|\mathbf{r}| \rightarrow \infty$ , and then we invert using

$$u(\mathbf{r}, t) = \frac{1}{2\pi i} \int_{\text{Br}} U(\mathbf{r}, s) e^{st} ds,$$

where Br is a Bromwich contour in the  $s$ -plane.

For an incident sound pulse, we know that, for any fixed  $\mathbf{r}$ ,  $u(\mathbf{r}, t) = 0$  for sufficiently large  $t$ . Consequently  $U(\mathbf{r}, s)$  is an analytic function of  $s$  for  $\text{Re } s > 0$ . When  $U(\mathbf{r}, s)$  is continued analytically into the other half-plane,  $\text{Re } s \leq 0$ , singularities will be encountered. These singularities are poles and they occur in complex-conjugate pairs (unless they are real and negative). The singularities are known as *natural frequencies*. Once they have been located, we can contemplate moving the Bromwich contour to the left, picking up residue contributions.

### 3 The sphere

For scattering by a sphere of radius  $a$ , we use spherical polar coordinates,  $r$ ,  $\theta$  and  $\phi$ . Separation of variables leads to

$$U = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} k_n(sr/c) P_n^m(\cos \theta) \mathcal{A}_n^m(\phi, s), \quad (3)$$

$$\mathcal{A}_n^m = A_n^m(s) \cos m\phi + B_n^m(s) \sin m\phi. \quad (4)$$

Here  $k_n$  is a modified spherical Bessel function,  $P_n^m$  is an associated Legendre function, and  $A_n^m$  and  $B_n^m$  are to be determined using the boundary condition. We note two things about the

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form of expansion (3). First, the angular functions  $P_n^m(\cos\theta) \frac{\cos}{\sin} m\phi$  do not depend on  $s$ . Second the radial function,  $k_n(sr/c)$ , does not depend on the mode number  $m$ . This structure is lost when we consider scattering by a spheroid.

As an example, suppose we have a Dirichlet boundary condition,  $u(a, \theta, \phi, t) = d(\theta, \phi, t)$ , a given function satisfying  $d(\theta, \phi, 0) = 0$ ; this constraint ensures that  $u$  is continuous. Suppose that  $D = \mathcal{L}\{d\}$  has the expansion

$$D(\theta, \phi, s) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} P_n^m(\cos\theta) \mathcal{D}_n^m(\phi, s), \quad (5)$$

where  $\mathcal{D}_n^m = D_{mn}^c(s) \cos m\phi + D_{mn}^s(s) \sin m\phi$  and  $D_{mn}^c$  and  $D_{mn}^s$  are coefficients. Then the boundary condition yields

$$U = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{k_n(sr/c)}{k_n(sa/c)} P_n^m(\cos\theta) \mathcal{D}_n^m(\phi, s). \quad (6)$$

This formula shows that the natural frequencies are those values of  $s$  for which  $k_n(sa/c) = 0$ . There may be additional singularities arising from the form of  $D_{mn}^c(s)$  and  $D_{mn}^s(s)$ .

All this is well known; the method outlined above was first used by J. Brillouin in 1950. See [5] for details and references.

#### 4 The prolate spheroid

We use prolate spheroidal coordinates  $\xi, \eta$  and  $\phi$ , defined by  $x = h\sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos\phi$ ,  $y = h\sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin\phi$ ,  $z = h\xi\eta$ , where  $h$  is a positive constant. The foci are at  $(x, y, z) = (0, 0, \pm h)$ . The surface  $\xi = \xi_0 > 1$  is a prolate spheroid with semi-major axis of length  $a = h\xi_0$  and semi-minor axis of length  $b = h\sqrt{\xi_0^2 - 1}$ . The exterior of the spheroid corresponds to  $\xi > \xi_0$ ,  $-1 \leq \eta \leq 1$  and  $-\pi \leq \phi < \pi$ .

To solve (2), we write

$$U = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} R_{mn}^{(3)}(ip, \xi) S_n^m(ip, \eta) \mathcal{A}_n^m(\phi, s) \quad (7)$$

for  $\xi > \xi_0$ . Here  $p = sh/c$ ,  $\mathcal{A}_n^m$  is defined by (4),  $R_{mn}^{(3)}$  is an outgoing radial spheroidal wavefunction (SWF) and  $S_n^m$  is an angular SWF [1]. Comparing (7) with (3), we see that the radial part  $R_{mn}^{(3)}(ip, \xi)$  depends on both  $m$  and  $n$ , and the angular part  $S_n^m(ip, \eta)$  depends on  $s$ .

For the Dirichlet boundary condition,  $u = d$  on  $\xi = \xi_0$ , we expand  $D = \mathcal{L}\{d\}$  as

$$D(\eta, \phi, s) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} S_{mn}(ip, \eta) \mathcal{D}_n^m(\phi, s),$$

see (5), whence

$$U = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{R_{mn}^{(3)}(ip, \xi)}{R_{mn}^{(3)}(ip, \xi_0)} S_n^m(ip, \eta) \mathcal{D}_n^m(\phi, s).$$

This formula should be compared with (6).

The natural frequencies are determined by the zeros of  $R_{mn}^{(3)}(ish/c, \xi_0)$  in the complex  $s$ -plane. It turns out that the relevant properties of radial SWFs are not in the literature on special functions, so we have developed some new asymptotic approximations that can be used to estimate the natural frequencies. There is literature on computing SWFs numerically [2, 3]; comparisons between asymptotics and numerics are being made.

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