

Chapter 35

Elastodynamic inverse obstacle scattering*

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Abstract

Time-harmonic elastic waves are incident upon a bounded cavity in three dimensions. The resulting scattered waves are characterized by their far-field patterns. We consider some simple questions concerning the determination of the shape of the cavity from information on the far-field patterns. We also prove two elastodynamic analogues of a theorem due to S.N. Karp (1962), giving sufficient conditions on the far-field patterns for the obstacle to be spherical. The proofs are indirect and are based on symmetry arguments, as used for scalar problems by A.G. Ramm (1991).

1 Introduction

Consider a single cavity in an otherwise unbounded elastic solid. When a time-harmonic elastic wave encounters the cavity, it is scattered to infinity in all directions. The scattered waves separate into a compressional wave (P -wave) and a shear wave (S -wave); each wave is characterized by an amplitude, called a *far-field pattern*. The *direct problem*, described in section 2, is concerned with the calculation of the two far-field patterns, given the incident field and the shape of the obstacle's surface, S . There is an extensive literature on direct problems; see, for example, [11] or [17].

In the corresponding *inverse problems*, the goal is to determine the shape of S , given some information on the far-field patterns for at least one known incident field. There is a considerable literature on these inverse problems in acoustics and in electromagnetism; see [5], [7], [20] [22] and the recent extensive review by Bates *et al.* [3]. However, the elastodynamic problem has received much less attention.

Most work in elastodynamics has been concerned with inclusions or cracks, where an inclusion is an obstacle composed of an elastic material differing from the surrounding solid, and a crack is defined to be an open smooth surface across which the elastic displacement vector is discontinuous; thus, a crack is a degenerate cavity. The motivation behind these studies comes mainly from ultrasonic nondestructive evaluation; for a general review in this area, see [26].

For inverse scattering by cavities, several approximate methods have been devised [10] in which only P -waves are used. For inverse scattering by cracks in a solid, we cite the papers of Gubernatis [12], [13], Achenbach [1], [2], [25] and their co-workers. For example, Gubernatis and Domany [12] note that, in the long-wavelength limit, an obstacle must be

*Partially supported by the EC under grant SC1-0079. Appeared in *2nd International Conference on Mathematical and Numerical Aspects of Wave Propagation* (ed. R. Kleinman, T. Angell, D. Colton, F. Santosa and I. Stakgold), SIAM, Philadelphia, 1993, pp. 330–337.

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a crack if the far-field patterns have equal magnitudes at all pairs of diametrically opposite directions.

All the elastodynamic work cited above involves some form of approximation. In the rest of this paper, we are concerned with exact results. One question to ask is: how much scattering information is sufficient to uniquely determine the obstacle, in principle? This question has been addressed by Wall [27]; his work is extended and refined in section 4.

In section 5, we consider elastodynamic analogues of Karp's theorem. Karp [16] proved the following result in two-dimensional acoustics for scattering of plane waves by a sound-soft (Dirichlet condition) obstacle, S : suppose that the far-field pattern $F(\theta; \alpha)$ is a function of $\theta - \alpha$,

$$F(\theta; \alpha) = f(\theta - \alpha),$$

say, for all θ and for all α , where θ is the angle of observation and α is the angle of incidence; then S is a circle. This provides the explicit solution to an inverse-scattering problem. New proofs were given later by Colton and Kirsch [4], in three dimensions and for sound-hard (Neumann condition) obstacles. Colton and Kress [6] have proved an analogous result for electromagnetic scattering by a perfectly conducting obstacle. We give proofs of two elastodynamic analogues of Karp's theorem in three dimensions, giving sufficient conditions on the far-field patterns for S to be a sphere. The proofs are based on some symmetry arguments, as used recently by Ramm [21] for several scalar problems. Thus, the proofs are indirect, whereas the proofs in [4], [6] and [16] are direct.

The work described in this paper was begun in January 1990, when the first author was visiting Patras. At that time, we tried to extend the direct proofs in [4], [6] and [16] to elastodynamics. We failed! Nevertheless, in [19], we sketch how a direct proof for elastodynamics might proceed, following the ideas in [6], and highlight where difficulties remain. The paper [19] also discusses other boundary conditions on S .

2 The direct problem

Let B_i denote a bounded, three-dimensional domain, with boundary S , and simply-connected unbounded exterior, B_e . We suppose that the surface S is *properly regular*, in the terminology of Gurtin [14, §5]; this means, roughly, that S is closed, connected and piecewise smooth (so that edges and corners are allowed).

The exterior domain B_e is filled with homogeneous isotropic elastic material, with Lamé moduli λ and μ , Poisson's ratio ν , and mass density ρ . A stress wave is incident upon the obstacle B_i ; this leads to the following scattering problem.

DIRECT PROBLEM. Find a displacement vector $\mathbf{v}(P)$ which satisfies

$$(1) \quad k^{-2} \text{grad div } \mathbf{v} - K^{-2} \text{curl curl } \mathbf{v} + \mathbf{v} = \mathbf{0}$$

for $P \in B_e$, radiation conditions at infinity (these are specified below in section 3) and the boundary condition

$$(2) \quad T\mathbf{u}(p) = \mathbf{0} \quad \text{for } p \in S,$$

where the total displacement in B_e is

$$(3) \quad \mathbf{u}(P) = \mathbf{v}(P) + \mathbf{u}_{\text{inc}}(P), \quad P \in B_e.$$

The given incident wave, \mathbf{u}_{inc} , is assumed to satisfy (1) everywhere. The wavenumbers k and K are defined by

$$\rho\omega^2 = (\lambda + 2\mu)k^2 = \mu K^2$$

and the time-dependence $e^{-i\omega t}$ is suppressed throughout. The traction operator T is defined on smooth parts of S by

$$(4) \quad (T\mathbf{u})_m(p) = \lambda n_m \frac{\partial u_j}{\partial x_j} + \mu n_j \left(\frac{\partial u_m}{\partial x_j} + \frac{\partial u_j}{\partial x_m} \right),$$

where $\mathbf{n}(p)$ is the unit normal at $p \in S$, pointing into B_e .

We shall use the following notation: capital letters P, Q denote points of $B_e \cup B_i$; lower-case letters p, q denote points of S ; \mathbf{r} is the position vector of P with respect to the origin O , which is chosen at some point in B_i ; $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$.

We shall also require some properties of the following related interior problem.

VIBRATION PROBLEM. Find a non-trivial displacement vector $\mathbf{u}(P)$ which satisfies (1) in the bounded domain B_i , together with the boundary condition (2) on the properly-regular surface S .

This eigenvalue problem only has non-trivial solutions for certain values of the frequency ω . It is known that these eigenfrequencies form an infinite, discrete set, and that each eigenfrequency has a finite multiplicity. For proofs of these results, see [14, §§75–78], [23, Chpt. 6, §III.3] and [24, Chpt. 2, §7].

3 Radiation conditions and far-field patterns

The formulation of radiation conditions is given in [17, pp. 124–130]. One formulation is the following: decompose the scattered field as

$$\mathbf{v}(P) = \mathbf{v}^P + \mathbf{v}^S$$

where

$$\mathbf{v}^P = -k^{-2} \text{grad div } \mathbf{v} \quad \text{and} \quad \mathbf{v}^S = \mathbf{v} - \mathbf{v}^P;$$

then, we require that

$$(5) \quad r \left(\frac{\partial \mathbf{v}^P}{\partial r} - ik\mathbf{v}^P \right) \rightarrow \mathbf{0} \quad \text{and} \quad r \left(\frac{\partial \mathbf{v}^S}{\partial r} - iK\mathbf{v}^S \right) \rightarrow \mathbf{0} \quad \text{as } r \rightarrow \infty,$$

uniformly with respect to all directions $\hat{\mathbf{r}}$. These are the radiation conditions. It is common to require also that both $\mathbf{v}^P \rightarrow \mathbf{0}$ and $\mathbf{v}^S \rightarrow \mathbf{0}$ as $r \rightarrow \infty$. However, these conditions are implied by (5).

The fields \mathbf{v}^P and \mathbf{v}^S are the longitudinal and transverse parts, respectively, of the scattered field \mathbf{v} ; they satisfy

$$(\nabla^2 + k^2)\mathbf{v}^P = \mathbf{0} \quad \text{and} \quad (\nabla^2 + K^2)\mathbf{v}^S = \mathbf{0},$$

and correspond to radiated P -waves and S -waves, respectively.

We can specify the behaviour of $\mathbf{v}(P)$ for large r more precisely. We have

$$(6) \quad \mathbf{v}(r\hat{\mathbf{r}}) = \mathbf{F}^P(\hat{\mathbf{r}}) \frac{e^{ikr}}{r} + \mathbf{F}^S(\hat{\mathbf{r}}) \frac{e^{iKr}}{r} + O(r^{-2})$$

as $r \rightarrow \infty$, uniformly with respect to all directions $\hat{\mathbf{r}}$. The vectors \mathbf{F}^P and \mathbf{F}^S are called the *far-field patterns* (or scattering amplitudes). It turns out that

$$(7) \quad \mathbf{F}^P(\hat{\mathbf{r}}) = F^P(\hat{\mathbf{r}})\hat{\mathbf{r}} \quad \text{and} \quad \hat{\mathbf{r}} \cdot \mathbf{F}^S(\hat{\mathbf{r}}) = 0.$$

Thus, the radiated P -wave propagates in the outward radial direction, whereas the radiated S -wave is polarized in a plane perpendicular to the radial direction.

The far-field patterns can be calculated in terms of the displacements and tractions on the surface of the obstacle. The starting point is the representation

$$(8) \quad \mathbf{v}(P) = \frac{1}{2} \int_S \{(T\mathbf{v}) \cdot \mathbf{G}(q; P) - \mathbf{v} \cdot T^q \mathbf{G}(q; P)\} ds_q$$

for $P \in B_e$, where T^q means T applied at $q \in S$ and $\mathbf{G}(P; Q)$ is the fundamental Green's tensor (Kupradze matrix), defined by

$$(\mathbf{G}(P; Q))_{ij} = \frac{1}{\mu} \left\{ \Psi \delta_{ij} + \frac{1}{K^2} \frac{\partial^2}{\partial x_i \partial x_j} (\Psi - \Phi) \right\},$$

where

$$\Phi = -e^{ikR}/(2\pi R), \quad \Psi = -e^{iKR}/(2\pi R),$$

and R is the distance between P and Q . Asymptotic approximation of (8) for large r yields (6) and (7), where, for example (see [9]),

$$(9) \quad F^P(\hat{\mathbf{r}}) = \frac{1}{4\pi(\lambda + 2\mu)} \int_S (\mathbf{v} \cdot T\mathbf{U} + \mathbf{U} \cdot T\mathbf{u}_{\text{inc}}) ds_q;$$

here, we have used the boundary condition (2), namely

$$T\mathbf{v} = -T\mathbf{u}_{\text{inc}} \quad \text{for } p \in S,$$

and we have defined a displacement field \mathbf{U} by

$$(10) \quad \mathbf{U} = (i/k) \text{grad}_Q \phi = \phi \hat{\mathbf{r}}$$

with $\phi(Q) = \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{q})$.

We can also obtain a formula for F^P in terms of the total field \mathbf{u} . Thus, since \mathbf{U} is regular everywhere, an application of the Betti reciprocal theorem in B_i to \mathbf{U} and \mathbf{u}_{inc} gives

$$\int_S (\mathbf{u}_{\text{inc}} \cdot T\mathbf{U} - \mathbf{U} \cdot T\mathbf{u}_{\text{inc}}) ds_q = 0.$$

Adding this result to (9), using (3), gives

$$F^P(\hat{\mathbf{r}}) = \frac{1}{4\pi(\lambda + 2\mu)} \int_S \mathbf{u} \cdot T\mathbf{U} ds_q.$$

Now, for any constant vector \mathbf{b} , we have

$$\mathbf{u}(q) \cdot T\{\phi \mathbf{b}\} = -ik\{\lambda(\mathbf{u} \cdot \mathbf{n})(\mathbf{b} \cdot \hat{\mathbf{r}}) + \mu(\mathbf{n} \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \mathbf{b}) + \mu(\mathbf{n} \cdot \mathbf{b})(\mathbf{u} \cdot \hat{\mathbf{r}})\}\phi(q).$$

Hence, using (10), we find that

$$(11) \quad \mathbf{F}^P(\hat{\mathbf{r}}) = \frac{-ik\hat{\mathbf{r}}}{4\pi(\lambda + 2\mu)} \int_S \{\lambda(\mathbf{u} \cdot \mathbf{n}) + 2\mu(\mathbf{n} \cdot \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})\} \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{q}) ds_q$$

where \mathbf{u} and \mathbf{n} are all evaluated at q , the integration point on S with position vector \mathbf{q} . Similar calculations for \mathbf{F}^S give

$$(12) \quad \mathbf{F}^S(\hat{\mathbf{r}}) = \frac{-iK}{4\pi} \hat{\mathbf{r}} \times \int_S \{(\mathbf{u} \times \hat{\mathbf{r}})(\mathbf{n} \cdot \hat{\mathbf{r}}) + (\mathbf{n} \times \hat{\mathbf{r}})(\mathbf{u} \cdot \hat{\mathbf{r}})\} \exp(-iK\hat{\mathbf{r}} \cdot \mathbf{q}) ds_q.$$

Wall [27] has given similar formulae for the far-field patterns, but his involve the scattered field \mathbf{v} rather than the total field \mathbf{u} . Similar formulae are also available for other boundary conditions on S [19].

4 The inverse problem

We are interested in the following *inverse problem*: given some information on the far-field patterns, determine the shape of S . To begin with, suppose that we know both \mathbf{F}^P and \mathbf{F}^S for all $\hat{\mathbf{r}} \in \Omega$, the unit sphere (since \mathbf{F}^P and \mathbf{F}^S are analytic functions of $\hat{\mathbf{r}}$, it is enough to know them on an open patch of Ω). We can then reconstruct the scattered field everywhere outside the smallest ball containing S , B_S [8]. This field can then be continued analytically into a portion of B_S (this portion certainly includes $B_S \setminus \overline{B_i}$).

In the rest of this section, we consider the general question of uniqueness: is it possible that two different obstacles can give rise to the same far-field patterns? Thus, let B_i^j denote the interior of the obstacle with boundary S_j and exterior B_e^j ($j = 1, 2$). Denote the corresponding scattered fields by \mathbf{v}_j . Wall [27] gives the following result.

THEOREM 4.1. *Suppose that B_i^1 and B_i^2 have the same non-zero far-field patterns, $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$, for all $\hat{\mathbf{r}} \in \Omega$. Then B_i^1 and B_i^2 are not disjoint, that is $B_i^1 \cap B_i^2 \neq \emptyset$.*

Proof. Following Jones [15], suppose that B_i^1 and B_i^2 are disjoint. We have $\mathbf{v}_1 \equiv \mathbf{v}_2$ everywhere outside $B_i^1 \cup B_i^2$. From (8), we have, for $P \in B_e^1$,

$$\begin{aligned} \mathbf{v}_1(P) &= \frac{1}{2} \int_{S_1} \{(T\mathbf{v}_1) \cdot \mathbf{G}(q; P) - \mathbf{v}_1 \cdot T^q \mathbf{G}(q; P)\} ds_q \\ &= \frac{1}{2} \int_{S_1} \{(T\mathbf{v}_2) \cdot \mathbf{G}(q; P) - \mathbf{v}_2 \cdot T^q \mathbf{G}(q; P)\} ds_q \\ &= \mathbf{0}, \end{aligned}$$

since \mathbf{v}_2 is a regular elastodynamic field in B_i^1 . This is a contradiction.

Wall [27] actually proves a more general result, allowing B_i^1 and B_i^2 to be inhomogeneous inclusions.

Theorem 4.1 does not require the incident field or the frequency of oscillation to be specified. However, it does require a knowledge of both \mathbf{F}^P and \mathbf{F}^S .

One way of making further progress is to suppose that we have information for a finite range of frequencies. This leads to an elastodynamic analogue of Schiffer's theorem. Before stating this theorem, we specify the allowable incident fields. Thus, we suppose that the incident field is a plane wave of unit amplitude, propagating in the direction of the unit vector $\hat{\boldsymbol{\alpha}}$. In particular, for an incident P -wave, we have

$$\mathbf{u}_{\text{inc}}(P) = \hat{\boldsymbol{\alpha}} \exp(ik\mathbf{r} \cdot \hat{\boldsymbol{\alpha}})$$

whereas for an incident S -wave, we have

$$\mathbf{u}_{\text{inc}}(P) = \hat{\boldsymbol{\beta}} \exp(iK\mathbf{r} \cdot \hat{\boldsymbol{\alpha}}),$$

where $\hat{\boldsymbol{\beta}}$ is any unit vector satisfying

$$\hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{\beta}} = 0.$$

For any of these incident fields, there will be, in general, a scattered P -wave and a scattered S -wave.

THEOREM 4.2. *Suppose that B_i^1 and B_i^2 have the same non-zero far-field patterns, $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$, for all $\hat{\mathbf{r}} \in \Omega$ and for all frequencies in the interval $\omega_1 \leq \omega \leq \omega_2$, with $\omega_1 < \omega_2$. Then $B_i^1 = B_i^2$.*

Proof. Theorem 4.1 implies that $B_i^1 \cap B_i^2 \neq \emptyset$ and that $\mathbf{v}_1 \equiv \mathbf{v}_2$ in the exterior of $B_i^1 \cup B_i^2$. Let B_0 be any connected component of $B_i^1 \setminus B_i^2$, with boundary S_0 . It is clear that S_0 is

properly regular, composed of a piece of S_1 and a piece of S_2 . Since $B_0 \subset B_e^2$, it follows that $\mathbf{u}_2(P) \equiv \mathbf{v}_2 + \mathbf{u}_{\text{inc}}$ solves the Vibration Problem in B_0 , for all ω with $\omega_1 \leq \omega \leq \omega_2$. There are now three possibilities: $\mathbf{u}_2 \not\equiv \mathbf{0}$ or $\mathbf{u}_2 \equiv \mathbf{0}$ or $B_0 = \emptyset$. The first possibility is excluded since the eigenfrequencies of the Vibration Problem are discrete, whereas the second is excluded since

$$|\mathbf{u}_2(P)| = |\mathbf{u}_{\text{inc}} + \mathbf{v}_2| \geq |\mathbf{u}_{\text{inc}}| - |\mathbf{v}_2| = 1 + O(r^{-1})$$

as $r \rightarrow \infty$. Thus, $B_0 = \emptyset$ and $B_1^1 = B_1^2$.

The above proof is basically Schiffer's proof [18, p. 173]; see also [20, §II.1.2]. Note that it is essential that B_0 is independent of ω . Thus, we cannot rephrase the result to assert that the given far-field information determines the shape of S uniquely. For, it may be possible that two different obstacles generate the same far-field patterns, but that these obstacles vary with ω ; see also [15, p. 187]. Note also that S_0 always has corners and edges, even if S_1 and S_2 are smooth; see also [27, p. 236].

The remarks in the previous paragraph are also applicable to the next theorem, in which the frequency ω is *fixed* but different incident waves are used. Here, 'different' means different angles of incidence (vary $\hat{\boldsymbol{\alpha}}$), different types (P -waves, S -waves or both) or different polarizations (vary $\hat{\boldsymbol{\beta}}$ for incident S -waves). We have the following result.

THEOREM 4.3. *Suppose that B_1^1 and B_1^2 have the same non-zero far-field patterns, $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$, for all $\hat{\mathbf{r}} \in \Omega$ and for an infinite number of different incident waves. Then $B_1^1 = B_1^2$.*

Proof. Let $\mathbf{u}_{\text{inc}}^m(P)$ denote the m -th incident wave, and let \mathbf{u}_j^m denote the corresponding total field exterior to B_1^j for $j = 1, 2$. Proceeding as in the proof of Theorem 4.2, we see that \mathbf{u}_2^m solves the Vibration Problem in B_0 . It can be shown [19] that the eigenfunctions \mathbf{u}_2^m are linearly independent, whence ω is an eigenfrequency of infinite multiplicity. However, the Vibration Problem only has eigenfrequencies of finite multiplicity; this contradiction implies that $B_1^1 = B_1^2$.

So far, we have placed only mild restrictions on S . If we tighten these restrictions, we can give a result that only requires information from a single incident wave at a single fixed frequency. Its proof relies on analyticity with respect to frequency.

THEOREM 4.4. *Suppose that the cavity B_1 with smooth boundary S has known non-zero far-field patterns, $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$, for all $\hat{\mathbf{r}} \in \Omega$. Then, the shape of S is uniquely determined.*

Proof. We proceed as in the proof of Theorem 4.2. Since S_1 and S_2 are assumed to be smooth, we can solve the direct problem using boundary integral equations [17]. It follows that the solutions $\mathbf{u}_j(P)$ are analytic functions of ω in B_e^j . In particular, $\mathbf{u}_2(P)$ is an analytic function of ω in $B_0 \subset B_e^2$ and so

$$\mathbf{w}(P) = \frac{\partial \mathbf{u}_2}{\partial \omega}$$

exists. The result now follows exactly as in Wall [27, Theorem 4.1] (which is modelled on Jones [15, Theorem 3]), by an application of the elastodynamic version of Green's second theorem in B_0 to \mathbf{w} and the complex conjugate of \mathbf{u}_2 .

We remark that the above proof will work for non-smooth S and other boundary conditions on S once the direct solution \mathbf{u} is known to be an analytic function of ω (in a neighbourhood of the positive real axis in the complex ω -plane).

5 Karp's theorem

The far-field patterns depend on the shape of the obstacle S , as well as on the incident wave. We make this dependence explicit with the notation

$$\mathbf{F}^P(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) \quad \text{and} \quad \mathbf{F}^S(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S).$$

Let R be a rotation matrix. Thus, R is a real, orthogonal matrix, which satisfies $R^{-1} = R^T$. Since the elastic material in B_e is isotropic, we have

$$(13) \quad R\mathbf{F}^Q(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) = \mathbf{F}^Q(R\hat{\mathbf{r}}; R\hat{\boldsymbol{\alpha}}, R\hat{\boldsymbol{\beta}}; RS),$$

where $Q = P, S$. This identity holds for all unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, and for all rotations R .

If the surface S is spherical, we have

$$RS = S \quad \text{for all rotations } R,$$

and (13) reduces to

$$(14) \quad R\mathbf{F}^Q(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) = \mathbf{F}^Q(R\hat{\mathbf{r}}; R\hat{\boldsymbol{\alpha}}, R\hat{\boldsymbol{\beta}}; S) \quad \text{for all rotations } R,$$

where $Q = P, S$. Under certain conditions, the converse is true, as we shall now show.

THEOREM 5.1. *Suppose that $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$ are both known for all $\hat{\mathbf{r}} \in \Omega$ and both satisfy the symmetry relation (14). Suppose further that they are both known for*

- (i) *one incident wave and all frequencies in the interval $\omega_1 \leq \omega \leq \omega_2$, with $\omega_1 < \omega_2$*
- or
- (ii) *one frequency and an infinite number of different incident waves.*

Then S is a sphere.

Proof. Replace S by $R^T S$ in (13) and subtract the result from (14), giving

$$\mathbf{F}^Q(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; S) = \mathbf{F}^Q(\hat{\mathbf{r}}; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}; R^T S)$$

for $Q = P, S$. By Theorem 4.2 or 4.3, the given information implies that

$$S = R^T S \quad \text{for all rotations } R,$$

and the result follows.

The basic idea of the above proof is due to Ramm [21]. We remark that case (ii) is the closest analogue of Karp's theorem in acoustic scattering. Similar results are available for other boundary conditions on S [19].

Note that it was assumed in Theorem 5.1 that S was in the class of properly-regular surfaces. If we make stronger assumptions, we can get the same result with weaker assumptions on the far-field patterns.

THEOREM 5.2. *Suppose that B_i is a cavity, with smooth boundary S . Suppose that $\mathbf{F}^P(\hat{\mathbf{r}})$ and $\mathbf{F}^S(\hat{\mathbf{r}})$ are both known for all $\hat{\mathbf{r}} \in \Omega$, for one frequency and for one incident wave. Suppose further that \mathbf{F}^P or \mathbf{F}^S satisfies the symmetry relation (14). Then S is a sphere.*

Proof. By Theorem 4.4, the given information on both \mathbf{F}^P and \mathbf{F}^S is sufficient to determine the shape of S , uniquely. However, we already know that both symmetry relations are satisfied if S is a sphere. Hence, the additional information on \mathbf{F}^P or \mathbf{F}^S implies that S must be a sphere.

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