# Use (and Misuse) of the Method of Images

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**Abstract**

Images are often used to replace flat boundaries. We do that in the context of antiplane elastic waves in an anisotropic half-space: the image of a singularity is not at the mirror-image point.

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## 1 Introduction

Mirror images are known to small children. Like them, we know that our image is located behind the mirror: if we place a source at a distance $h$ from a flat mirror, the image is at a distance $h$ behind the mirror, and the image itself is a faithful copy of the source.

This notion is used (and sometimes misused) when one wants to replace a flat boundary by appropriate images. For example, consider the Helmholtz equation $(\nabla^2+k^2)u=0$ in the semi-infinite region $y>-h$ with a sound-hard wall at $y=-h$ (so that $\partial u/\partial y=0$ at $y=-h$) together with a source at the origin. Then, quoting Pierce [4, p. 208], 'the original boundary-value problem of source plus wall is replaced by one with two sources (original source and image source) but no wall.' Exploiting and extending this simple idea leads to useful methods for solving scattering problems when one has scatterers in the presence of flat boundaries.

## 2 Antiplane anisotropic elasticity

Time-harmonic antiplane motions of a homogeneous anisotropic elastic solid are governed by

$$C_{55}\frac{\partial^2 u}{\partial x^2}+2C_{45}\frac{\partial^2 u}{\partial x\,\partial y}+C_{44}\frac{\partial^2 u}{\partial y^2}+\rho\omega^2 u=0, \quad (1)$$

where $u(x,y)$ is the out-of-plane displacement, $C_{55}$, $C_{45}$ and $C_{44}$ are stiffnesses, $\rho$ is the density and $\omega$ is the frequency. Suppose we want to solve (1) in a half-space $y>-h$ ($h>0$) with a traction-free boundary condition at $y=-h$ and a scatterer of some kind (such as a circular cavity) within the half-space. For simplicity, assume that the origin is inside the scatterer.

The simplest case is *isotropy.* Then $C_{44}=C_{55}$, $C_{45}=0$ and (1) reduces to the Helmholtz equation with $k^2=\rho\omega^2/C_{55}$. The basic singular solution is $H_0(kr)$ where $r^2=x^2+y^2$ and $H_0$ is a Hankel function. The boundary condition, $\partial u/\partial y=0$ at $y=-h$, can be incorporated by adding an image term,

$$u(x,y)=H_0(kr)+H_0(k\hat{r}), \quad (2)$$

where $\hat{r}^2=x^2+(y+2h)^2$. The extra term is singular at the image point $(x,y)=(0,-2h)$, which is the mirror image of the origin in the 'mirror' at $y=-h$.

A slightly more complicated case is *orthotropy*, for which $C_{44}\neq C_{55}$ and $C_{45}=0$. Then we can reduce (1) to the Helmholtz equation by scaling $x$, $y$ or both. For example, putting $x'=x/\alpha$ with $\alpha=\sqrt{C_{55}/C_{44}}$ gives $\partial^2u/\partial x'^2+\partial^2u/\partial y^2+(\alpha k)^2u=0$ with $k$ as before. This scaling does not move the flat boundary at $y=-h$ but it does deform the shape of the scatterer. Alternatively, put $y'=\alpha y$ giving $\partial^2u/\partial x^2+\partial^2u/\partial y'^2+k^2u=0$. Stretching $y$ is closer to what is often done in the context of anisotropic elasticity but it moves the flat boundary to $y'=-\alpha h$. Once the stretching has been done, we can reuse known solutions for the Helmholtz equation. In particular, for a solution singular at the origin, we can incorporate the boundary condition at $y=-h$ by adding an appropriate solution that is singular at the mirror-image point.

For the general *anisotropic* case, governed by (1) with $C_{45}\neq 0$, we could transform (1) into the Helmholtz equation using an appropriate scaling and rotation of coordinates, the rotation being needed so as to eliminate the mixed-derivative term in (1). The implication is that solutions involving Hankel (or Bessel) functions of certain arguments will appear. This approach is convenient for full-space problems but less so for half-space problems because the required transformation will also move the boundary of the half-space.

Instead, we first construct the full-space solutions (as has been done by others) and express them in terms of the original independent variables, $x$ and $y$. We then introduce corresponding solutions singular at an appropriate image point. The location of this point was found by Ting [6, §3.5]: *it is not the mirror-image point unless* $C_{45} = 0$, it is at

$$(x, y) = (2qh, -2h) \quad \text{with } q = -C_{45}/C_{44}.$$

Ting was concerned with static problems, but introducing dynamics does not change the *location* of the image point, just the kind of solutions that are to be singular at that point. Once this observation has been made, the rest is mere calculation [3]. For the simplest example, we find the fundamental solution

$$G(x, y) = H_0(kR) + H_0(k\widehat{R}), \tag{3}$$

where $R$ and $\widehat{R}$ are given by

$$\begin{aligned} R^2 &= \left(C_{44}x^2 - 2C_{45}xy + C_{55}y^2\right)/C_{44}, \\ \widehat{R}^2 &= \big\{C_{44}x^2 - 2C_{45}xy + C_{55}(y + 2h)^2 \\ &\qquad - 4h(y + h)C_{45}^2/C_{44}\big\}/C_{44}. \end{aligned}$$

The function $G$ satisfies (1) everywhere in the anisotropic half-space $y > -h$ except for a logarithmic singularity at the origin, and it satisfies the traction-free boundary condition at $y = -h$. In the special case of orthotropy, (3) reduces to a formula in Kausel's book [1, eqn (5.10)]. In fact, the formula (3) itself can be found in a paper by Stevenson [5, §6]. He considers a three-dimensional (but scalar) version of (1).

Multipole solutions (with higher-order singularities) can also be constructed [3]. Note that the curve $R =$ constant is a material-dependent ellipse in the $xy$-plane. Consequently, infinite series of multipoles will converge in regions bounded by certain confocal ellipses (i.e., not concentric circles).

## 3 Discussion

Obviously, $G$ could be used to derive boundary integral equations for scattering problems posed in an anisotropic half-space. The corresponding multipoles may also have their uses; see [3] for some discussion.

More generally, the whole subject of images and their use may be worth further investigation and systemisation. (For some examples of misuse, see references given in [2,3].) Evidently, one difficulty with (1) stems from the mixed derivative term, but that cannot be the whole story: for example, with linear water waves, the velocity potential $u$ satisfies Laplace's equation in the water, $y > 0$, with the boundary condition $Ku + \partial u/\partial y = 0$ on the mean free surface at $y = 0$, where $K$ is a positive constant. The corresponding fundamental solution is known explicitly but it is quite complicated. One can say the same about plane-strain elastic waves in an isotropic half-space.

Fundamental solutions for half-space problems can often be constructed but an interesting question remains: when can the effect of a (flat) boundary be replaced by (a finite number of) image singularities? This is probably a difficult question, in general. For example, Ting [6, §8.7] shows that, for some problems involving an anisotropic elastic half-space, 9 distinct image singularities are needed!

## References


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