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Diffraction of elastic waves by a penny-shaped crack

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Consider an infinite elastic solid containing a penny-shaped crack. We suppose that time-harmonic elastic waves are incident on the crack and are required to determine the scattered displacement field u_i. In this paper, we describe a new method for solving the corresponding linear boundary-value problem for u_i, which we denote by S. We begin by defining an 'elastic double layer'; we prove that any solution of S can be represented by an elastic double layer whose 'density' satisfies certain conditions. We then introduce various Green functions and define a new crack Green function, G_{ij}, that is discontinuous across the crack. Next, we use G_{ij} to derive a Fredholm integral equation of the second kind for the discontinuity in u_i across the crack. We prove that this equation always has a unique solution. Hence, we are able to prove that the original boundary-value problem S always possesses a unique solution, and that this solution has an integral representation as an elastic double layer whose density solves an integral equation of the second kind.

1. INTRODUCTION

Consider a three-dimensional elastic solid of unbounded extent, containing a finite crack that occupies a surface γ; the two faces of the crack are labelled γ^+ and γ^-. Suppose that time-harmonic stress waves, of frequency ω, are incident on the crack. We are required to determine the scattered waves when the faces of the crack are free from applied tractions. Let us denote the scattered displacements and stresses by u_i and τ_{ij}, respectively, where a harmonic time-dependent of $e^{-i\omega t}$ will be suppressed throughout. We can formulate the following boundary-value problem for u_i.

Boundary-value problem $S(u_i^{(i)})$. Determine $u_i(P)$, $P \in D$, the region exterior to γ, satisfying

(S1) elastodynamic equations of motion in the solid,

$$\partial\tau_{ij}(P)/\partial x_j + \rho_0\omega^2 u_i(P) = 0, \quad P \in D;$$

(S2) boundary conditions on the crack faces,

$$n_j(\mathrm{q}^\pm)\tau_{ij}(\mathrm{q}) = -n_j(\mathrm{q}^\pm)\tau_{ij}^{(\mathrm{i})}(\mathrm{q}), \quad \mathrm{q}^\pm \in \gamma^\pm,$$

where $u_i^{(\mathrm{i})}$ and $\tau_{ij}^{(\mathrm{i})}$ denote the incident displacements and stresses, respectively;†

(S3) radiation conditions (we shall specify these later); and

(S4) edge conditions,

$$u_i(\mathrm{P}) = O(1) \quad \text{as} \quad s \to 0,$$

where s is the shortest distance between the point P and the crack edge.

We use the following notation: capital letters P, Q denote points of D; small letters p, q denote points of γ; q^+, q^- and q^e denote points on γ^+, γ^- and $\partial\gamma$, respectively, where $\partial\gamma$ is the crack edge; the origin of Cartesian coordinates O is taken at a point of γ; an arbitrary point $\mathrm{P} \in D$ has coordinates $(x_1, x_2, x_3) = (x, y, z)$; and r_P is the length OP.

The stress tensor τ_{ij} is related to u_i by

$$\tau_{ij}(\mathrm{P}) = c_{ijkl}\,\partial u_l(\mathrm{P})/\partial x_k, \tag{1.1}$$

c_{ijkl} are the material moduli, ρ_0 is the mass density of the solid, and $\boldsymbol{n}$ is the unit normal vector, which is assumed to point into D. Note that $u_i^{(\mathrm{i})}(\mathrm{P})$ must satisfy (S1) in $D \cup \gamma$. Also, the total displacement at a point of D is $u_i(\mathrm{P}) + u_i^{(\mathrm{i})}(\mathrm{P})$. Henceforth, we shall always consider the elastic solid to be homogeneous and isotropic, whence the material moduli are given by

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

and (S1) becomes

$$k^{-2}\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - K^{-2}\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u} + \boldsymbol{u} = 0, \tag{1.2}$$

where the wavenumbers k and K are defined by

$$\rho_0\omega^2 = (\lambda + 2\mu)k^2 = \mu K^2,$$

and λ and μ are the Lamé constants, related to Poisson's ratio ν by

$$2\nu = \lambda/(\lambda+\mu) = (K^2 - 2k^2)/(K^2 - k^2).$$

To give a precise statement of the radiation conditions, we follow Kupradze (1963) and decompose u_i by writing

$$u_i(\mathrm{P}) = u_i^\mathrm{p}(\mathrm{P}) + u_i^\mathrm{s}(\mathrm{P}),$$

where u_i^p and u_i^s are determined uniquely from (1.2) as

$$u_i^\mathrm{p}(\mathrm{P}) = -\frac{1}{k^2}\frac{\partial^2 u_j(\mathrm{P})}{\partial x_i \partial x_j}, \quad u_i^\mathrm{s}(\mathrm{P}) = u_i(\mathrm{P}) - u_i^\mathrm{p}(\mathrm{P}), \tag{1.3}$$

† In linear elastodynamics, these boundary conditions are applied to the reference state of the solid. Therefore, it is not possible in this formulation to ensure that the crack faces do not intersect during the motion, unless the crack can be opened sufficiently by superposing a suitable static loading at infinity; for a flat crack, a simple tension perpendicular to the crack plane is adequate.

and satisfy

$$(\nabla^2+k^2)\,u_i^{\mathrm{p}}(\mathrm{P}) = 0, \quad (\nabla^2+K^2)\,u_i^{\mathrm{s}}(\mathrm{P}) = 0. \tag{1.4}$$

We take the radiation conditions to be (Eringen & Suhubi 1975, §5.13)

$$\left.\begin{aligned} r_{\mathrm{P}}\{\partial u_i^{\mathrm{p}}(\mathrm{P})/\partial r_{\mathrm{P}} - \mathrm{i}k u_i^{\mathrm{p}}(\mathrm{P})\} \to 0, \quad u_i^{\mathrm{p}}(\mathrm{P}) \to 0, \\ r_{\mathrm{P}}\{\partial u_i^{\mathrm{s}}(\mathrm{P})/\partial r_{\mathrm{P}} - \mathrm{i}K u_i^{\mathrm{s}}(\mathrm{P})\} \to 0, \quad u_i^{\mathrm{s}}(\mathrm{P}) \to 0, \end{aligned}\right\} \tag{1.5}$$

as $r_{\mathrm{P}} \to \infty$. Kupradze has shown that these are sufficient to ensure that S has at most one solution when γ is a (closed) Lyapunov surface. The corresponding uniqueness theorem for regular surfaces (in the sense of Kellogg (1929)) has recently been proved by Wickham (1981). In particular, Wickham's uniqueness theorem holds for open, smooth surfaces, across which the displacement may be discontinuous, provided that the edge conditions (S4) are satisfied.

In what follows, we shall be concerned solely with flat, circular cracks ('penny-shaped' cracks); for this particular geometry, we define cylindrical polar coordinates (r, θ, z), where the region γ is $z = 0$, $r < 1$, $0 \leqslant \theta < 2\pi$, and an arbitrary point $\mathrm{q} \in \gamma$ will always be assigned plane polar coordinates (r, θ). However, parts of the sequel may be generalized to arbitrary (smooth) cracks without difficulty.

In this paper, we shall present a new method for solving $\mathrm{S}(u_i^{(\mathrm{i})})$, rigorously. This is an extension to three-dimensional elastodynamics of the work of Cole (1977), who constructed a Green function for solving the two-dimensional exterior Neumann problem of acoustics for an open arc. We begin, in the next section, by describing the properties of an 'elastic double layer', which is the elastodynamic analogue of the well known harmonic double-layer potential. In particular, we prove that the solution of $\mathrm{S}(u_i^{(\mathrm{i})})$, if it exists, can be represented as an elastic double layer whose 'density' vanishes on $\partial\gamma$, is suitably differentiable on γ, and satisfies a certain integro-differential equation. In §3, we introduce various Green functions, including the exact static Green function which is described, briefly, in Appendix A. We then use plausible, physical arguments and define a new Green function, G_{ij}. Next, we demonstrate, rigorously, that G_{ij} has properties that allow us to derive a Fredholm integral equation of the second kind for the discontinuity of u_i across the crack. This equation, which is derived in §4, has a continuous kernel and a continuous free term. Expressions for the kernel are given in Appendix B. In §5, we show that our integral equation always has a unique solution, and hence prove an existence theorem for the original boundary-value problem $\mathrm{S}(u_i^{(\mathrm{i})})$; the solution of S has an integral representation as an elastic double layer whose density is the solution of our integral equation. Finally, we mention two recent review articles by Kraut (1976) and Datta (1978). These contain brief surveys of the existing literature on the diffraction of elastic waves by a penny-shaped crack.

2. Integral representations

Consider a (closed) Lyapunov surface B, and denote the corresponding boundary-value problem to S by S_B. (S_B corresponds to the scattering of elastic waves by a cavity with a smooth boundary.) Kupradze (1963) has shown that the displacement exterior to B may be written as

$$u_k(\mathrm{P}) = \int_B u_i(\mathrm{q})\,\Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}} - \int_B G^{\mathrm{f}}_{ik}(\mathrm{q};\mathrm{P})\,\tau_{ij}(\mathrm{q})\,n_j\,\mathrm{d}s_{\mathrm{q}}, \tag{2.1}$$

where

$$G^{\mathrm{f}}_{ij}(\mathrm{P};\mathrm{Q}) = \mu^{-1}\left\{\delta_{ij}\Psi + \frac{1}{K^2}\frac{\partial^2}{\partial x_i \partial x_j}(\Psi - \Phi)\right\},$$

$$\Phi = \mathrm{e}^{\mathrm{i}kR}/4\pi R, \quad \Psi = \mathrm{e}^{\mathrm{i}KR}/4\pi R,$$

$R = |\boldsymbol{r}_{\mathrm{P}} - \boldsymbol{r}_{\mathrm{Q}}|$ and $\Sigma^{\mathrm{f}}_{ijm}$ is the stress tensor corresponding to G^{f}_{im}. $G^{\mathrm{f}}_{ij}(\mathrm{P};\mathrm{Q})$ is called the fundamental Green function; it represents the ith component of the displacement at P, due to an oscillating point force (with time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$), acting at Q, in the jth direction. The corresponding stress tensor is given by

$$\begin{aligned}\Sigma^{\mathrm{f}}_{ijm}(\mathrm{P};\mathrm{Q}) &= c_{ijkl}\,\partial G^{\mathrm{f}}_{lm}(\mathrm{P};\mathrm{Q})/\partial x_k \\ &= \frac{2}{K^2}\frac{\partial^3}{\partial x_i \partial x_j \partial x_m}(\Psi - \Phi) + \frac{K^2 - 2k^2}{K^2}\delta_{ij}\frac{\partial \Phi}{\partial x_m} + \delta_{jm}\frac{\partial \Psi}{\partial x_i} + \delta_{im}\frac{\partial \Psi}{\partial x_j}.\end{aligned} \tag{2.2}$$

Kupradze (1963), drawing an analogy with classical potential theory, calls the first term on the right-hand side of (2.1) a 'potential of a double layer of the first kind with density u_i', and the second term a 'potential of a single layer with density $\tau_{ij}n_j$'; we shall call them 'elastic double layer' and 'elastic single layer', respectively. Kupradze has shown that these potentials have the same properties as the classical single and double layers, when P approaches the boundary B. If we make use of these properties in (2.1), we obtain

$$\tfrac{1}{2}u_k(\mathrm{p}) - \int_B u_i(\mathrm{q})\,\Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{p})\,n_j\,\mathrm{d}s_{\mathrm{q}} = \int_B G^{\mathrm{f}}_{ik}(\mathrm{q};\mathrm{p})\,\tau^{(\mathrm{i})}_{ij}(\mathrm{q})\,n_j\,\mathrm{d}s_{\mathrm{q}}, \tag{2.3}$$

where we have used the boundary condition (S2). This is an integral equation of the second kind for the unknown boundary values of u_i. Equations of this type have been considered by Tan (1975); see also the modified integral equation derived by Ahner & Hsiao (1975).

A second method for obtaining a solution to S_B has been used by Kupradze (1963). He represents the displacement as an elastic single layer,

$$u_k(\mathrm{P}) = \int_B \rho_i(\mathrm{q})\,G^{\mathrm{f}}_{ik}(\mathrm{q};\mathrm{P})\,\mathrm{d}s_{\mathrm{q}}, \tag{2.4}$$

where $\rho_i(\mathrm{q})$ is an unknown density. If we apply the stress operator to (2.4) and then use the property of the elastic single layer as P approaches B, we obtain

$$\tfrac{1}{2}\rho_k(\mathrm{p}) - \int_B \rho_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijk}(\mathrm{p};\mathrm{q})\, n_j\, \mathrm{d}s_{\mathrm{q}} = -n_j \tau^{(\mathrm{i})}_{kj}(\mathrm{p}), \tag{2.5}$$

where we have again used (S2). This is an integral equation of the second kind for the boundary values of ρ_i. Once $\rho_i(\mathrm{q})$ has been found, the displacement at any point in D is given by (2.4). Kupradze (1963, p. 166) has shown that (2.5) always has a unique solution, except at a discrete set of frequencies (these are called *irregular* frequencies in the corresponding exterior problem of acoustics). Since the kernel of (2.3) is the transpose of the kernel in (2.5), it follows that (2.3) also has a unique solution except at the same irregular frequencies.

Let us now return to the boundary-value problem S, for a crack γ. Since τ_{ij}, G^{f}_{ij} and $\Sigma^{\mathrm{f}}_{ijk}$ are all continuous across γ, (2.1) reduces to

$$u_k(\mathrm{P}) = \int_\gamma [u_i(\mathrm{q})]\, \Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_{\mathrm{q}}, \tag{2.6}$$

where we use square brackets to denote the discontinuity across γ, i.e.

$$[u_i(\mathrm{q})] = u_i(\mathrm{q}^+) - u_i(\mathrm{q}^-). \tag{2.7}$$

Equation (2.6) suggests that the solution of S may be represented as an elastic double layer. So, we look for a solution

$$u_k(\mathrm{P}) = \int_\gamma \rho_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_{\mathrm{q}}, \tag{2.8}$$

for $\mathrm{P} \in D$. In the remainder of this section, we shall investigate the properties of (2.8) when the (unknown) vector density function $\rho_i(\mathrm{q})$ possesses

Properties $\mathscr{P}(\mathrm{q})$. For $i = 1$, 2, and 3,

(i) $\rho_i(\mathrm{q}^{\mathrm{e}}) = 0$ for all $\mathrm{q}^{\mathrm{e}} \in \partial\gamma$; and

(ii) $t_j \partial\rho_i(\mathrm{q})/\partial x_j$ exist and are Hölder continuous for all $\mathrm{q} \in \gamma\, (\mathrm{q} \notin \partial\gamma)$, where $\boldsymbol{t}$ is an arbitrary vector in the tangent plane to γ at the point q.

It is easily shown that the integrals obtained by repeated differentiation with respect to x_i under the integral sign in (2.8) are absolutely convergent for all $\mathrm{P} \in D$.

Let us introduce the auxiliary function ϕ_i and ψ_i, defined by

$$\phi_i(\mathrm{P}) = \int_\gamma \rho_i(\mathrm{q})\, \Phi(\mathrm{q};\mathrm{P})\, \mathrm{d}s_{\mathrm{q}}, \quad \psi_i(\mathrm{P}) = \int_\gamma \rho_i(\mathrm{q})\, \Psi(\mathrm{q};\mathrm{P})\, \mathrm{d}s_{\mathrm{q}}. \tag{2.9a, b}$$

Then, (2.8) gives

$$u_i(\mathrm{P}) = \frac{-2}{K^2} \frac{\partial^3}{\partial x_i \partial z \partial x_j}(\psi_j - \phi_j) - \frac{K^2 - 2k^2}{K^2} \frac{\partial \phi_z}{\partial x_i} - \delta_{iz} \frac{\partial \psi_j}{\partial x_j} - \frac{\partial \psi_i}{\partial z}, \tag{2.10}$$

where we note that $\Phi(\mathrm{P};\mathrm{Q})$ and $\Psi(\mathrm{P};\mathrm{Q})$ are functions of $|\boldsymbol{r}_{\mathrm{P}}-\boldsymbol{r}_{\mathrm{Q}}|$. From (1.3), we have

$$u_i^{\mathrm{p}}(\mathrm{P}) = \frac{2}{K^2}\frac{\partial^3\phi_j}{\partial x_i\partial z\partial x_j} - \frac{K^2-2k^2}{K^2}\frac{\partial\phi_z}{\partial x_i}, \tag{2.11a}$$

$$u_i^{\mathrm{s}}(\mathrm{P}) = -\frac{2}{K^2}\frac{\partial^3\psi_j}{\partial x_i\partial z\partial x_j} - \delta_{iz}\frac{\partial\psi_j}{\partial x_j} - \frac{\partial\psi_i}{\partial z}. \tag{2.11b}$$

Since

$$(\nabla^2+k^2)\phi_i(\mathrm{P}) = \int_\gamma \rho_i(\mathrm{q})\,(\nabla^2+k^2)\,\Phi(\mathrm{q};\mathrm{P})\,\mathrm{d}s_{\mathrm{q}} = 0,$$

and $(\nabla^2+K^2)\psi_i(\mathrm{P}) = 0$, for $\mathrm{P}\in D$, it follows that (2.11) satisfy (1.4) and hence (2.8) satisfies the equations of motion, (S1), in D.

Let us now determine the behaviour of (2.8) at large distances from the crack. We introduce spherical polar coordinates to locate points $\mathrm{P}\in D$, and write

$$x = r_{\mathrm{P}}\sin\phi_{\mathrm{P}}\cos\theta_{\mathrm{P}}, \quad y = r_{\mathrm{P}}\sin\phi_{\mathrm{P}}\sin\theta_{\mathrm{P}} \quad\text{and}\quad z = r_{\mathrm{P}}\cos\phi_{\mathrm{P}}.$$

In addition, we have

$$R = |\boldsymbol{r}_{\mathrm{P}}-\boldsymbol{r}_{\mathrm{q}}| = r_{\mathrm{P}} - r\cos\psi + O(r_{\mathrm{P}}^{-1}),$$

as $r_{\mathrm{P}}\to\infty$, where ψ is the angle between $\boldsymbol{r}_{\mathrm{P}}$ and $\boldsymbol{r}_{\mathrm{q}}$, i.e. $\cos\psi = \sin\phi_{\mathrm{P}}\cos(\theta_{\mathrm{P}}-\theta)$. Then, from (2.9) and (2.11), we have

$$\begin{aligned} u_m^{\mathrm{p}}(\mathrm{P}) &= \frac{\mathrm{i}k}{K^2}\int_\gamma \rho_i(\mathrm{q})\,\{2k^2R_iR_z-(2k^2-K^2)\,\delta_{iz}\}\,R_m\,\Phi(\mathrm{q};\mathrm{P})\,\mathrm{d}s_{\mathrm{q}} \\ &= \frac{-\mathrm{i}k}{4\pi K^2}\{2k^2\bar{R}_i\bar{R}_m\cos\phi_{\mathrm{P}}+(2k^2-K^2)\,\bar{R}_m\,\delta_{iz}\}\,\bar{\rho}_i(k)\,\frac{\mathrm{e}^{\mathrm{i}kr_{\mathrm{P}}}}{r_{\mathrm{P}}}+O(r_{\mathrm{P}}^{-2}), \end{aligned}$$

as $r_{\mathrm{P}}\to\infty$, where

$$\bar{\rho}_i(\lambda) = \int_\gamma \rho_i(r,\theta)\,\mathrm{e}^{-\mathrm{i}\lambda r\cos\psi}\,r\,\mathrm{d}r\,\mathrm{d}\theta,$$

$R_i(\mathrm{q}) = \partial R/\partial x_i = \bar{R}_i + O(r_{\mathrm{P}}^{-1})$ and $\bar{R}_i = -x_i/r_{\mathrm{P}}$. Similarly,

$$u_m^{\mathrm{s}}(\mathrm{P}) = \frac{\mathrm{i}k}{4\pi}(2\bar{R}_i\bar{R}_m\cos\phi_{\mathrm{P}}+\bar{R}_i\delta_{mz}-\delta_{im}\cos\phi_{\mathrm{P}})\,\bar{\rho}_i(K)\,\frac{\mathrm{e}^{\mathrm{i}Kr_{\mathrm{P}}}}{r_{\mathrm{P}}}+O(r_{\mathrm{P}}^{-2}),$$

as $r_{\mathrm{P}}\to\infty$. It is now straightforward to show that radiation conditions (1.5) are satisfied.

It remains to investigate the properties of the elastic double layer (2.8) near to and on the crack. Consider first the auxiliary functions ϕ_i and ψ_i; using the power series expansion for Φ, we find

$$\phi_i(\mathrm{P}) = \frac{1}{4\pi}\int_\gamma \rho_i(\mathrm{q})\,\{R^{-1}+\mathrm{i}k-\tfrac{1}{2}k^2R+4\pi\Phi^*(k;R)\}\,\mathrm{d}s_{\mathrm{q}}, \tag{2.12}$$

where

$$4\pi\Phi^*(k;R) = (\mathrm{e}^{\mathrm{i}kR}-1)/R-\mathrm{i}k+\tfrac{1}{2}k^2R = -\tfrac{1}{6}\mathrm{i}k^3R^2+O(R^3) \quad\text{as}\quad R\to 0.$$

To obtain the corresponding expressions for $\psi_i(\mathrm{P})$, we simply replace k by K in (2.12). We may now calculate the displacement components from (2.10); after some simple manipulations, we obtain the following expressions

$$u_x(\mathrm{P}) = \frac{-1}{4\pi}\frac{\partial}{\partial z}\int_\gamma \frac{\rho_x(\mathrm{q})}{R}\mathrm{d}s_\mathrm{q} - \frac{\partial}{\partial x}\int_\gamma \frac{\sigma_z(\mathrm{q},\mathrm{P})}{R}\mathrm{d}s_\mathrm{q} + \int_\gamma \rho_i(\mathrm{q})\,U_{ix}(\mathrm{q};\mathrm{P})\,\mathrm{d}s_\mathrm{q}, \tag{2.13}$$

$$u_y(\mathrm{P}) = \frac{-1}{4\pi}\frac{\partial}{\partial z}\int_\gamma \frac{\rho_y(\mathrm{q})}{R}\mathrm{d}s_\mathrm{q} - \frac{\partial}{\partial y}\int_\gamma \frac{\sigma_z(\mathrm{q},\mathrm{P})}{R}\mathrm{d}s_\mathrm{q} + \int_\gamma \rho_i(\mathrm{q})\,U_{iy}(\mathrm{q},\mathrm{P})\,\mathrm{d}s_\mathrm{q}, \tag{2.14}$$

$$u_z(\mathrm{P}) = \frac{-1}{4\pi}\frac{\partial}{\partial z}\int_\gamma \frac{\rho_z(\mathrm{q})}{R}\mathrm{d}s_\mathrm{q} + \int_\gamma \rho_i(\mathrm{q})\,U_{iz}(\mathrm{q},\mathrm{P})\,\mathrm{d}s_\mathrm{q} - \frac{\partial}{\partial x}\int_\gamma \frac{\sigma_x(\mathrm{q},\mathrm{P})}{R}\,\mathrm{d}s_\mathrm{q} - \frac{\partial}{\partial y}\int_\gamma \frac{\sigma_y(\mathrm{q},\mathrm{P})}{R}\mathrm{d}s_\mathrm{q}, \tag{2.15}$$

where

$$4\pi K^2\sigma_x(\mathrm{q},\mathrm{P}) = \rho_x(\mathrm{q})\{(K^2-k^2)\cos^2\Theta + k^2\} - \tfrac{1}{2}\rho_z(\mathrm{q})\,(K^2-k^2)\sin 2\Theta\cos\Omega,$$

$$4\pi K^2\sigma_y(\mathrm{q},\mathrm{P}) = \rho_y(\mathrm{q})\{(K^2-k^2)\cos^2\Theta + k^2\} - \tfrac{1}{2}\rho_z(\mathrm{q})\,(K^2-k^2)\sin 2\Theta\sin\Omega,$$

$$4\pi K^2\sigma_z(\mathrm{q},\mathrm{P}) = \rho_z(\mathrm{q})\{(K^2-k^2)\cos^2\Theta - k^2\} + \tfrac{1}{2}(K^2-k^2)\{\rho_x(\mathrm{q})\cos\Omega + \rho_y(\mathrm{q})\sin\Omega\}\sin 2\Theta,$$

$$K^2U_{ij}(\mathrm{q},\mathrm{P}) = -2\frac{\partial^3}{\partial x_i\partial x_j\partial z}\{\Phi^*(K;R) - \Phi^*(k;R)\} - (K^2-2k^2)\,\delta_{iz}\frac{\partial}{\partial x_j}\left\{\Phi^*(k;R) - \frac{k^2R}{8\pi}\right\} - K^2\left(\delta_{ij}\frac{\partial}{\partial z} + \delta_{jz}\frac{\partial}{\partial x_i}\right)\left\{\Phi^*(K;R) - \frac{K^2R}{8\pi}\right\},$$

$\partial R/\partial x = \sin\Theta\cos\Omega$, $\partial R/\partial y = \sin\Theta\sin\Omega$ and $\partial R/\partial z = \cos\Theta$. The first and second terms in (2.13) are normal and tangential derivatives, respectively, of a harmonic single-layer potential. The general properties of such potentials are well known and are described by, for example, Kellogg (1929, ch. 6). In particular, since $\rho_i(\mathrm{q})$ possesses properties $\mathscr{P}(\mathrm{q})$, it is known that the first derivatives of single layers are continuous and bounded in any finite neighbourhood of γ, that the tangential derivative is continuous across γ, and that the normal derivative is discontinuous across γ, i.e. if we write

$$V(\mathrm{P}) = \int_\gamma \frac{\rho(\mathrm{q})}{R}\mathrm{d}s_\mathrm{q},$$

then (Kellogg 1929, p. 164)

$$\frac{\partial V(\mathrm{p}^+)}{\partial n} - \frac{\partial V(\mathrm{p}^-)}{\partial n} = -4\pi\rho(\mathrm{p}). \tag{2.16}$$

The third term in (2.13) is clearly continuous and bounded everywhere. Similar remarks apply to (2.14) and (2.15). Consequently, we see that the displacement vector given by (2.8) is continuous from both sides of the crack and satisfies the edge condition (S4). Moreover, using (2.7) and (2.16), we find that

$$[u_i(\mathrm{q})] = \rho_i(\mathrm{q}). \tag{2.17}$$

We can now go on to examine the stress components in D, using (1.1) and (2.8). Using the same methods as before, we can show that the stresses are continuous everywhere in D, except possibly at the edge of γ. In particular, the traction vector on γ^+, say, is given by

$$n_k(\mathrm{p}^+)\,\tau_{kl}(\mathrm{p}) = n_k c_{klmn}\frac{\partial}{\partial x_m}\int_\gamma \rho_i(\mathrm{q})\,\Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}\bigg|_{\mathrm{P}=\mathrm{p}}. \tag{2.18}$$

(We remark that if $\rho_i(\mathrm{q})$ has properties $\mathscr{P}(\mathrm{q})$, then this is sufficient to ensure the existence (and continuity) of the right-hand side of (2.18); cf. Günter 1967, pp. 71–76.) We have thus proved the following theorem.

THEOREM 1. *If the vector $\rho_i(\mathrm{q})$ possesses properties $\mathscr{P}(\mathrm{q})$, then* (2.8) *solves the boundary-value problem* $\mathrm{S}(u_i^{(\mathrm{i})})$ *provided also that $\rho_i(\mathrm{q})$ is a solution of the integro-differential equation*

$$-n_k\tau^{(\mathrm{i})}_{kl}(\mathrm{p}) = n_k c_{klmn}\frac{\partial}{\partial x_m}\int_\gamma \rho_i(\mathrm{q})\,\Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}\bigg|_{\mathrm{P}=\mathrm{p}}. \tag{2.19}$$

As an immediate consequence, we can prove the following theorem.

THEOREM 2. *If there exists a vector $\rho_i(\mathrm{q})$ that has properties $\mathscr{P}(\mathrm{q})$ and satisfies the integro-differential equation* (2.19), *then it is unique.*

Proof. Suppose that the homogeneous form of (2.19) has a non-trivial solution, i.e. there exists a function $v_i(\mathrm{q})$ that satisfies

$$0 = n_k c_{klmn}\frac{\partial}{\partial x_m}\int_\gamma v_i(\mathrm{q})\,\Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}\bigg|_{\mathrm{P}=\mathrm{p}} \tag{2.20}$$

for all $\mathrm{p}\in\gamma$. We can also use $v_i(\mathrm{q})$ as the density in the integral representation (2.8) to define an elastodynamic displacement vector $u_i(\mathrm{P})$ which, by virtue of theorem 1 and (2.20), solves S(0). However, we know from Wickham's uniqueness theorem (1981) that the only solution of S(0) is identically zero; in particular, $[u_i(\mathrm{q})] = 0$. It follows from (2.17) that $v_i(\mathrm{q}) = 0$, which is contrary to hypothesis.

We have now shown that there is at most one vector $\rho_i(\mathrm{q})$ with properties $\mathscr{P}(\mathrm{q})$, which solves the integro-differential equation (2.19); in the sequel, we shall prove that such a vector exists.

This section is closed with some remarks on the solution of the integro-differential equation (2.19). We cannot immediately rewrite (2.19) as an integral equation by applying the differential operator and then letting P approach the crack, because the resulting integrals do not exist in the limit, owing to the highly singular kernel. However, we can integrate by parts first and then let P approach γ. This procedure has been followed by Budiansky & Rice (1979) for an arbitrary flat crack. They obtained a system of equations involving $[u_i]$ and tangential derivatives of $[u_i]$. However, they did not attempt to solve their equations for any particular crack geometry.

3. Green functions for diffraction by a crack

At the beginning of § 2, we introduced the fundamental Green function, $G^{\mathrm{f}}_{ij}(\mathrm{P};\mathrm{Q})$, which can be used to solve $\mathrm{S}_B(u_i^{\mathrm{i})})$ by deriving an integral equation of the second kind for $u_i(\mathrm{q})$, namely (2.3), and then using the integral representation (2.1). For $\mathrm{S}(u_i^{(\mathrm{i})})$, (2.1) reduces to (2.6), and so we need an integral equation for $[u_i(\mathrm{q})]$; this cannot be derived directly by using G^{f}_{ij}, since this function is continuous across the crack. However, G^{f}_{ij} can be replaced by any other function that satisfies (S1) everywhere in D except at $\mathrm{P}=\mathrm{Q}$, where it has a suitable singularity, and satisfies the radiation and edge conditions. There is an infinite number of these *fundamental solutions*, the simplest of which is G^{f}_{ij}.

Let us now consider a second fundamental solution, $G^{*}_{ij}(\mathrm{P};\mathrm{P}_0)$, defined by

$$G^{*}_{ij}(\mathrm{P};\mathrm{P}_0)=G^{\mathrm{f}}_{ij}(\mathrm{P};\mathrm{P}_0)+G^{\mathrm{e}}_{ij}(\mathrm{P};\mathrm{P}_0),$$

where $G^{\mathrm{e}}_{ij}(\mathrm{P};\mathrm{P}_0)$ is the solution of $\mathrm{S}(G^{\mathrm{f}}_{ij}(\mathrm{P};\mathrm{P}_0))$, i.e. the ith component of the scattered field at P when the incident waves are produced by an oscillatory point force at P_0, acting in the jth direction. We shall call G^{e}_{ij} the exact Green function for S, because it can be used in an explicit formula for the solution of our scattering boundary-value problem. This formula is given in the next theorem, which has been proved by Wickham (1981).

Theorem 3. *If G^{e}_{ij} exists and $u_i^{(\mathrm{i})}$ is any incident field giving rise to bounded stresses on the crack faces, then the solution of $\mathrm{S}(u_i^{(\mathrm{i})})$ is given by the formula*

$$u_k(\mathrm{P}_0)=\int_\gamma [G^{\mathrm{e}}_{ik}(\mathrm{q};\mathrm{P}_0)]\,\tau^{(\mathrm{i})}_{ij}(\mathrm{q})\,n_j\,\mathrm{d}s_{\mathrm{q}}. \tag{3.1}$$

Of course, G^{e}_{ij} is not known explicitly, and the problem of finding this function is essentially the same as solving the original boundary-value problem. However, let us proceed as if we know G^{e}_{ij}; from theorem 1, we know that if $\rho_i(\mathrm{q})=[u_i(\mathrm{q})]$ is given, then the solution of $\mathrm{S}(u^{(\mathrm{i})}_{ij})$ is (2.8), whence (3.1) gives

$$\rho_k(\mathrm{p}_0)=\int_\gamma [H_{ik}(\mathrm{q};\mathrm{p}_0)]\,\tau^{(\mathrm{i})}_{ij}(\mathrm{q})\,n_j\,\mathrm{d}s_{\mathrm{q}}, \tag{3.2}$$

where

$$H_{ij}(\mathrm{Q};\mathrm{p}_0)=G^{\mathrm{e}}_{ij}(\mathrm{Q};\mathrm{p}_0^+)-G^{\mathrm{e}}_{ij}(\mathrm{Q};\mathrm{p}_0^-).$$

In addition, theorem 1 asserts that, if it exists, G^{e}_{ij} satisfies

$$G^{\mathrm{e}}_{kl}(\mathrm{P};\mathrm{P}_0)=\int_\gamma [G^{\mathrm{e}}_{il}(\mathrm{q};\mathrm{P}_0)]\,\Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}$$

and hence, on taking the appropriate limits,

$$H_{kl}(\mathrm{P};\mathrm{p}_0)=\int_\gamma [H_{il}(\mathrm{q};\mathrm{p}_0)]\,\Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}, \tag{3.3}$$

where it is assumed that $[H_{ij}(q;p_0)]$ has properties $\mathscr{P}(q)$, except possibly near $q = p_0$. Moreover, from (2.19), it follows that $[H_{ij}]$ satisfies the following integro-differential equation on γ,

$$-\delta_{ls}\delta(p-p_0) = n_k c_{klmn}\frac{\partial}{\partial x_m}\int_\gamma [H_{is}(q;p_0)]\Sigma^{\mathrm{f}}_{ijn}(q;P)\,n_j\,\mathrm{d}s_q\bigg|_{P=p}. \tag{3.4}$$

Thus, $\rho_k(p_0)$ is given by (3.2) in terms of the displacement discontinuity maintained by equal and opposite oscillatory point forces acting at p_0^+ and p_0^-; this quantity, which we have denoted by $[H_{ij}(q;p_0)]$, satisfies the integro-differential equation (3.4). Solving this equation for $[H_{ij}]$ is certainly not easier than solving (2.19) for arbitrary incident waves. However, we may be able to use physical arguments to infer something about the analytical properties of this particular solution. For any fixed value of the wavenumber K, we expect that there is a neighbourhood of the points p_0^+ and p_0^- where the motion of the elastic solid may be regarded as quasistatic, i.e., apart from the time factor $e^{-i\omega t}$, the stresses and displacements are dominated by those produced by static point forces at p_0^+ and p_0^-.

So we introduce the exact static Green function, which we denote by $\tilde{G}_{ij}$. In particular, we shall require $[\tilde{G}_{ij}(p;p_0)]$, which represents the discontinuity in the ith component of the elastostatic displacement vector at p, when the crack is opened by equal and opposite point forces at p_0^+ and p_0^-, acting in the jth direction. The explicit construction of $[\tilde{G}_{ij}]$ for a penny-shaped crack has been given by Martin (1979, 1981); in Appendix A, we describe this Green function and some of its properties.

Suppose now that we replace the unknown function $[H_{il}]$ in (3.3) by the known function $[\tilde{G}_{il}]$. We expect, from the preceding physical arguments, that this will define a new fundamental solution; the mathematical properties of this function are contained in the next theorem.

THEOREM 4. *Define a function G_{ij} by*

$$G_{lm}(P;p_0) = \int_\gamma [\tilde{G}_{im}(q;p_0)]\Sigma^{\mathrm{f}}_{ijl}(q;P)\,n_j\,\mathrm{d}s_q, \tag{3.5}$$

with corresponding stress components given by

$$\Sigma_{ijm}(P;p_0) = c_{ijkl}\frac{\partial}{\partial x_k}G_{lm}(P;p_0). \tag{3.6}$$

Then G_{ij} is a fundamental solution that has the following properties.

(i) For any fixed $p_0 \in \gamma$, and for $j = 1, 2$, and 3, $G_{ij}(P;p_0)$ satisfies the equations of motion in D, the radiation conditions and the edge conditions;

(ii) $$[G_{ij}(p;p_0)] = [\tilde{G}_{ij}(p;p_0)]; \tag{3.7}$$

(iii) $$\lim_{\epsilon\to 0}\int_{H_\epsilon^\pm}\Sigma_{ijk}(q;p_0)\,n_j\,\mathrm{d}s_q = \mp\delta_{ik}, \tag{3.8}$$

where H_ϵ^+ and H_ϵ^- are small hemispherical surfaces of radius ϵ, centred on p_0^+ and p_0^-, respectively; and

(iv) $\Sigma_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j$ is continuous for all $\mathrm{q} \neq \mathrm{p}_0$.

Proof. From (A 13), we see that $[\tilde{G}_{ij}(\mathrm{q};\mathrm{p}_0)]$ has a singularity at $\mathrm{q} = \mathrm{p}_0$, i.e. $[\tilde{G}_{ij}(\mathrm{q};\mathrm{p}_0)]$ does not have properties $\mathscr{P}(\mathrm{q}) \times \mathscr{P}(\mathrm{p}_0)$. (If it did, the proofs of (i), (ii) and (iv) would simply repeat most of the arguments leading to theorem 1.) However, $[\tilde{G}_{ij}(\mathrm{q};\mathrm{p}_0)]$ is absolutely integrable over γ and so it follows that $G_{ij}(\mathrm{P};\mathrm{p}_0)$ satisfies the equations of motion and the radiation conditions. To prove that G_{ij} also satisfies the edge condition, we note first that the exact static Green function, $\tilde{G}_{ij}$, satisfies an equation similar to (3.5), namely (A 3). Subtracting this equation from (3.5), we obtain

$$G_{kl}(\mathrm{P};\mathrm{p}_0) - \tilde{G}_{kl}(\mathrm{P};\mathrm{p}_0) = \int_\gamma [\tilde{G}_{il}(\mathrm{q};\mathrm{p}_0)]\{\Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P}) - \tilde{\Sigma}^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\}\,n_j\,\mathrm{d}s_{\mathrm{q}}, \tag{3.9}$$

where $\tilde{\Sigma}^{\mathrm{f}}_{ijk}$ is given by (A 5). The expression inside the braces is easily seen to be a continuous function of q and P, and so the right-hand side of (3.9) is continuous and bounded. Therefore, since $\tilde{G}_{ij}(\mathrm{P};\mathrm{p}_0)$ satisfies the edge condition (see Appendix A), it follows that G_{ij} must also satisfy the same edge condition. Moreover, if we let P approach each side of γ in turn, and then subtract the two results, we obtain (3.7). Again from (3.9), we see that

$$\Sigma_{ijk}(\mathrm{P};\mathrm{p}_0) - \tilde{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0) = \Sigma'_{ijk}(\mathrm{P};\mathrm{p}_0) \tag{3.10}$$

say, is a continuous function of P and p_0; this may be shown by using arguments similar to those used in Appendix A. In particular, $\Sigma_{ijk}(\mathrm{P};\mathrm{p}_0)$ has the same singularity at $\mathrm{P} = \mathrm{p}_0$ as $\tilde{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0)$, and so (3.8) is a consequence of (A 14) and (A 15). Since, by definition, $\tilde{\Sigma}_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j$ vanishes everywhere on γ except at $\mathrm{q} = \mathrm{p}_0$, it follows that $\Sigma_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j$ is continuous except at $\mathrm{q} = \mathrm{p}_0$. This completes the proof of theorem 4.

From the preceding analysis, it is clear that we can generalize theorem 4 to describe the properties of any function $\hat{G}_{ij}$, defined by

$$\hat{G}_{lm}(\mathrm{P};\mathrm{p}_0) = \int_\gamma \hat{\rho}_{im}(\mathrm{q};\mathrm{p}_0)\,\Sigma^{\mathrm{f}}_{ijl}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}},$$

where

$$\hat{\rho}_{ij} = [\tilde{G}_{ij}] + \rho^{+}_{ij}$$

and $\rho^{+}_{ij}(\mathrm{q};\mathrm{p}_0)$ has properties $\mathscr{P}(\mathrm{q}) \times \mathscr{P}(\mathrm{p}_0)$. Thus we may write

$$\hat{G}_{ij}(\mathrm{P};\mathrm{p}_0) = G_{ij}(\mathrm{P};\mathrm{p}_0) + G^{+}_{ij}(\mathrm{P};\mathrm{p}_0),$$

where

$$G^{+}_{lm}(\mathrm{P};\mathrm{p}_0) = \int_\gamma \rho^{+}_{im}(\mathrm{q};\mathrm{p}_0)\,\Sigma^{\mathrm{f}}_{ijl}(\mathrm{q};\mathrm{P})\,n_j\,\mathrm{d}s_{\mathrm{q}}$$

is an elastodynamic displacement vector that satisfies the radiation and edge conditions.

We shall call $G_{ij}(\mathrm{P};\mathrm{p}_0)$ and $\hat{G}_{ij}(\mathrm{P};\mathrm{p}_0)$ *crack Green functions*. These functions satisfy the equation of motion in D, and the radiation and edge conditions. The fundamental Green function, G^{f}_{ij}, also has these properties. However, unlike G^{f}_{ij}, the crack Green functions are discontinuous across the crack and have a singularity at $P = p_0$, which enables us to derive an integral equation of the second kind for $[u_i(q)]$. In the next section, we shall use G_{ij} to derive our integral equation.

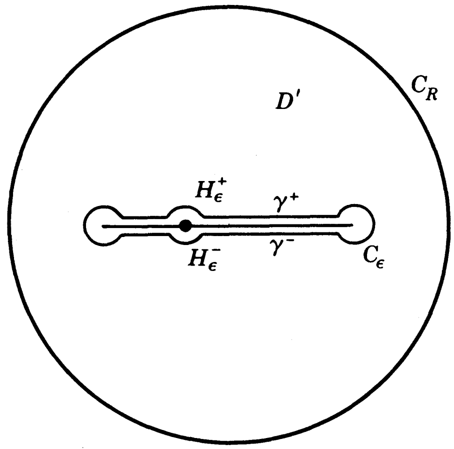

FIGURE 1. The region D'.

4. An integral equation of the second kind

Let u_i and v_i be two elastodynamic displacement vectors, defined in a closed region bounded by a regular surface S. If τ_{ij} and σ_{ij} are the stress tensors corresponding to u_i and v_i, respectively, then, in the absence of the body forces,

$$\int_S \{u_i(q)\,\sigma_{ij}(q) - v_i(q)\,\tau_{ij}(q)\}\,n_j\,ds_q = 0; \tag{4.1}$$

this is called the *reciprocal theorem* (see, for example, Eringen & Suhubi 1975, p. 432).

Let us apply the reciprocal theorem to $u_i(P)$ and the crack Green function $G_{ij}(P; p_0)$; these are both elastodynamic displacement vectors defined in the region D' (see figure 1), with boundary S defined by

$$S = \gamma^+ \cup \gamma^- \cup C_R \cup C_\epsilon \cup H_\epsilon^+ \cup H_\epsilon^-.$$

Here, C_ϵ is a torus-like surface enclosing the edge of γ, and C_R is a large sphere of radius R. Since u_i and G_{ij} both satisfy the radiation conditions, there is no contribution from integrating over C_R as $R \to \infty$. Similarly, let us assume for now that u_i and G_{ij} behave near the crack edge in such a way that there is no contribution from integrating over C_ϵ as this surface shrinks to the edge.

Write

$$\Sigma_{ijk} = \tilde{\Sigma}_{ijk} + \Sigma'_{ijk},$$

where Σ'_{ijk} is defined by (3.10); $\Sigma'_{ijk}(q; p_0)\,n_j$ is continuous everywhere on γ while $\tilde{\Sigma}_{ijk}(q; p_0)\,n_j$ vanishes except when $q = p_0$. Thus if u_i and $\tau_{ij}n_j$ are continuous, say, then all the contributions from integrating over the small hemispheres, $H_\epsilon^\pm$, are zero as $\epsilon \to 0$, except those arising from $\tilde{\Sigma}_{ijk}$, and these are given by (A 17). Combining all these results, we obtain

$$\mu[u_k(\mathrm{p}_0)] - \int_\gamma [u_i(\mathrm{q})]\, \Sigma'_{ijk}(\mathrm{q};\mathrm{p}_0)\, n_j \,\mathrm{d}s_\mathrm{q} = -\int_\gamma \tau_{ij}(\mathrm{q})\, [G_{ik}(\mathrm{q};\mathrm{p}_0)]\, n_j \,\mathrm{d}s_\mathrm{q},$$

since Σ'_{ijk} and τ_{ij} are both continuous across the crack. The right-hand side of this equation can be simplified, by using the boundary condition (S2) and the relation (3.7), and so we have proved the following theorem.

THEOREM 5. *If the boundary-value problem* $\mathrm{S}(u_i^{(\mathrm{i})})$ *possesses a solution* $u_i(\mathrm{P})$, *then* $[u_i(\mathrm{q})]$ *satisfies*

$$[u_k(\mathrm{p}_0)] - \int_\gamma [u_i(\mathrm{q})]\, K_{ik}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_\mathrm{q} = [\tilde{u}_k(\mathrm{p}_0)], \tag{4.2}$$

where the kernel and free term are given by

$$K_{ik}(\mathrm{p};\mathrm{p}_0) = (1/\mu)\{\Sigma_{ijk}(\mathrm{P};\mathrm{p}_0)\, n_j - \tilde{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0)\, n_j\}\Big|_{\mathrm{P}=\mathrm{p}} \tag{4.3}$$

and

$$[\tilde{u}_k(\mathrm{p}_0)] = (1/\mu)\int_\gamma \tau_{ij}^{(\mathrm{i})}(\mathrm{q})\, [\tilde{G}_{ik}(\mathrm{q};\mathrm{p}_0)]\, n_j\,\mathrm{d}s_\mathrm{q}, \tag{4.4}$$

respectively.

Equation (4.2) is a system of three coupled two-dimensional Fredholm integral equations of the second kind for $[u_i(\mathrm{q})]$, the discontinuity in the displacement vector across the crack. For a crack lying in the plane $z = 0$, these equations partially decouple: in cylindrical polar coordinates, we have the following two problems.

Normal problem (symmetric about $z = 0$).

$$[u_z(\mathrm{p}_0)] - \int_\gamma [u_z(\mathrm{q})]\, K_{zz}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_\mathrm{q} = [\tilde{u}_z(\mathrm{p}_0)], \tag{4.5}$$

Shear problem (antisymmetric about $z = 0$).

$$[u_\beta(\mathrm{p}_0)] - \int_\gamma [u_\alpha(\mathrm{q})]\, K_{\alpha\beta}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_\mathrm{q} = [\tilde{u}_\beta(\mathrm{p}_0)]; \tag{4.6}$$

this is a pair of coupled equations: $\alpha, \beta = r$ or θ and, as usual, the summation convention has been used.

Let us now examine the properties of the integral equation (4.2). From theorem 4, we see that the kernel $K_{ij}(\mathrm{q};\mathrm{p}_0)$ is continuous everywhere on γ, and that $K_{ij}(\mathrm{q};\mathrm{p}^\mathrm{e}) = 0$. If we seek a continuous solution of the integral equation, then it follows that the quantity

$$\int_\gamma [u_i(\mathrm{q})]\, K_{ij}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_\mathrm{q}$$

has properties $\mathscr{P}(\mathrm{p}_0)$. (Note that the kernel itself is not differentiable at $\mathrm{q} = \mathrm{p}_0$: terms such as $r \ln\{\max(r, r_0)\}$ occur; see Martin (1979) for further details.) The free term in (4.2), $[\tilde{u}_i(\mathrm{p}_0)]$, is precisely the displacement discontinuity that would be maintained by *static* stresses $-\tau_{iz}^{(\mathrm{i})}$ on the crack faces; for quite general loads, $[\tilde{u}_i]$ may be calculated by using the formulae given by Martin (1981). Clearly, $[\tilde{u}_i(\mathrm{p}^{\mathrm{e}})] = 0$, and so we see that $[\tilde{u}_i(\mathrm{p}_0)]$ has properties $\mathscr{P}(\mathrm{p}_0)$ if we assume that $\tau_{ij}^{(\mathrm{i})}(\mathrm{q})$ is Hölder continuous, say (Kellogg 1929). Combining all these results, we obtain

THEOREM 6. *Suppose that there exists a continuous function* $v_i(\mathrm{q})$ *that solves the integral equation* (4.2). *Then* $v_i(\mathrm{q})$ *has properties* $\mathscr{P}(\mathrm{q})$.

In Appendix B, we give some expressions for the kernel K_{ij}. It can be shown from these that K_{ij} may be expanded as a power series in the wavenumber K and that $K_{ij} = O(K^2)$ as $K \to 0$; these expansions will be given elsewhere. This implies, together with the interpretation given for $[\tilde{u}_i]$, that the integral equation (4.2) may be solved by iteration for sufficiently small K. In other words, the solution of (4.2) may be obtained, rigorously, for $K \leqslant K_0$, say, by constructing its Liouville–Neumann series (see, for example, Smithies 1958, p. 29).

In the next section, we shall prove that the integral equation (4.2) has a unique solution for all values of K, and that this solution can be used to solve $\mathrm{S}(u_i^{(\mathrm{i})})$. However, let us first consider the crack Green function $\hat{G}_{ij}(\mathrm{P};\mathrm{p})$, defined by (3.11). This may also be used to derive an integral equation for $[u_i]$, namely

$$[u_k(\mathrm{p}_0)] - \int_\gamma [u_i(\mathrm{q})]\, \hat{K}_{ik}(\mathrm{q};\mathrm{p}_0)\, \mathrm{d}s_{\mathrm{q}} = \frac{1}{\mu}\int_\gamma \tau_{ij}^{(\mathrm{i})}(\mathrm{q})\, [\hat{G}_{ik}(\mathrm{q};\mathrm{p}_0)]\, n_j\, \mathrm{d}s_{\mathrm{q}}, \qquad (4.7)$$

where

$$\mu \hat{K}_{ik}(\mathrm{p};\mathrm{p}_0) = \{\hat{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0)\, n_j - \tilde{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0)\, n_j\}\Big|_{\mathrm{P}=\mathrm{p}}.$$

From (3.7), (3.12) and (4.3), we have

$$[\hat{G}_{ij}(\mathrm{q};\mathrm{p}_0)] = [\tilde{G}_{ij}(\mathrm{q};\mathrm{p}_0)] + [G_{ij}^{+}(\mathrm{q};\mathrm{p}_0)]$$

and

$$\mu \hat{K}_{ik}(\mathrm{p};\mathrm{p}_0) = \mu K_{ik}(\mathrm{p};\mathrm{p}_0) + \Sigma_{ijk}^{+}(\mathrm{p};\mathrm{p}_0)\, n_j,$$

whence (4.7) may be written as

$$[u_k(\mathrm{p}_0)] - \int_\gamma [u_i(\mathrm{q})]\, K_{ik}(\mathrm{q};\mathrm{p}_0)\, \mathrm{d}s_{\mathrm{q}} = [\tilde{u}_k(\mathrm{p}_0)] + f_k(\mathrm{p}_0),$$

where

$$\mu f_k(\mathrm{p}_0) = \int_\gamma \{[u_i(\mathrm{q})]\, \Sigma_{ijk}^{+}(\mathrm{q};\mathrm{p}_0) + \tau_{ij}^{(\mathrm{i})}(\mathrm{q})\, [G_{ik}^{+}(\mathrm{q};\mathrm{p}_0)]\}\, n_j\, \mathrm{d}s_{\mathrm{q}}.$$

But, by using the reciprocal theorem (4.1), it quickly follows that $f_k(\mathrm{p}_0) = 0$ for all p_0, and so the integral equation (4.7) reduces to the original integral equation (4.2).

5. An existence theorem

Let us assume that there exists a continuous function $\rho_i(\mathrm{q})$ that solves the integral equation of the second kind (4.2). Then, by theorem 6, $\rho_i(\mathrm{q})$ has properties $\mathscr{P}(\mathrm{q})$. Hence, we can use $\rho_i(\mathrm{q})$ as the density in an elastic double layer, i.e. we can define a function $u_k(\mathrm{P})$ by

$$u_k(\mathrm{P}) = \int_\gamma \rho_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijk}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_\mathrm{q}. \tag{5.1}$$

By theorem 1, $u_k(\mathrm{P})$ satisfies the equations of motion in D, the radiation conditions and the edge condition, while (2.17) gives

$$[u_i(\mathrm{q})] = \rho_i(\mathrm{q}). \tag{5.2}$$

Finally, if $\rho_i(\mathrm{q})$ also solves the integro-differential equation,

$$-n_k \tau^{(\mathrm{i})}_{kl}(\mathrm{p}) = n_k c_{klmn} \frac{\partial}{\partial x_m} \int_\gamma \rho_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_\mathrm{q} \bigg|_{\mathrm{P}=\mathrm{p}}, \tag{5.3}$$

then (by theorem 1), $u_k(\mathrm{P})$ solves the boundary-value problem $\mathrm{S}(u^{(\mathrm{i})}_i)$.

To show that $\rho_i(\mathrm{q})$ satisfies (5.3), we apply the reciprocal theorem in D' (see figure 1) to $u_k(\mathrm{P})$ and $G_{kl}(\mathrm{P};\mathrm{p}_0)$, defined by (5.1) and (3.5), respectively. We proceed as in §4, except that we know *a priori* (from the properties of (5.1) and (3.5)) that the contribution from integrating over C_ϵ must vanish as $\epsilon \to 0$. Hence, we obtain

$$[u_k(\mathrm{p}_0)] - \int_\gamma [u_i(\mathrm{q})]\, K_{ik}(\mathrm{q};\mathrm{p}_0)\, \mathrm{d}s_\mathrm{q} = -\int_\gamma [G_{ik}(\mathrm{q};\mathrm{p}_0)]\, \tau_{ij}(\mathrm{q})\, n_j\, \mathrm{d}s_\mathrm{q},$$

where $\tau_{ij}(\mathrm{P})$ is the stress tensor corresponding to $u_i(\mathrm{P})$, and is given by (1.1). By using (3.7) and (5.2), this reduces to

$$\rho_k(\mathrm{p}_0) - \int_\gamma \rho_i(\mathrm{q})\, K_{ik}(\mathrm{q};\mathrm{p}_0)\, \mathrm{d}s_\mathrm{q} = -\int_\gamma [\tilde{G}_{ik}(\mathrm{q};\mathrm{p}_0)]\, \tau_{ij}(\mathrm{q})\, n_j\, \mathrm{d}s_\mathrm{q}.$$

But, since $\rho_i(\mathrm{q})$ satisfies (4.2), the left-hand side of this equation is equal to $[\tilde{u}_k(\mathrm{p}_0)]$. If we now use (1.1), (4.4) and (5.1), we obtain

$$\int_\gamma [\tilde{G}_{ik}(\mathrm{q};\mathrm{p}_0)]\, T_i(\mathrm{q})\, \mathrm{d}s_\mathrm{q} = 0, \tag{5.4}$$

for all $\mathrm{p}_0 \in \gamma$, where

$$T_l(\mathrm{p}) = \tau^{(\mathrm{i})}_{kl}(\mathrm{p})\, n_k + n_k c_{klmn} \frac{\partial}{\partial x_m} \int_\gamma \rho_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_\mathrm{q} \bigg|_{\mathrm{P}=\mathrm{p}}.$$

It remains to prove from (5.4) that $T_i(\mathrm{p}) = 0$; we require the following

Lemma. *If $T_i(\mathrm{p})$ is a Hölder continuous function that satisfies* (5.4) *for all* p_0, *then $T_i(\mathrm{p})$ vanishes identically.*

Proof. Consider the integro-differential equation

$$t_l(\mathrm{p}) = n_k c_{klmn} \frac{\partial}{\partial x_m} \int_\gamma v_i(\mathrm{q})\, \Sigma^{\mathrm{f}}_{ijn}(\mathrm{q};\mathrm{P})\, n_j\, \mathrm{d}s_{\mathrm{q}} \bigg|_{\mathrm{P}=\mathrm{p}},$$

which is the elastostatic equivalent of (5.3). The unique solution of this equation is given by (Martin 1979, 1981)

$$v_k(\mathrm{p}_0) = -\frac{1}{\mu} \int_\gamma [\tilde{G}_{ik}(\mathrm{q};\mathrm{p}_0)]\, t_i(\mathrm{q})\, \mathrm{d}s_{\mathrm{q}},$$

where $t_i(\mathrm{q})$ is a given Hölder continuous function and $\tilde{G}_{ij}$ is the exact static Green function. Combining these two equations yields the identity

$$t_l(\mathrm{p}') = -n'_k c_{klmn} \frac{\partial}{\partial x'_m} \int_\gamma \Sigma^{\mathrm{f}}_{sjn}(\mathrm{q};\mathrm{P}') \int_\gamma [\tilde{G}_{is}(\mathrm{p};\mathrm{q})]\, t_i(\mathrm{p})\, \mathrm{d}s_{\mathrm{p}}\, n_j\, \mathrm{d}s_{\mathrm{q}} \bigg|_{\mathrm{P}'=\mathrm{p}'}.$$

When $t_i = T_i$, the inner integral vanishes for all values of q, by hypothesis, and so the lemma follows.

Applying the lemma to (5.4), we see that if a solution of the integral equation (4.2) exists, then it also satisfies the integro-differential equation (5.3). Consequently, we have proved

THEOREM 7. *Suppose that there exists a continuous function* $\rho_i(\mathrm{q})$ *that satisfies the integral equation* (4.2). *Then, the elastic double layer* (5.1), *with density* $\rho_i(\mathrm{q})$, *solves the original boundary-value problem*, $\mathrm{S}(u_i^{(\mathrm{i})})$.

The existence of a solution to $\mathrm{S}(u_i^{(\mathrm{i})})$ is assured by the next theorem.

THEOREM 8. *There exists a* (*unique*) *solution to the boundary-value problem* $\mathrm{S}(u_i^{(\mathrm{i})})$, *for a penny-shaped crack situated in an unbounded, homogeneous, isotropic, elastic solid.*

Proof. Suppose that there exists a non-trivial solution of the homogeneous form of the Fredholm integral equation of the second kind (4.2), i.e. there is a function $v_i(\mathrm{q})$, which is not identically zero, and which satisfies

$$v_k(\mathrm{p}_0) - \int_\gamma v_i(\mathrm{q})\, K_{ik}(\mathrm{q};\mathrm{p}_0)\, \mathrm{d}s_{\mathrm{q}} = 0. \tag{5.5}$$

By theorem 7, $v_i(\mathrm{q})$ also solves the homogeneous form of the integro-differential equation (5.3), namely (2.20). It then follows, from theorem 2, that (5.5) has only the trivial solution $v_i(\mathrm{q}) \equiv 0$, which is contrary to hypothesis. Hence, by the Fredholm alternative (see, for example, Smithies 1958, p. 51), the inhomogeneous integral equation (4.2) always possesses a solution, and this solution is evidently unique (since the difference between any two solutions of (4.2) must satisfy (5.5) and is hence zero). Moreover, this solution leads to the unique solution of $\mathrm{S}(u_i^{(\mathrm{i})})$ when combined with the integral representation (5.1). This completes the proof of our existence theorem.

6. Conclusion

In this paper, we have constructed a new crack Green function, G_{ij}, for the diffraction of elastic waves by a penny-shaped crack. We have shown that the corresponding boundary-value problem always possesses a unique solution. Moreover, this solution can be represented as the elastic double layer whose density solves a Fredholm integral equation of the second kind, namely (4.2). In principle, it is straightforward to solve this integral equation. In practice, difficulties arise because the equation is two-dimensional and the kernel is complicated (see Appendix B). However, suppose we write

$$[u_i(r,\theta)] = \tfrac{1}{2}w_0^i(r) + \sum_{n=1}^{\infty} (w_n^i(r)\cos n\theta + \tilde{w}_n^i(r)\sin n\theta).$$

Then, we can replace our two-dimensional equation by an infinite system of one-dimensional integral equations for the unknown Fourier components of $[u_i]$. These equations can be considerably simplified by exploiting the structure of the kernel and introducing new unknown functions. In the future, it is hoped to present these simpler integral equations, their low-frequency asymptotic solutions and some numerical results.

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Appendix A. The exact static Green function

Let q and p_0 be two points on the surface of a penny-shaped crack with polar coordinates (r, θ) and (r_0, θ_0), respectively. The following expressions for $[\tilde{G}_{ij}(q; p_0)]$ may be obtained from (Martin 1981):

$$[\tilde{G}_{zz}] = \frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n f_n^z(r; r_0) \cos n(\theta - \theta_0), \tag{A 1}$$

$$[\tilde{G}_{rr}] = \frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n f_n^+(r; r_0) \cos n(\theta - \theta_0), \quad [\tilde{G}_{\theta r}] = \sum_{n=1}^{\infty} g_n^+(r; r_0) \sin n(\theta - \theta_0),$$

$$[\tilde{G}_{r\theta}] = -\sum_{n=1}^{\infty} f_n^-(r; r_0) \sin n(\theta - \theta_0), \quad [\tilde{G}_{\theta\theta}] = \frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n g_n^-(r; r_0) \cos n(\theta - \theta_0),$$

$$[\tilde{G}_{rz}] = [\tilde{G}_{\theta z}] = [\tilde{G}_{zr}] = [\tilde{G}_{z\theta}] = 0,$$

$$\tfrac{1}{4}\pi^2 f_n^z(r; r_0) = (1-\nu) w^n g^n(r, r_0), \tag{A 2}$$

$$\begin{aligned}
\tfrac{1}{4}\pi^2(2-\nu) f_n^+(r; r_0) &= (1-\nu-n\nu)^2 w^{n+1} g^{n+1} + (1-\nu+n^2\nu^2) w^{n-1} g^{n-1} \\
&\quad + n\nu(1-\nu-n\nu)(r^2+r_0^2) w^{n-1} g^n, \\
\tfrac{1}{4}\pi^2(2-\nu) f_n^-(r; r_0) &= (1+n\nu)(1-\nu-n\nu) w^{n+1} g^{n+1} - (1-\nu+n^2\nu^2) w^{n-1} g^{n-1} \\
&\quad + n\nu\{(1+n\nu) r_0^2 - (1-\nu-n\nu) r^2\} w^{n-1} g^n, \\
\tfrac{1}{4}\pi^2(2-\nu) g_n^-(r; r_0) &= (1+n\nu)^2 w^{n+1} g^{n+1} + (1-\nu+n^2\nu^2) w^{n-1} g^{n-1} \\
&\quad - n\nu(1+n\nu)(r^2+r_0^2) w^{n-1} g^n,
\end{aligned}$$

$g_n^+(r; r_0) = f_n^-(r_0; r)$, $w = rr_0$, ϵ_n is the Neumann factor, defined by $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n > 0$, and the function $g^n(r, r_0)$ is defined by

$$g^n(r, r_0) = \int_{\bar{r}}^{1} \frac{\mathrm{d}t}{t^{2n}(t^2-r^2)^{\frac{1}{2}}(t^2-r_0^2)^{\frac{1}{2}}},$$

with $\bar{r} = \max(r, r_0)$; $g^n(r, r_0)$ may be expressed in terms of incomplete elliptic integrals and also satisfies an inhomogeneous recurrence relation (Martin 1981).

The exact static Green function is defined in D by

$$\tilde{G}_{kl}(P; p_0) = \int_\gamma [\tilde{G}_{il}(q; p_0)] \tilde{\Sigma}^f_{ijk}(q; P) n_j \, \mathrm{d}s_q, \tag{A 3}$$

where

$$\tilde{G}^f_{ik}(P; Q) = \frac{1}{16\pi\mu(1-\nu)R}\left\{(3-4\nu)\delta_{ik} + \frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_k}\right\} \tag{A 4}$$

is the fundamental static Green function (Kelvin's point-load solution), $\tilde{\Sigma}^f_{ijk}$ are the corresponding stress components given by

$$\tilde{\Sigma}^f_{ijk}(P; Q) = c_{ijlm} \partial \tilde{G}^f_{mk}(P; Q)/\partial x_l \tag{A 5}$$

and $R = |\boldsymbol{r}_P - \boldsymbol{r}_Q|$. We see immediately that $\tilde{G}^f_{ij} = O(R^{-1})$ and $\tilde{\Sigma}^f_{ijk} = O(R^{-2})$ as $R \to 0$.

Let us examine the stress components corresponding to $\tilde{G}_{ij}$, which we denote by $\tilde{\Sigma}_{ijk}$; these are given by

$$\tilde{\Sigma}_{ijk}(\mathrm{P};\mathrm{p}_0) = c_{ijlm}\,\partial\tilde{G}_{mk}(\mathrm{P};\mathrm{p}_0)/\partial x_l.$$

As an example, we shall evaluate $\tilde{\Sigma}_{zzz}$; using (A 3), (A 4) and (A 5), we obtain

$$4\pi(1-\nu)\,\tilde{\Sigma}_{zzz}(\mathrm{P};\mathrm{p}_0) = \mu\left(z\frac{\partial}{\partial z}-1\right)\int_\gamma [\tilde{G}_{zz}(\mathrm{q};\mathrm{p}_0)]\,\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}\right)\mathrm{d}s_{\mathrm{q}}.$$

Let P have cylindrical polar coordinates (ρ,ϕ,z). Then, for $z>0$,

$$R^{-1} = \int_0^\infty J_0(\xi b)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi$$

$$= \sum_{m=0}^{\infty}\epsilon_m\int_0^\infty J_m(\xi\rho)\,J_m(\xi r)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi\cos m(\theta-\phi), \tag{A 6}$$

where we have used equations from Erdélyi *et al.* (1953, 7.15(30); 1954, 8.2(18)) and

$$b^2 = r^2+\rho^2-2r\rho\cos(\theta-\phi).$$

Substituting for $[\tilde{G}_{zz}]$ and integrating over θ, we find that

$$4\pi(1-\nu)\,\tilde{\Sigma}_{zzz}(\mathrm{P};\mathrm{p}_0)$$

$$= -\mu\sum_{n=0}^{\infty}\epsilon_n\int_0^\infty \xi^2(1+\xi z)\,J_n(\xi\rho)\,M_n(\xi,r_0)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi\cos n(\theta_0-\phi), \tag{A 7}$$

where

$$M_n(\xi,r_0) = \int_0^1 f_n^z(r;r_0)\,J_n(\xi r)\,r\,\mathrm{d}r.$$

If we now substitute for f_n^z from (A 2), interchange the order of integration and integrate over r (Erdélyi *et al.* 1954, 8.5(33)), we obtain

$$M_n(\xi,r_0) = \frac{4(1-\nu)}{\pi^2}\,r_0^n\int_{r_0}^1 \frac{j_n(\xi t)\,\mathrm{d}t}{t^{n-1}(t^2-r_0^2)^{\frac{1}{2}}}, \tag{A 8}$$

where $j_n(x) = (\frac{1}{2}\pi/x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x)$. By using this in (A 7), and integrating over the crack, it is straightforward to show that

$$\frac{1}{\mu}\int_\gamma \tilde{\Sigma}_{zzz}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_{\mathrm{q}} = -1. \tag{A 9}$$

Let us now examine $\tilde{\Sigma}_{zzz}(\mathrm{P};\mathrm{p}_0)$ for P near p_0. For small z, the most significant contribution to the integral in (A 7) will come from large ξ; write

$$M_n = M_n^{\mathrm{P}} + M_n'', \tag{A 10}$$

where

$$M_n^{\mathrm{P}}(\xi,r_0) = \frac{4(1-\nu)}{\pi^2}\,r_0^n\int_{r_0}^\infty \frac{j_n(\xi t)\,\mathrm{d}t}{t^{n-1}(t^2-r_0^2)^{\frac{1}{2}}}$$

and $M_n'' = M_n - M_n^{\mathrm{P}}$. We can evaluate M_n^{P} explicity as (Erdélyi *et al.* 1954, 8.5(32))

$$M_n^{\mathrm{P}}(\xi,r_0) = \{2(1-\nu)/\pi\xi\}\,J_n(\xi r_0), \tag{A 11}$$

while $M''_n = O(\xi^{-2})$ as $\xi \to \infty$ (integrate by parts). Using (A 10) and an obvious notation, we can write (A 7) as

$$\tilde{\Sigma}_{zzz} = \Sigma^{\mathrm{P}}_{zzz} + \Sigma''_{zzz};$$

$\Sigma^{\mathrm{P}}_{zzz}$ is given by

$$\Sigma^{\mathrm{P}}_{zzz}(\mathrm{P};\mathrm{p}_0) = \frac{-\mu}{2\pi}\int_0^\infty \xi(1+\xi z)\, J_0(\xi b_0)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi,$$

where we have used (A 6) and (A 11), and

$$b_0^2 = r_0^2 + \rho^2 - 2r_0\rho\cos(\theta_0 - \phi).$$

This is precisely the stress component τ_{zz} due to a unit point force, acting normally at p_0, on the otherwise free surface of a semi-infinite elastic solid (this result may be proved by using Muki's (1960) solution); the corresponding normal stress components for this boundary-value problem are given by

$$(\Sigma^{\mathrm{P}}_{xzz}, \Sigma^{\mathrm{P}}_{yzz}, \Sigma^{\mathrm{P}}_{zzz}) = (-3\mu z^2/2\pi R_0^5)\,(x - x_0, y - y_0, z),$$

where $R_0^2 = b_0^2 + z^2$. Furthermore, it is elementary to show that

$$\frac{1}{\mu}\int_{H_a} \Sigma^{\mathrm{P}}_{izz}(\mathrm{q};\mathrm{p}_0)\, n_i\,\mathrm{d}s_{\mathrm{q}} = -1,$$

where H_a is a hemispherical surface, of radius a and centre p_0, and $\boldsymbol{n}$ is the unit normal on H_a pointing away from p_0. We can also show that Σ''_{zzz} is continuous and bounded in a neighbourhood of the crack, including the crack faces but not the edge.

Thus we see that

$$\tilde{\Sigma}_{izz}(\mathrm{P};\mathrm{p}_0) = O(R_0^{-2}) \quad \text{as} \quad R_0 \to 0 \tag{A 12}$$

and

$$[\tilde{G}_{zz}(\mathrm{p};\mathrm{p}_0)] = O(b_0^{-1}) \quad \text{as} \quad b_0 \to 0, \tag{A 13}$$

where p has cylindrical polar coordinates $(\rho, \phi, 0)$. Moreover, using (A 9) and (A 12), we have

$$\frac{1}{\mu}\int f_i(\mathrm{q})\,\tilde{\Sigma}_{izz}(\mathrm{q};\mathrm{p}_0)\,\mathrm{d}s_{\mathrm{q}} = -f_z(\mathrm{p}_0),$$

provided that $f_z(\mathrm{q})$ is Hölder continuous at $\mathrm{q} = \mathrm{p}_0$, since $\tilde{\Sigma}_{xzz}(\mathrm{q};\mathrm{p}_0) = \tilde{\Sigma}_{yzz}(\mathrm{q};\mathrm{p}_0) = 0$; thus, our constructed solution has the required 'δ-function' property at $\mathrm{q} = \mathrm{p}_0$.

Similar results may be obtained for the other components of $\tilde{\Sigma}_{ijk}$; the detailed analysis is more complicated and is given in Martin (1979). In particular, we have

$$\tilde{\Sigma}_{ijk} = \Sigma^{\mathrm{P}}_{ijk} + \Sigma''_{ijk}, \tag{A 14}$$

where $n_j\Sigma''_{ijk}$ is continuous on the crack faces,

$$\frac{1}{\mu}\int_{H_a} \Sigma^{\mathrm{P}}_{ijk}(\mathrm{q};\mathrm{p}_0)\, n_j\,\mathrm{d}s_{\mathrm{q}} = -\delta_{ik}, \tag{A 15}$$

and

$$\frac{1}{\mu}\int_{\gamma} f_i(\mathrm{q})\,\tilde{\Sigma}_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j\,\mathrm{d}s_{\mathrm{q}} = -f_k(\mathrm{p}_0). \tag{A 16}$$

Also, the particular singular behaviour at p_0, given by (A 12) and (A 13), is typical of all the other components.

We shall now prove a result that is required in §4. Let H_ϵ^+ and H_ϵ^- be small hemispheres of radius ϵ, centred on p_0^+ and p_0^-, respectively. Then, if $u_i(\mathrm{p})$ is Hölder continuous, we have

$$\lim_{\epsilon\to 0}\frac{1}{\mu}\int_{H_\epsilon^+\cup H_\epsilon^-} u_i(\mathrm{q})\,\tilde{\Sigma}_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j\,\mathrm{d}s_{\mathrm{q}} = -[u_k(\mathrm{p}_0)], \tag{A 17}$$

where the discontinuity in u_i across the crack, $[u_i]$, is defined by (2.7). To prove this result, we write

$$\int_{H_\epsilon^\pm} u_i(\mathrm{q})\,\tilde{\Sigma}_{ijk}(\mathrm{q};\mathrm{p}_0)\,n_j\,\mathrm{d}s_{\mathrm{q}} = u_i(\mathrm{p}_0^\pm)\int_{H_\epsilon^\pm}\tilde{\Sigma}_{ijk}\,n_j\,\mathrm{d}s_{\mathrm{q}} + \int_{H_\epsilon^\pm}\{u_i(\mathrm{q})-u_i(\mathrm{p}_0^\pm)\}\,\tilde{\Sigma}_{ijk}\,n_j\,\mathrm{d}s_{\mathrm{q}}.$$

The second integral vanishes as $\epsilon\to 0$ if we note the continuity properties of u_i and the singular behaviour of $\tilde{\Sigma}_{ijk}$; by using (A 14), the first integral may be written as

$$\int_{H_\epsilon^\pm}\Sigma_{ijk}^{\mathrm{P}}(\mathrm{q};\mathrm{p}_0)\,n_j\,\mathrm{d}s_{\mathrm{q}} + \int_{H_\epsilon^\pm}\Sigma_{ijk}''(\mathrm{q};\mathrm{p}_0)\,n_j\,\mathrm{d}s_{\mathrm{q}}.$$

The second of these also vanishes as $\epsilon\to 0$, while the first (which is independent of ϵ) may be evaluated by using (A 15). Collecting together the various terms, we obtain (A 17).

Finally, let us show that $\tilde{G}_{ij}(\mathrm{P};\mathrm{p}_0)$ satisfies the edge condition (S4). Let p^{e} denote the point on the crack edge which is nearest to P, i.e. P and p^{e} have cylindrical polar coordinates (ρ,ϕ,z) and $(1,\phi,0)$, respectively. Then it can be shown that (see, for example, Krenk 1979)

$$\tilde{G}_{ij}(\mathrm{P};\mathrm{p}_0)-\tilde{G}_{ij}(\mathrm{p}^{\mathrm{e}};\mathrm{p}_0) = s^{\frac{1}{2}}k_{ij}+O(s)$$

as $s\to 0$, where k_{ij} can be determined from $[\tilde{G}_{ij}]$, and $s^2 = (\rho-1)^2+z^2$, i.e. s is the distance of P from the crack edge. Hence, $\tilde{G}_{ij}(\mathrm{P};\mathrm{p}_0)$ is bounded if $\tilde{G}_{ij}(\mathrm{p}^{\mathrm{e}};\mathrm{p}_0)$ is bounded. Suppose now that p_0 does not lie on the crack edge; for if it does, it follows that $\tilde{G}_{ij}(\mathrm{P};\mathrm{p}_0) = 0$. Thus we can partition γ, such that $\gamma = \gamma_1\cup\gamma_2$ with $\mathrm{p}_0\in\gamma_1$ and $\mathrm{p}^{\mathrm{e}}\in\gamma_2$, and write

$$\tilde{G}_{kl}(\mathrm{p}^{\mathrm{e}};\mathrm{p}_0) = I_1+I_2,$$

where

$$I_m(\mathrm{p}^{\mathrm{e}};\mathrm{p}_0) = \int_{\gamma_m}[\tilde{G}_{il}(\mathrm{q};\mathrm{p}_0)]\,\tilde{\Sigma}_{ijk}^{\mathrm{f}}(\mathrm{q};\mathrm{p}^{\mathrm{e}})\,n_j\,\mathrm{d}s_{\mathrm{q}},$$

$m = 1, 2$. On γ_1, $\tilde{\Sigma}_{ijk}^{\mathrm{f}}(\mathrm{q};\mathrm{p}^{\mathrm{e}})$ is continuous while $[\tilde{G}_{il}(\mathrm{q};\mathrm{p}_0)]$ has an integrable singularity at $\mathrm{q} = \mathrm{p}_0$ given by (A 13), and so I_1 is bounded. On γ_2, $[\tilde{G}_{il}(\mathrm{q};\mathrm{p}_0)]$ is continuous and vanishes when $q \in \partial\gamma$. In particular, $[\tilde{G}_{il}(p^e; p_0)] = 0$. It follows that I_2 is bounded, even though $\tilde{\Sigma}^f_{ijk}(p; q) = O(R^{-2})$ as $R \to 0$. Combining these results, we see that $\tilde{G}_{ij}(p^e; p_0)$ is bounded and hence $\tilde{G}_{ij}(P; p_0)$ satisfies the edge condition.

Appendix B. Expressions for the kernel

In this Appendix, we shall give expressions for $K_{ij}(q; p_0)$ in (4.2). We begin by obtaining an expansion for Φ analogous to (A 6). For $z > 0$, we have

$$\begin{aligned} 4\pi\Phi(q; P) &= e^{ikR}/R \\ &= \sum_{m=0}^{\infty} \epsilon_m \int_0^{\infty} J_m(\xi r)\, J_m(\xi\rho)\, \frac{\xi}{\gamma}\, e^{-\gamma z}\, d\xi \cos m(\theta - \phi), \end{aligned}$$

where, to satisfy the radiation conditions, $\gamma(\xi)$ must be interpreted as

$$\begin{aligned} \gamma(\xi) &= (\xi^2 - k^2)^{\frac{1}{2}}, \quad \xi > k, \\ &= -i(k^2 - \xi^2)^{\frac{1}{2}}, \quad 0 \leqslant \xi \leqslant k, \end{aligned}$$

and we have used equation 8.2 (26) from Erdélyi *et al.* (1954); a similar expansion may be found for $\Psi(q; P)$, involving $\beta(\xi) = (\xi^2 - K^2)^{\frac{1}{2}}$. Using (A 1), we have

$$\begin{aligned} &\int_{\gamma} [\tilde{G}_{zz}(q; p)]\, \Phi(q; P)\, ds_q \\ &\qquad = \frac{1}{4} \sum_{n=0}^{\infty} \epsilon_n \int_0^{\infty} J_n(\xi\rho)\, M_n(\xi, r_0)\, \frac{\xi}{\gamma}\, e^{-\gamma z}\, d\xi \cos n(\phi - \theta_0), \end{aligned}$$

where M_n is defined by (A 8) and we have integrated over θ. If we now use (2.2), (3.5) and (3.6), we obtain

$$\frac{1}{\mu} \Sigma_{zzz}(P; p_0) = \sum_{n=0}^{\infty} \epsilon_n \int_0^{\infty} J_n(\xi\rho)\, M_n(\xi, r_0) \hat{f}(\xi; z)\, d\xi \cos n(\phi - \theta_0), \tag{B 1}$$

where

$$4K^2 \hat{f}(\xi; z) = (\xi/\gamma)\{(2\xi^2 - K^2)^2 e^{-\gamma z} - 4\xi^2 \beta\gamma e^{-\beta z}\}.$$

As $K \to 0$ ($\xi \to \infty$), we find that

$$4(1-\nu) \hat{f}(\xi; z) = -(1 + \xi z)\, \xi^2 e^{-\xi z} + O(K^2).$$

Subtracting $\tilde{\Sigma}_{zzz}(P; p_0)$, as given by (A 7), from (B 1), we find that the singularities at $P = p_0$ cancel and we can then put $z = 0$, yielding

$$K_{zz}(p; p_0) = \sum_{n=0}^{\infty} \epsilon_n \int_0^{\infty} J_n(\xi\rho)\, M_n(\xi, r_0) f(\xi)\, d\xi \cos n(\phi - \theta_0),$$

where

$$4K^2 f(\xi) = (\xi/\gamma)\{(2\xi^2 - K^2)^2 - 4\xi^2 \beta\gamma\} - 2\xi^2(k^2 - K^2).$$

Note that f and M_n are independent of r_0 and K, respectively.

The analysis for the shear problem is similar, but more lengthy. We shall only quote the results here (the details may be found in Martin (1979)):

$$K_{rr}(\mathrm{p};\mathrm{p}_0) = \int_0^\infty (h_1-h_2)\,J_1(\xi\rho)\,\mathrm{P}_0^+(\xi;r_0)\,\mathrm{d}\xi - \sum_{n=1}^\infty \int_0^\infty \{A_n^+ J_{n+1}(\xi\rho)+B_n^+ J_{n-1}(\xi\rho)\}\,\mathrm{d}\xi \cos n(\phi-\theta_0)$$

$$K_{\theta r}(\mathrm{p};\mathrm{p}_0) = -\sum_{n=1}^\infty \int_0^\infty \{A_n^+ J_{n+1}(\xi\rho)-B_n^+ J_{n-1}(\xi\rho)\}\,\mathrm{d}\xi \sin n(\phi-\theta_0),$$

$$K_{r\theta}(\mathrm{p};\mathrm{p}_0) = \sum_{n=1}^\infty \int_0^\infty \{A_n^- J_{n+1}(\xi\rho)+B_n^- J_{n-1}(\xi\rho)\}\,\mathrm{d}\xi \sin n(\phi-\theta_0),$$

$$K_{\theta\theta}(\mathrm{p};\mathrm{p}_0) = -\int_0^\infty (h_1+h_2)\,J_1(\xi\rho)\,\mathrm{P}_0^-(\xi;r_0)\,\mathrm{d}\xi - \sum_{n=1}^\infty \int_0^\infty \{A_n^- J_{n+1}(\xi\rho)-B_n^- J_{n-1}(\xi\rho)\}\,\mathrm{d}\xi \cos n(\phi-\theta_0),$$

where

$$K^2h_1(\xi) = \xi^3(4\xi^2-3K^2-4\gamma\beta)/\beta+\xi^2(K^2-2k^2),$$

$$h_2(\xi) = -h_1(\xi)+2\xi(\beta-\xi),$$

$$A_n^\pm(\xi;r_0) = h_2\mathrm{P}_n^\pm+h_1\mathrm{Q}_n^\pm, \quad B_n^\pm(\xi;r_0) = h_1\mathrm{P}_n^\pm+h_2\mathrm{Q}_n^\pm,$$

$$\mathrm{Q}_n^+ + \mathrm{Q}_n^- = \alpha(\mathrm{P}_n^+ + \mathrm{P}_n^-) = \frac{\nu r_0^{n+1}}{2\pi^2}\int_{r_0}^1 \frac{j_{n+1}(\xi t)\,\mathrm{d}t}{t^n(t^2-r_0^2)^{\frac{1}{2}}},$$

$$\mathrm{Q}_n^+ - \mathrm{Q}_n^- = \alpha(\mathrm{P}_n^+ - \mathrm{P}_n^-) + \frac{2(1-\nu)r_0^{n+1}}{(2-\nu)\pi^2}\int_{r_0}^1 \frac{j_{n-1}(\xi t)}{t^n(t^2-r_0^2)^{\frac{1}{2}}}\frac{t^2}{r_0^2}\,\mathrm{d}t,$$

$$\mathrm{P}_n^+ - \mathrm{P}_n^- = \frac{-\nu r_0^{n+1}}{2\pi^2}\int_{r_0}^1 \frac{j_{n+1}(\xi t)}{t^n(t^2-r_0^2)^{\frac{1}{2}}}\left(2n+1-2n\frac{t^2}{r_0^2}\right)\mathrm{d}t,$$

and $\alpha = \nu/(2-\nu)$.