Diffraction by Non-Planar Cracks*

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Abstract

A classic problem in elastodynamics is the diffraction of time-harmonic stress waves by a penny-shaped crack. It is also a benchmark problem, in that it is the simplest three-dimensional problem for bounded cracks. What happens if the flat circular crack is perturbed? This question motivates the present paper. Recent work is discussed, in which a perturbation theory is developed, based on the governing hypersingular integral equation.

1 Introduction

Consider a thin flat circular disc D, embedded in a three-dimensional medium. Various physical settings lead to similar exterior Neumann problems for this geometry. Examples are potential flow past a rigid disc (Laplace's equation), acoustic scattering by a sound-hard disc (Helmholtz's equation) or elasticity problems for penny-shaped cracks. Such problems can be treated using more-or-less classical methods; exact solutions are available for static problems.

Now, imagine that the disc is perturbed in some way, into Ω (assumed to be a smooth, open, simply-connected surface with a smooth edge). The disc might be enlarged in its own plane, or it might be wrinkled. How does this small change in the geometry affect the corresponding fields? For example, if Ω is a crack, how will the stress-intensity factors change?

In one sense, this is a singular perturbation: the places where the fields are singular (namely the edges) have moved. But in another sense, the perturbation is regular: the fields on Ω will not be too different from those on D. This means that trying to perturb the boundary-value problem (as is done in water-wave theory, for example, when the free-surface boundary conditions on the actual free surface are linearized about a mean flat surface) is not advisable, but working with the associated boundary integral equations is relatively straightforward.

For problems involving thin plates or cracks, we inevitably encounter hypersingular integral equations. For wrinkled discs, our strategy is as follows:

1. derive an exact integral representation for the field u exterior to Ω;
2. derive an exact hypersingular integral equation for $[u]$, the discontinuity in u across Ω;
3. project this integral equation onto a reference surface D; if Ω is given by $z = F(x, y)$ with $(x, y) \in D$, projection amounts to substitution, and leads to a new exact integral equation over D;
4. now, at this late stage, assume that $F = \varepsilon f$, where $0 < \varepsilon \ll 1$, and look for solutions for $[u]$ in the form of a regular perturbation expansion;

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5. group terms leading to a sequence of hypersingular integral equations over D, one for each term, which can then be treated by one of the known methods for flat discs.

We have developed this method for potential flow past a wrinkled disc in [5]. Here, we outline a similar method for acoustic scattering by a wrinkled disc; this is a scalar prototype for the more important elastic problems. We have previously treated in-plane perturbations (so that Ω is flat but non-circular) for pressurized [4] and shear-loaded [3] cracks. The elastic problems are important because of their relevance to fracture prediction.

2 Formulation

We consider acoustic scattering by a thin rigid screen Ω surrounded by a compressible fluid; we model the screen as a smooth simply-connected bounded surface with a smooth edge $\partial\Omega$. Thus, the problem is to solve the Helmholtz equation in three dimensions, $\nabla^2 u + k^2 u = 0$, subject to

$$\partial u/\partial n + \partial u_{\text{inc}}/\partial n = 0 \quad \text{on } \Omega \tag{1}$$

and the Sommerfeld radiation condition at infinity, where $k \geq 0$, u_{inc} is the given incident field and $\partial/\partial n$ denotes normal differentiation. For an incident plane wave, we have

$$u_{\text{inc}}(x, y, z) = \exp\{\mathrm{i}k(x \sin\beta - z\cos\beta)\}, \tag{2}$$

where β is a given constant.

Denote the two sides of Ω by Ω^+ and Ω^-, and define the unit normal vector on Ω, $\mathbf{n}$, to point from Ω^+ into the fluid. Finally, define the discontinuity in u across Ω by

$$[u(q)] = \lim_{Q\to q^+} u(Q) - \lim_{Q\to q^-} u(Q)$$

where $q \in \Omega$, $q^\pm \in \Omega^\pm$ and Q is a point in the fluid. Then, we have the integral representation

$$u(P) = \frac{1}{4\pi}\int_\Omega [u(q)]\,\frac{\partial}{\partial n_q} G(P, q)\,\mathrm{d}S_q, \tag{3}$$

where $G(P, q) = \mathcal{R}^{-1}\exp(\mathrm{i}k\mathcal{R})$ is the usual free-space Green's function, $\mathcal{R} = |\mathbf{r} - \mathbf{q}|$, $q \in \Omega$ has position vector $\mathbf{q}$ with respect to an origin O, and P has position vector $\mathbf{r}$ and spherical polar coordinates (r, θ, φ).

To be more explicit, we suppose that the surface Ω is given by

$$\Omega \;:\; z = F(x, y), \qquad (x, y) \in D,$$

where D is the *unit disc* in the xy-plane. We define a normal vector to Ω by

$$\mathbf{N} = (-\partial F/\partial x,\, -\partial F/\partial y,\, 1),$$

and then $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ is a unit normal vector. Suppose that P and $q \in \Omega$ are at (x_0, y_0, z_0) and (x, y, z), respectively. Let

$$[u(q)] = w(x, y).$$

Then, we find that (3) becomes, exactly,

$$u(x_0, y_0, z_0) = \frac{1}{4\pi}\int_D w(x, y)\,(\mathbf{N}(q)\cdot\mathbf{R}_2)(1 - \mathrm{i}kR_2)\,\mathrm{e}^{\mathrm{i}kR_2}\,\frac{\mathrm{d}A}{R_2^3}$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, $\mathbf{R}_2 = (x_0 - x,\, y_0 - y,\, z_0 - F(x,y))$ and $R_2 = |\mathbf{R}_2|$.

In the far field, we have

$$u(P) \sim r^{-1} \mathrm{e}^{\mathrm{i}kr}\, \mathcal{F}(\hat{\mathbf{r}}) \qquad \text{as } r \to \infty,$$

where $\mathbf{r} = r\hat{\mathbf{r}}$, $\hat{\mathbf{r}} = (\sin\theta\, \cos\varphi,\, \sin\theta\, \sin\varphi,\, \cos\theta)$,

$$\mathcal{F}(\hat{\mathbf{r}}) = \frac{-\mathrm{i}k}{4\pi} \int_D w(x,y)\, \{\hat{\mathbf{r}} \cdot \mathbf{N}(q)\}\, \exp\left(-\mathrm{i}k\mathbf{q} \cdot \hat{\mathbf{r}}\right) \mathrm{d}A \tag{4}$$

and $\mathbf{q} = (x, y, F(x,y))$; $\mathcal{F}$ is the *far-field pattern.* The formula (4) is exact. Although the integration is over a flat disc, the geometry enters through w, $\mathbf{N}$ and $\mathbf{q}$.

The next stage is to obtain an approximation to w; $\mathcal{F}$ can then be found from (4) without further approximation.

3 A Hypersingular Integral Equation

Application of the boundary condition (1) to (3) gives

$$\frac{1}{4\pi} \,⨎_{\Omega} [u(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p,q)\, \mathrm{d}S_q = -\frac{\partial u_{\mathrm{inc}}}{\partial n_p}, \qquad p \in \Omega, \tag{5}$$

where the integral must be interpreted in the finite-part sense. Equation (5) is the governing hypersingular integral equation for $[u]$; it is to be solved subject to the edge condition

$$[u(q)] = 0 \text{ for all } q \in \partial\Omega.$$

Projecting onto D, (5) becomes

$$\frac{1}{4\pi} \,⨎_D K(x_0, y_0;\, x, y)\, w(x,y)\, \mathrm{d}A = b(x_0, y_0), \qquad (x_0, y_0) \in D, \tag{6}$$

where

$$\begin{aligned} K = {} & R_1^{-3}(1 - \mathrm{i}kR_1)\, \mathrm{e}^{\mathrm{i}kR_1} \{\mathbf{N}(p) \cdot \mathbf{N}(q)\} \\ & - 3R_1^{-5}(1 - \mathrm{i}kR_1 - \tfrac{1}{3}k^2R_1^2)\, \mathrm{e}^{\mathrm{i}kR_1} (\mathbf{N}(p) \cdot \mathbf{R}_1)(\mathbf{N}(q) \cdot \mathbf{R}_1), \end{aligned}$$

$\mathbf{R}_1 = (x - x_0,\, y - y_0,\, F(x,y) - F(x_0,y_0))$, $R_1 = |\mathbf{R}_1|$ and

$$b(x,y) = -\partial u_{\mathrm{inc}}/\partial N = \mathrm{i}k \left\{\cos\beta + (\partial F/\partial x)\sin\beta\right\} \mathrm{e}^{\mathrm{i}k(x \sin\beta - F\cos\beta)} \tag{7}$$

when u_{inc} is given by (2). Equation (6) is to be solved subject to the edge condition

$$w(x,y) = 0 \text{ for } r = \surd(x^2 + y^2) = 1.$$

Let

$$F_1 = \partial F/\partial x \quad \text{and} \quad F_2 = \partial F/\partial y \quad \text{evaluated at } (x,y), \tag{8}$$

with F_1^0 and F_2^0 being the corresponding quantities at (x_0, y_0). Then $\mathbf{N}(q) = (-F_1, -F_2, 1)$ and $\mathbf{N}(p) = (-F_1^0, -F_2^0, 1)$. Let $R = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}$ and $\Lambda = \{F(x,y) - F(x_0,y_0)\}/R$. Also, define the angle Θ by

$$x - x_0 = R\cos\Theta \quad \text{and} \quad y - y_0 = R\sin\Theta,$$

whence $\mathbf{R}_1 = R(\cos\Theta,\, \sin\Theta,\, \Lambda)$. Hence

$$K = \frac{\mathrm{e}^{\mathrm{i}kRX}}{R^3}\left\{\frac{1-\mathrm{i}kRX}{X^3}(1+F_1F_1^0+F_2F_2^0) - \frac{3Y}{X^5}(1-\mathrm{i}kRX-\tfrac{1}{3}(kRX)^2)\right\}, \qquad (9)$$

where $X = \sqrt{1+\Lambda^2}$ and

$$Y = (F_1\cos\Theta + F_2\sin\Theta - \Lambda)(F_1^0\cos\Theta + F_2^0\sin\Theta - \Lambda).$$

This formula for K is exact. If we expand K for small R about p, we find that

$$K \sim R^{-3}\sigma(p;\Theta)$$

where

$$\sigma(p;\Theta) = \frac{1+(F_1^0)^2+(F_2^0)^2}{1+(F_1^0\cos\Theta+F_2^0\sin\Theta)^2}.$$

In particular, $\sigma \equiv 1$ when F is constant. Thus, for non-constant F, the singularity in the kernel of the integral equation (6) is essentially different from that occurring in the integral equation for constant F. A similar phenomenon was noted previously [3, 4], when the integral equation for a *flat* but *non-circular* Ω was mapped onto the unit disc D. In that case, the difficulty was resolved by using a *conformal mapping*. Here, we are projecting onto D, so that the mapping from Ω onto D is prescribed. However, we can make progress by supposing that Ω is *almost* flat.

4 Wrinkled Discs

Suppose that

$$F(x,y) = \varepsilon f(x,y)$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda \quad \text{with} \quad \lambda = \{f(x,y)-f(x_0,y_0)\}/R, \qquad (10)$$

we find that

$$K = R^{-3}\mathrm{e}^{\mathrm{i}kR}\{1-\mathrm{i}kR+\varepsilon^2K_2+O(\varepsilon^4)\} \quad \text{as } \varepsilon \to 0,$$

where

$$\begin{aligned} K_2 = {}&(1-\mathrm{i}kR)(f_1f_1^0+f_2f_2^0-\tfrac{3}{2}\lambda^2)+\tfrac{1}{2}\lambda^2(kR)^2 \\ &-3(1-\mathrm{i}kR-\tfrac{1}{3}(kR)^2)(f_1\cos\Theta+f_2\sin\Theta-\lambda)(f_1^0\cos\Theta+f_2^0\sin\Theta-\lambda) \end{aligned}$$

and f_j, f_j^0 are defined similarly to F_j; see (8).

We expand b similarly. For an incident plane wave, (7) gives

$$b(x,y) = \mathrm{i}k(b_0(x,y)+\varepsilon b_1(x,y)+\cdots),$$

where, for example, $b_0(x,y) = \mathrm{e}^{\mathrm{i}kx\sin\beta}\cos\beta$.

Then, if we expand w as

$$w(x,y) = \mathrm{i}k(w_0+\varepsilon w_1+\varepsilon^2 w_2+\cdots),$$

we find from (6) that

$$H_k w_0 = b_0, \quad H_k w_1 = b_1 \quad \text{and} \quad H_k w_2 = b_2 - \mathcal{K}_2 w_0,$$

where

$$(H_k w)(x_0, y_0) = \frac{1}{4\pi} \oint_D w(x,y)\,(1 - \mathrm{i}kR)\,\mathrm{e}^{\mathrm{i}kR} \frac{\mathrm{d}A}{R^3}$$

is the basic hypersingular operator for acoustic scattering by a sound-hard circular disc and

$$(\mathcal{K}_2 w)(x_0, y_0) = \frac{1}{4\pi} \oint_D K_2(x, y;\, x_0, y_0)\, w(x,y)\, \mathrm{e}^{\mathrm{i}kR} \frac{\mathrm{d}A}{R^3}.$$

Thus, we have a sequence of hypersingular integral equations, $H_k w_n = f_n$, to solve.

5 Discussion

When $k = 0$ (potential flow), we have obtained w_0, w_1 and w_2 explicitly for particular geometries, namely inclined elliptical discs and spherical caps [5]. The results obtained agree with known exact results. Extension to elastostatic problems is in progress.

For $k > 0$, we can see that w_0 is simply the solution for a flat disc; however, the far-field pattern will be different, it being given by (4) with w replaced by w_0. It should be possible to obtain w_1 without too much difficulty, as $f_1 = b_1$ is simple. For higher-order terms, one would have to evaluate $\mathcal{K}_2 w$. Nevertheless, low-frequency results may be accessible.

There are few results to compare with. Jansson [2] has given some results for scattering by wrinkled discs, but his method is defective [5]. For spherical caps, there are the exact results of Thomas [6]. Numerical results for elastic-wave scattering by cracks in the shape of spherical and spheroidal caps have been given by Boström and Olsson [1]. It remains to be seen whether the present method is effective for such elastodynamic problems.

References

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