

# Multiple Scattering: an Invitation\*

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## Abstract

Multiple scattering is an important and interesting topic, of relevance in a wide variety of physical contexts. In this short paper, we attempt to review some of the theory and applications with the aim of encouraging further investigations. For simplicity, we consider the scattering of two-dimensional time-harmonic acoustic waves by hard parallel cylinders. Methods described include separation of variables,  $T$ -matrix methods and integral-equation methods. Our point of view is that the theory of multiple scattering has a rich literature which can be profitably studied to the benefit of current computational goals.

## 1 Introduction: What is Multiple Scattering?

*‘The mathematics of the full treatment may be altogether beyond human power in a reasonable time; nevertheless. . .’*  
(Heaviside, 1893, p. 324)

When a wave meets an obstacle, it is scattered. The scattered field can be calculated in various well-known ways, such as by separation of variables,  $T$ -matrix methods or integral-equation methods.

If there are several obstacles, the field scattered from one obstacle will induce further scattered fields from all the other obstacles, which will induce further scattered fields from all the other obstacles, and so on. This recursive way of thinking about how to calculate the total field leads to the notion of *multiple scattering*; it can be used to actually compute the total scattered field – each step is called an *order of scattering*. Heaviside [11, p. 323] gives a clear qualitative description of this process.

The simplest approximation, called *single scattering*, is to ignore multiple scattering completely: ‘the total scattered field is just the sum of the fields scattered by the individual [obstacles], each of which is acted on by the [incident] field in isolation from the other [obstacles]’ [4, p. 9]. This approximation is used widely; it is only expected to be valid when the spacing is large compared to both the size of the obstacles and the length of the incident waves. Indeed, with these assumptions, higher-order approximations can be derived [38]. However, there are many instances where multiple scattering is important; for some natural examples, see Bohren’s fascinating book [3].

The exact multiple-scattering problem is easily formulated: it is an exterior boundary-value problem (with a radiation condition at infinity) where the boundary is not simply-connected. Suppose that the boundary has  $N$  components  $S_j$ ,  $j = 1, 2, \dots, N$  and set

$$S = \bigcup_{j=1}^N S_j.$$

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Then, it is straightforward to reduce the boundary-value problem to a boundary integral equation over  $S$ ; see § 4.2.1. Computationally, this direct approach can be expensive, especially for problems involving many three-dimensional obstacles. Thus, in the context of hydrodynamics (where water waves interact with immersed structures, such as neighbouring ships, wave-power devices or elements of a single larger structure), Ohkusu [19] wrote: ‘For the purpose of calculating hydrodynamic forces. . . , it is essential that only the hydrodynamic properties of each element be given. A method having such a merit will facilitate the calculation for a body having many elements and may be applied to the design arrangement of the elements’. In other words, assuming that we know everything about scattering by each obstacle in isolation, how can we use this knowledge to solve the multi-obstacle problem?

For simple geometries, such as circular cylinders or spheres, one way is to combine separated solutions of the Helmholtz equation (multipoles); a necessary ingredient is an addition theorem for expanding multipoles centred at one origin in terms of similar multipoles centred on a different origin. This old but useful method is discussed in § 2.

A more powerful method, which can be viewed as a generalization of the method of separation of variables to non-separable geometries, involves the so-called  $T$ -matrix. This method is described briefly in § 3.

Finally, we discuss integral-equation methods. In order to incorporate knowledge about scattering by one obstacle in isolation, we introduce the exact Green’s function for that obstacle; this can be constructed by solving an integral equation, and can be used as a fundamental solution for the multi-body problem. Aspects of this approach are described in § 4.

Multiple scattering is a huge subject with a huge literature. We limit ourselves here to scattering of two-dimensional acoustic waves by sound-hard cylinders with bounded cross-sections. For an extensive review up to 1964, see [28], [7].

## 2 Methods Based on Separation of Variables

*‘Today, the separation of variables derivations . . . are only of academic interest.’*  
(Burke & Twersky, 1964, p. 501)

Separation of variables can be used to study acoustic scattering by a single obstacle, provided its surface coincides with a coordinate surface. In fact, the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0, \tag{1}$$

separates in eleven three-dimensional coordinate systems [1]. Of these eleven, only six are useful for bounded obstacles: circular and elliptic cylinders in two dimensions; spheres, prolate spheroids, oblate spheroids and ellipsoids in three dimensions.

For two, or more, obstacles, we can proceed by combining separable solutions appropriate to each obstacle with an appropriate addition theorem. This method was used by Závı́ška in 1913 for two-dimensional scattering by circular cylinders [37]. It is exact, and leads to an infinite system of simultaneous algebraic equations.

Despite the opening quotation, the method is widely used, probably because it is both conceptually simple and numerically effective. Consequently, we give a brief derivation for acoustic scattering by several circular cylinders, and then mention various extensions.

## 2.1 Separation of Variables for One Circular Cylinder

Consider a circular cylinder of radius  $a$ . Choose Cartesian coordinates  $(x, y)$ , with the origin  $O$  at the centre of a typical cross-section,  $S$ , and plane polar coordinates  $(r, \theta)$ , so that  $x = r \cos \theta$  and  $y = r \sin \theta$ .

We suppose that a plane wave is incident upon the cylinder, so that

$$u_{\text{inc}} = e^{ik(x \cos \alpha + y \sin \alpha)} = e^{ikr \cos(\theta - \alpha)},$$

where  $\alpha$  is the angle of incidence. Using the Jacobi expansion [33, p. 22], we obtain

$$u_{\text{inc}} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in(\theta - \alpha)}. \quad (2)$$

Now, in plane polar coordinates, the Helmholtz equation has separated solutions

$$J_n(kr) e^{\pm in\theta} \quad \text{and} \quad Y_n(kr) e^{\pm in\theta}.$$

In order to satisfy the radiation condition at infinity, we take the combination

$$\{J_n(kr) + iY_n(kr)\} e^{\pm in\theta} \equiv H_n^{(1)}(kr) e^{\pm in\theta},$$

giving rise to a cylindrical wave that propagates outwards (this is where we take note of the assumed time dependence,  $e^{-i\omega t}$ ). Furthermore, in order to have the same symmetries as the incident wave, we take the combination

$$u_{\text{sc}} = \sum_{n=-\infty}^{\infty} A_n H_n(kr) e^{in(\theta - \alpha)}, \quad (3)$$

where, from now on, we write  $H_n$  for  $H_n^{(1)}$ . So, for any reasonable choice of the coefficients  $A_n$ , the expression on the right-hand side of (3) satisfies the Helmholtz equation and the radiation condition.

To complete the problem, we use the sound-hard boundary condition on the cylinder:

$$\partial u / \partial n = 0 \quad \text{on } S, \quad (4)$$

where  $u = u_{\text{inc}} + u_{\text{sc}}$  is the total field and  $\partial / \partial n$  denotes normal differentiation from  $S$  into the exterior; here,  $\partial / \partial n = \partial / \partial r$ . Differentiating (2) and (3), and setting  $r$  equal to  $a$  gives

$$\sum_{n=-\infty}^{\infty} k \{i^n J'_n(ka) + A_n H'_n(ka)\} e^{in(\theta - \alpha)} = 0,$$

for  $0 \leq \theta < 2\pi$ . Then, orthogonality of  $\{e^{in\theta}\}$  implies that the expression inside the braces must be zero for each  $n$ , whence

$$A_n = \frac{-i^n J'_n(ka)}{H'_n(ka)}$$

and then the scattered field is given everywhere in  $r \geq a$  by (3).

## 2.2 Notation

For multiple-scattering problems, good notation is important. Thus, it is convenient to make the following definitions:

$$\psi_n(\mathbf{r}) = H_n(kr) e^{in\theta}, \quad \hat{\psi}_n(\mathbf{r}) = J_n(kr) e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \dots$$

Here,  $\mathbf{r}$  is the position vector of the point at  $(x, y)$  with respect to the origin. The functions  $\psi_n(\mathbf{r})$  and  $\hat{\psi}_n(\mathbf{r})$  are separated solutions of the Helmholtz equation in plane polar coordinates.  $\psi_n(\mathbf{r})$  satisfies the radiation condition at infinity and is singular at  $r = |\mathbf{r}| = 0$ : we call  $\psi_n$  a *radiating partial wave*.  $\hat{\psi}_n(\mathbf{r})$  is regular at  $r = 0$ : we call  $\hat{\psi}_n$  a *regular partial wave*. Note that

$$\psi_{-n}(\mathbf{r}) = (-1)^n \psi_n(\mathbf{r}) = \psi_n(-\mathbf{r}) \quad \text{and} \quad \hat{\psi}_{-n}(\mathbf{r}) = (-1)^n \hat{\psi}_n(\mathbf{r}) = \hat{\psi}_n(-\mathbf{r}).$$

With the above notation, we can reconsider the problem of scattering by one circular cylinder. Thus, the incident wave can be expanded as

$$u_{\text{inc}} = \sum_m d_m \hat{\psi}_m(\mathbf{r}), \quad (5)$$

where the notation implies that the summation is from  $m = -\infty$  to  $m = \infty$ , and

$$d_m = i^m e^{-im\alpha}.$$

Similarly, the scattered field can be expanded as

$$u_{\text{sc}} = \sum_m c_m \psi_m(\mathbf{r}). \quad (6)$$

Application of the boundary condition (4) on  $r = a$  shows that  $c_m$  and  $d_m$  are related by

$$c_m H'_m(ka) + d_m J'_m(ka) = 0, \quad m = 0, \pm 1, \pm 2, \dots;$$

this relation determines the coefficients  $c_m$  in terms of the known coefficients  $d_m$ :

$$c_m = \sum_n T_{mn} d_n \quad (7)$$

where

$$T_{mn} = - [J'_m(ka) / H'_m(ka)] \delta_{mn} \quad (8)$$

and  $\delta_{ij}$  is the Kronecker delta ( $\delta_{ij} = 0$  if  $i \neq j$  and  $\delta_{ij} = 1$  if  $i = j$ ). Although the matrix  $T$  used to be called the *transition matrix*, it is now invariably called the *T-matrix*! Note that, for this simple example,  $T$  is a diagonal matrix. In general, if the cylinder is not circular,  $T$  will be a full matrix; see § 3.

## 2.3 Multipole Method for Two Circular Cylinders

Consider two circular cylinders,  $S_j$ ,  $j = 1, 2$ . The circle  $S_j$  has radius  $a_j$  and centre  $O_j$  at  $(x, y) = (\xi_j, \eta_j)$ . Define plane polar coordinates  $(r_j, \theta_j)$  at  $O_j$ , so that  $x = \xi_j + r_j \cos \theta_j$  and  $y = \eta_j + r_j \sin \theta_j$ .

The given incident field  $u_{\text{inc}}$  is scattered by the cylinders. We assume that, in the neighbourhood of each cylinder (including the interior of each cylinder),  $u_{\text{inc}}$  is a regular solution of the Helmholtz equation, so that

$$u_{\text{inc}} = \sum_m d_m^j \hat{\psi}_m(\mathbf{r}_j), \quad j = 1, 2. \quad (9)$$

The coefficients  $d_m^j$  are known in terms of  $u_{\text{inc}}$ .

Generalizing (6), we express the scattered field as an infinite sum of multipoles at the centre of each circle,

$$u_{\text{sc}} = \sum_m c_m^1 \psi_m(\mathbf{r}_1) + \sum_m c_m^2 \psi_m(\mathbf{r}_2). \quad (10)$$

This representation can be justified; clearly it provides a radiating solution of the Helmholtz equation for any reasonable choice of the coefficients  $c_m^1$  and  $c_m^2$ . These coefficients will now be determined by applying the boundary condition on each cylinder.

Consider the cylinder  $S_1$ . In order to use the boundary condition on  $r_1 = a_1$ , we must express each term as a function of  $\theta_1$ . Thus, we must express  $\psi_m(\mathbf{r}_2)$  in terms of functions of  $\mathbf{r}_1$ . Now, in the neighbourhood of  $O_1$ ,  $\psi_m(\mathbf{r}_2)$  is a regular solution of the Helmholtz equation; hence, it can be expanded in terms of  $\hat{\psi}_n(\mathbf{r}_1)$ . Let  $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{b}$ , where  $\mathbf{b}$  is the position vector of  $O_1$  with respect to  $O_2$ . Then, we have

$$\psi_m(\mathbf{r}_2) = \sum_n S_{mn}(\mathbf{b}) \hat{\psi}_n(\mathbf{r}_1) \quad \text{for } r_1 < b, \quad (11)$$

this inequality coming from the fact that the left-hand side of (11) is singular when  $\mathbf{r}_1 = -\mathbf{b}$ .

The matrix  $S(\mathbf{b})$  is known explicitly; its entries are given by

$$S_{mn}(\mathbf{b}) = \psi_{m-n}(\mathbf{b}) = H_{m-n}(kb) e^{i(m-n)\beta},$$

where  $\mathbf{b} = (b \cos \beta, b \sin \beta)$ . The formula (11) is Graf's *addition theorem* in disguise [33, §11.3]. Note that  $S$  has the property

$$S_{mn}(\mathbf{b}) = S_{nm}(-\mathbf{b}) \quad \text{for all } m \text{ and } n.$$

Returning to (10), we use the addition theorem (11) to expand the scattered field near  $S_1$ . When combined with (9), we obtain

$$\begin{aligned} u &= \sum_m \left\{ d_m^1 \hat{\psi}_m(\mathbf{r}_1) + c_m^1 \psi_m(\mathbf{r}_1) \right\} + \sum_m c_m^2 \sum_n S_{mn}(\mathbf{b}) \hat{\psi}_n(\mathbf{r}_1) \\ &= \sum_m \left\{ d_m^1 \hat{\psi}_m(\mathbf{r}_1) + c_m^1 \psi_m(\mathbf{r}_1) + \hat{\psi}_m(\mathbf{r}_1) \sum_n S_{nm}(\mathbf{b}) c_n^2 \right\} \quad \text{for } r_1 < b. \end{aligned}$$

Next, we apply the sound-hard boundary condition on  $S_1$ : differentiate with respect to  $r_1$ , set  $r_1$  equal to  $a_1$  and use orthogonality of  $\{e^{im\theta_1}\}$  to give

$$c_m^1 H'_m(ka_1) + J'_m(ka_1) \sum_n S_{nm}(\mathbf{b}) c_n^2 = -d_m^1 J'_m(ka_1), \quad m = 0, \pm 1, \pm 2, \dots \quad (12)$$

Given that the other cylinder,  $S_2$ , is also sound-hard, a similar argument gives

$$c_m^2 H'_m(ka_2) + J'_m(ka_2) \sum_n S_{nm}(-\mathbf{b}) c_n^1 = -d_m^2 J'_m(ka_2), \quad m = 0, \pm 1, \pm 2, \dots \quad (13)$$

Equations (12) and (13) form a coupled infinite system of simultaneous linear algebraic equations for the coefficients  $c_m^1$  and  $c_m^2$ ,  $m = 0, \pm 1, \pm 2, \dots$

## 2.4 Multipole Method for $N$ Circular Cylinders

The method described in § 2.3 extends easily to  $N$  circular cylinders,  $S_j$ ,  $j = 1, 2, \dots, N$ . The circle  $S_j$  has radius  $a_j$  and centre  $O_j$  at  $(x, y) = (\xi_j, \eta_j)$ . As before, plane polar coordinates  $(r_j, \theta_j)$  are defined at  $O_j$ , with  $\theta_j = 0$  being in the  $x$ -direction.

The scattered field is expressed as

$$u_{\text{sc}} = \sum_{j=1}^N \sum_m c_m^j \psi_m(\mathbf{r}_j), \quad (14)$$

which is an infinite number of radiating partial waves at the centre of each circle, with unknown coefficients  $c_m^j$ .

Using (11), we find that the total field in the vicinity of  $S_l$  is given by

$$u = \sum_m \left\{ d_m^l \hat{\psi}_m(\mathbf{r}_l) + c_m^l \psi_m(\mathbf{r}_l) + \hat{\psi}_m(\mathbf{r}_l) \sum_{\substack{j=1 \\ j \neq l}}^N \sum_n S_{nm}(\mathbf{b}_{lj}) c_n^j \right\} \quad (15)$$

for  $r_l < b_l$ , where  $\mathbf{b}_{lj}$  is the position vector of  $O_l$  relative to  $O_j$  (so that  $\mathbf{b}_{jl} = -\mathbf{b}_{lj}$ ) and

$$b_l = \min_{\substack{1 \leq j \leq N \\ j \neq l}} |\mathbf{b}_{lj}|.$$

Application of the sound-hard boundary condition on  $S_l$  gives

$$c_m^l H_m^l(ka_l) + J_m^l(ka_l) \sum_{\substack{j=1 \\ j \neq l}}^N \sum_n S_{nm}(\mathbf{b}_{lj}) c_n^j = -d_m^l J_m^l(ka_l), \quad \begin{array}{l} m = 0, \pm 1, \pm 2, \dots, \\ l = 1, 2, \dots, N. \end{array} \quad (16)$$

This is an algebraic system of equations for the coefficients  $c_m^l$ . It reduces to (12) and (13) when  $N = 2$ . If the system (16) can be solved, its solution will yield an exact solution for multiple scattering by  $N$  circular cylinders — no approximations have been made.

Similar systems can be derived for sound-soft cylinders ( $u = 0$  on  $S_l$ ), and for any combination of soft and hard cylinders. Indeed, the method works for impedance boundary conditions and also for penetrable cylinders. The latter situation leads to a transmission problem, in which acoustic fields also exist inside each cylinder and are coupled to the exterior field through appropriate transmission conditions on the surfaces of each cylinder.

### 2.4.1 History of the System (16).

The multipole method, leading to the algebraic system (16), was apparently devised by Závíška [37]. In fact, Závíška treated the corresponding transmission problem. Row [21] derived the system (16) (for sound-soft cylinders) by specialising an integral equation for cylinders of arbitrary cross-section to circular cylinders.

Závíška's method has been rediscovered by many subsequent authors; see, for example, [26] and [36].

### 2.4.2 Properties of the System (16).

From (16), we see that  $c_m^l$  must be proportional to  $J'_m(ka_l)$ ; in particular,  $c_m^l$  and  $c_{-m}^l$  both vanish whenever  $ka_l$  is such that  $J'_m(ka_l) = 0$ . This suggests defining modified coefficients  $\tilde{c}_m^l$  by

$$c_m^l = \tilde{c}_m^l J'_m(ka_l),$$

whence (16) becomes

$$\tilde{c}_m^l H'_m(ka_l) + \sum_{\substack{j=1 \\ j \neq l}}^N \sum_n S_{nm}(\mathbf{b}_{lj}) J'_n(ka_j) \tilde{c}_n^j = -d_m^l, \quad \begin{matrix} m = 0, \pm 1, \pm 2, \dots, \\ l = 1, 2, \dots, N. \end{matrix} \quad (17)$$

Similarly, from (15), the total field in the vicinity of  $S_l$  is given by

$$u = \sum_m \left\{ d_m^l \hat{\psi}_m(\mathbf{r}_l) + \tilde{c}_m^l J'_m(ka_l) \psi_m(\mathbf{r}_l) + \hat{\psi}_m(\mathbf{r}_l) \sum_{\substack{j=1 \\ j \neq l}}^N \sum_n S_{nm}(\mathbf{b}_{lj}) J'_n(ka_j) \tilde{c}_n^j \right\} \quad (18)$$

for  $r_l < b_l$ . It was observed by Linton and Evans [15] that the double sum inside the braces in (18) can be eliminated, using the system (17); the result is

$$u = \sum_m \tilde{c}_m^l \left\{ J'_m(ka_l) \psi_m(\mathbf{r}_l) - H'_m(ka_l) \hat{\psi}_m(\mathbf{r}_l) \right\} \quad (19)$$

for  $r_l < b_l$ . This formula makes it much simpler to evaluate the field near  $S_l$ , once the coefficients  $\tilde{c}_m^l$  have been found. In particular, when  $r_l = a_l$ , we have

$$u(a_l, \theta_l) = \frac{-2i}{\pi ka_l} \sum_m \tilde{c}_m^l e^{im\theta_l}$$

for  $0 \leq \theta_l < 2\pi$ , after using the Wronskian relation for Bessel functions.

### 2.4.3 Numerical Solution of the System (16).

*'Without the use of large-scale automatic computing machinery, it would be impractical to compute the solutions to the system (16) [for two cylinders] for any appreciable range of radii and spacings.'*

(Row, 1955, p. 674)

Despite this pessimistic quotation, it turns out that Závřiska's method is very efficient for numerical calculations. Of course, the infinite system (16), or (17), must be truncated: for example, (17) should be truncated to the system

$$\tilde{c}_m^l H'_m(ka_l) + \sum_{\substack{j=1 \\ j \neq l}}^N \sum_{n=-M}^M S_{nm}(\mathbf{b}_{lj}) J'_n(ka_j) \tilde{c}_n^j = -d_m^l, \quad \begin{matrix} m = 0, \pm 1, \pm 2, \dots, \pm M, \\ l = 1, 2, \dots, N. \end{matrix}$$

This is a system of  $N(2M + 1)$  equations in  $N(2M + 1)$  unknowns. However, simplifications are possible if symmetry can be exploited.

Row [21] was the first to solve (16) numerically. He considered two identical cylinders, of radius  $a$ , centred at  $(x, y) = (0, \pm \frac{1}{2}b)$ ; the incident field was generated by a line-source at  $(x, y) = (x_0, 0)$ . For this symmetric configuration, we have

$$c_n^1 = c_{-n}^2.$$

Thus, the truncated form of (12) appropriate to sound-soft cylinders reduces to

$$c_m^1 H_m(ka) + J_m(ka) \sum_{n=-M}^M S_{nm}(\mathbf{b}) c_{-n}^1 = -d_m^1 J_m(ka), \quad m = 0, \pm 1, \pm 2, \dots, \pm M. \quad (20)$$

This is a system of  $(2M + 1)$  equations in  $(2M + 1)$  unknowns. Row [21] solved it for  $ka = 2$  and  $b/a = \pi$  with  $M = 6$  (his Fig. 6). As the quotation above suggests, this was a formidable calculation (using desk calculators and three assistants) in 1953. Moreover, the results were shown to be in excellent agreement with experiments.

Linton and Evans [15] found that taking  $M = 6$  gave results accurate to four significant figures, except when the cylinders were very close together.

#### 2.4.4 Extensions of the Method.

The general idea of combining multipole expansions with appropriate addition theorems has been used in many physical contexts, involving configurations of cylinders and spheres. To give some flavour of this work, we mention the paper by Levine and Olaofe [14], correcting the earlier work of Trinks [27] on electromagnetic scattering by two dielectric spheres, and papers on the acoustics of bubbly liquids [23], interactions in a thermoviscous fluid [10], and gas bubbles in a solid [35].

### 3 Methods based on the $T$ -matrix

Consider one cylinder with non-circular cross-section  $\Omega$ ; let  $S$  be the boundary of  $\Omega$ . Choose an origin  $O$  in  $\Omega$ , and let  $C_+$  and  $C_-$  be the escribed and inscribed circles, respectively, to  $S$ , centred on  $O$ . Outside  $C_+$ , the expansion (6) for  $u_{sc}$  is valid, whereas the expansion (5) for  $u_{inc}$  is certainly valid inside  $C_-$ . Then, the  $T$ -matrix for the scatterer is defined by (7): it maps the (known) coefficients  $\{d_n\}$  into the (unknown) coefficients  $\{c_m\}$ .

The  $T$ -matrix has certain properties (such as symmetry) which can be either used as a check on any numerical scheme for its computation or incorporated into the numerical scheme itself [31] [32].

The  $T$ -matrix depends on the shape of  $S$  and on the frequency, but it is independent of the incident field. It is this property that makes it useful as a ‘building block’ for multiple-scattering problems; the other ingredient is the matrix  $S$  occurring in the addition theorem (11). The basic reference is [20]. Many subsequent applications are listed in [29]. We also mention some applications in hydrodynamics [16], [12] and geophysics [6]. There is no doubt that  $T$ -matrix methods are very effective for multiple-scattering problems, even when there are many scatterers; for example, Wang and Chew [30] have used an iterative variant for electromagnetic scattering by 6859 dielectric spheres.

### 4 Methods based on Integral Equations

Integral-equation methods for scattering by a single obstacle are well known [8]. Here, we outline such methods for two obstacles, using either the free-space Green’s function or an exact Green’s function.

#### 4.1 One Non-Circular Cylinder, Two Approaches

We consider one cylinder with non-circular cross-sectional boundary  $S$ . Thus, the problem is to solve (1) in  $D$ , the unbounded region exterior to  $S$ , subject to a radiation condition at infinity and

the sound-hard boundary condition on  $S$  (4); we write the latter as

$$\partial u_{\text{sc}}/\partial n = f \quad \text{on } S, \quad (21)$$

where  $f = -\partial u_{\text{inc}}/\partial n$  is known.

#### 4.1.1 Standard Boundary Integral Equation.

We start with a fundamental solution for (1). The simplest is the free-space Green's function,

$$G(P, Q) = G(Q, P) = -\frac{1}{2}iH_0^{(1)}(kR),$$

where  $R$  is the distance between the two points  $P$  and  $Q$ . Then, an application of Green's theorem to  $u_{\text{sc}}(Q)$  and  $G(Q, P)$ , for fixed  $P$ , gives

$$2u_{\text{sc}}(P) = \int_S \left( f(q) G(q, P) - u_{\text{sc}}(q) \frac{\partial}{\partial n_q} G(q, P) \right) ds_q. \quad (22)$$

This is an integral representation for  $u_{\text{sc}}(P)$ , with  $P \in D$ , in terms of the boundary values of  $u_{\text{sc}}$ . To determine  $u_{\text{sc}}(q)$  for  $q \in S$ , let  $P$  go to  $p$  on  $S$  giving the familiar integral equation,

$$u_{\text{sc}}(p) + \int_S u_{\text{sc}}(q) \frac{\partial}{\partial n_q} G(q, p) ds_q = \int_S f(q) G(q, p) ds_q.$$

This is a Fredholm integral equation of the second kind with a continuous kernel; we write it concisely as

$$Au_{\text{sc}} = S_0 f, \quad (23)$$

where  $A$  and  $S_0$  are integral operators. One can also obtain a similar equation for the total field on  $S$ , namely

$$Au = 2u_{\text{inc}}. \quad (24)$$

It is known that (23) and (24) are uniquely solvable, except at certain irregular values of  $k^2$  [8]. We disregard these here (several methods for their elimination, leading to different integral equations are available [8]). Thus, formally, we can solve (23):

$$u_{\text{sc}} = A^{-1} S_0 f. \quad (25)$$

In practice, we cannot find  $A^{-1}$  analytically. However, we can solve (23) numerically using a boundary-element method. This gives a discrete approximation to  $A^{-1}$ .

#### 4.1.2 The Exact Green's Function.

Let us introduce a different fundamental solution  $G^{\text{E}}$ , defined as follows. Fix the point  $P$ . Then, write

$$G^{\text{E}}(Q; P) = G(Q, P) + w(Q; P),$$

and choose  $w$  so that it (i) satisfies (1) for all  $Q \in D$ , (ii) satisfies the radiation condition, and (iii) is such that  $G^{\text{E}}$  satisfies

$$\frac{\partial}{\partial n_q} G^{\text{E}}(q; P) = 0 \quad \text{for } q \in S. \quad (26)$$

We call  $G^{\text{E}}$  the *exact Green's function*; Bergman and Schiffer [2] call it the *Neumann function*. One can find detailed discussions of (exact) Green's functions in older books on partial differential

equations, such as those of Kellogg [13, Chpt. 9], Webster [34, §66] or Garabedian [9, Chpt. 7]. Like  $G$ ,  $G^E$  is *symmetric*:

$$G^E(P; Q) = G^E(Q; P) \quad \text{for all } P \text{ and } Q \text{ in } D \cup S, P \neq Q. \quad (27)$$

Since  $G^E$  is a fundamental solution, we can use it to derive the integral representation (22), with  $G$  replaced by  $G^E$ . However, the representation simplifies because of (26), giving

$$2u_{sc}(P) = \int_S f(q) G^E(q; P) ds_q. \quad (28)$$

This is an explicit formula for the solution of the one-obstacle problem. In particular, for  $p \in S$ , it gives

$$u_{sc}(p) = \frac{1}{2} \int_S f(q) G^E(q; p) ds_q = \frac{1}{2} S_0^E f,$$

say. Comparison of this formula with (25) gives

$$A^{-1} S_0 f = \frac{1}{2} S_0^E f. \quad (29)$$

As this holds for every  $f$ , we deduce that

$$A^{-1} S_0 = \frac{1}{2} S_0^E. \quad (30)$$

Moreover, (29) gives  $AS_0^E f = 2S_0 f$ , which implies that  $G^E$  solves the integral equation

$$G^E(q; p) + \int_S G^E(l; p) \frac{\partial}{\partial n_l} G(l, q) ds_l = 2G(q, p). \quad (31)$$

In this equation, the point  $p$  occurs as a parameter; indeed, the same equation holds when  $p$  is replaced by  $P \in D$ . The idea of constructing  $G^E$  by solving a boundary integral equation can be found in a paper by Boley [5].

In summary, if we want to find  $G^E$  for a particular geometry, we typically have to solve a boundary integral equation such as (31): we have shown above that this is equivalent to calculating  $A^{-1}$ .

## 4.2 Multiple Scattering by Two Obstacles

In this section, we consider the same scattering problem as in the preceding section but with two cylinders. Thus, the problem is to solve (1) in  $D$ , the unbounded region exterior to  $S_1$  and  $S_2$ , subject to a radiation condition and the boundary conditions

$$\partial u_{sc} / \partial n = f_j \quad \text{on } S_j, j = 1, 2, \quad (32)$$

where  $f_j = -\partial u_{inc} / \partial n$  on  $S_j$ .

We describe two methods for solving this problem. First, we derive a pair of coupled boundary integral equations using  $G$ , in a standard way; these equations are weighted equally between  $S_1$  and  $S_2$ . In practice, we may already have information on how to scatter by one of the cylinders ( $S_2$ , say) in isolation, such as  $A^{-1}$ ,  $G^E$  or one of their discrete approximations. One way to use this information is to ‘partition’ the pair of integral equations; another is to replace  $G$  by  $G^E$ . We can prove that these two approaches lead to exactly the same equations. A third approach is to assume that we have two exact Green’s functions, one for each scatterer; this leads naturally to the generalized Born series, discussed later.

### 4.2.1 Partitioning.

The method used above to derive the boundary integral equation (23) works equally well for two obstacles: simply replace  $S$  by  $S_1 \cup S_2$ . The resulting equation can be written in the following form:

$$A_{11}u_1 + A_{12}u_2 = S_{11}f_1 + S_{12}f_2, \quad (33)$$

$$A_{21}u_1 + A_{22}u_2 = S_{21}f_1 + S_{22}f_2. \quad (34)$$

Here,  $u_j = u_{\text{sc}}(p_j)$  where  $p_j \in S_j$  for  $j = 1, 2$ ,

$$A_{jk}u_k = \delta_{jk}u_{\text{sc}}(p_k) + \int_{S_k} u_{\text{sc}}(q_k) \frac{\partial}{\partial n_q} G(q_k, p_j) ds_q \quad (35)$$

$$S_{jk}f_k = \int_{S_k} f_k(q_k) G(q_k, p_j) ds_q \quad (36)$$

and  $\delta_{ij}$  is the Kronecker delta. We note that  $A_{jj}$  is simply the operator  $A$  for  $S_j$  (both the field point  $p_j$  and the source point (integration point)  $q_j$  are on  $S_j$ ), whereas  $A_{12}$  and  $A_{21}$  give the interactions between  $S_1$  and  $S_2$  (these integral operators have smooth kernels, as the field and source points are on different surfaces).

It is known that the pair of integral equations (33) and (34) suffers from irregular frequencies. Again, we disregard these here; for a discussion on methods for eliminating irregular frequencies from such equations, and for references to numerical work, see [17].

Now, suppose we already have (a discrete approximation to)  $A_{22}^{-1}$ ; for example, we may have solved (23) using an accurate boundary-element method. Then, (34) gives

$$u_2 = A_{22}^{-1} \{S_{21}f_1 + S_{22}f_2 - A_{21}u_1\}. \quad (37)$$

Eliminating  $u_2$  from (33), we obtain

$$\mathcal{A}_{11}u_1 = S_{11}f_1 + S_{12}f_2, \quad (38)$$

where

$$\mathcal{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad (39)$$

$$S_{1j} = S_{1j} - A_{12}A_{22}^{-1}S_{2j}, \quad j = 1, 2. \quad (40)$$

Equation (38) is an integral equation to solve for  $u$  on  $S_1$ . We shall return to it later.

We could view partitioning as merely a method for solving systems of linear algebraic equations. However, we shall argue later that it is profitable to view partitioning as arising directly from partitions of the boundary.

### 4.2.2 Use of the Exact Green's Function.

Suppose that  $G^{\text{E}}$  is the exact Green's function for  $S_2$  (in isolation). Proceeding as for the one-obstacle problem, using  $G^{\text{E}}$  for our chosen fundamental solution, we obtain the following integral representation for the two-obstacle problem:

$$2u_{\text{sc}}(P) = \int_{S_1} \left( f_1(q_1) G^{\text{E}}(q_1; P) - u_{\text{sc}}(q_1) \frac{\partial}{\partial n_q} G^{\text{E}}(q_1; P) \right) ds_q + \int_{S_2} f_2(q_2) G^{\text{E}}(q_2; P) ds_q.$$

This representation does not involve the unknown boundary values of  $u$  on  $S_2$ . To find  $u$  on  $S_1$ , we let  $P \rightarrow p_1 \in S_1$ , as usual; the result can be written as

$$A_{11}^E u_1 = S_{11}^E f_1 + S_{12}^E f_2, \quad (41)$$

where

$$A_{11}^E u_1 = u_{sc}(p_1) + \int_{S_1} u_{sc}(q_1) \frac{\partial}{\partial n_q} G^E(q_1, p_1) ds_q \quad (42)$$

$$S_{1j}^E f_j = \int_{S_j} f_j(q_j) G^E(q_j, p_1) ds_q, \quad j = 1, 2. \quad (43)$$

### 4.2.3 Comparison.

We have two boundary integral equations for  $u$  on  $S_1$ , namely (38) and (41). It turns out that these equations are identical: Martin and Rizzo [18] give direct proofs that

$$\mathcal{A}_{11} = A_{11}^E, \quad \mathcal{S}_{11} = S_{11}^E, \quad \text{and} \quad \mathcal{S}_{12} = S_{12}^E. \quad (44)$$

Thus, the two methods are equivalent, although the introduction of exact Green's functions has pedagogical advantages, at least.

### 4.2.4 Generalized Born Series.

The standard integral equations, (33) and (34), can be written as

$$\begin{aligned} A_{11} u_1 &= F_1 - A_{12} u_2, \\ A_{22} u_2 &= F_2 - A_{21} u_1, \end{aligned}$$

where  $F_j = S_{j1} f_1 + S_{j2} f_2$ . The generalized Born series [25], [24] is a method for solving this pair iteratively in the context of multiple scattering by two obstacles (actually, it can be recognised as the block Jacobi method for linear algebraic equations): assuming that  $A_{11}$  and  $A_{22}$  are non-singular, construct  $u_j^{(m)}$  according to

$$\begin{aligned} u_1^{(m+1)} &= A_{11}^{-1} \{F_1 - A_{12} u_2^{(m)}\}, \\ u_2^{(m+1)} &= A_{22}^{-1} \{F_2 - A_{21} u_1^{(m)}\}, \end{aligned}$$

with  $u_1^{(0)} = u_2^{(0)} = 0$ . Eliminating  $u_2^{(m)}$ , we obtain

$$u_1^{(m+1)} = g_1 + B_{11} u_1^{(m-1)} \quad (45)$$

for  $m = 1, 2, \dots$ , where

$$g_1 = A_{11}^{-1} \{F_1 - A_{12} A_{22}^{-1} F_2\} \quad \text{and} \quad B_{11} = A_{11}^{-1} A_{12} A_{22}^{-1} A_{21}.$$

Hence,

$$u_1^{(2M)} = \sum_{m=0}^{M-1} (B_{11})^m g_1 \quad \text{for } M = 1, 2, \dots \quad (46)$$

Also, since  $u_1^{(1)} = A_{11}^{-1}F_1$ , (45) gives

$$u_1^{(2M+1)} = u_1^{(2M)} + (B_{11})^M A_{11}^{-1}F_1 \quad \text{for } M = 1, 2, \dots \quad (47)$$

The (geometric) series in (46) and (47) converges if

$$\|B_{11}\| < 1, \quad (48)$$

with any reasonable norm. Under this condition (which will be satisfied if the two scatterers are sufficiently far apart), the last term in (47) tends to zero and both sequences ( $\{u_1^{(2M)}\}$  and  $\{u_1^{(2M+1)}\}$ ) converge to  $u_1^{(\infty)}$ , say, where

$$\begin{aligned} u_1^{(\infty)} &= (I - B_{11})^{-1}g_1 = \mathcal{A}_{11}^{-1}A_{11}g_1 \\ &= \mathcal{A}_{11}^{-1} \{F_1 - A_{12}A_{22}^{-1}F_2\} \\ &= \mathcal{A}_{11}^{-1} \{\mathcal{S}_{11}f_1 + \mathcal{S}_{12}f_2\}, \end{aligned}$$

which is the solution of the partitioning equation (38). This latter equation is not subject to the condition (48).

Note that the generalized Born series requires a knowledge of both  $A_{11}^{-1}$  and  $A_{22}^{-1}$ ; this is equivalent to knowing two exact Green's functions, one for each obstacle. Rudgers [22] has used the sum of these two exact Green's functions as a fundamental solution, followed by an iterative method.

## 5 Conclusions

There are no conclusions! We have tried to describe several methods for solving basic multiple-scattering problems. Of course, there are many problems that we have not touched, such as scattering by rough surfaces or the determination of effective dynamic properties of composite materials. We have pointed out that the theory of multiple scattering has a century of literature. It is our contention that this provides a rich source of information that should be exploited for the benefit of current computational goals.

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