ACOUSTIC SCATTERING AND RADIATION PROBLEMS, AND THE NULL-FIELD METHOD

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The best known methods for solving the scattering and radiation problems of acoustics are integral-equation methods. However, it is also known that the simplest of these methods yield equations which are not uniquely solvable at certain discrete sets of frequencies (the irregular frequencies). In this paper, we shall analyse an alternative method (the null-field method, or *T*-matrix method). We prove that the infinite system of null-field equations always has precisely one solution, i.e. the unphysical irregular frequencies do not occur with this method. Moreover, we also prove that the solution of the original boundary-value problem can always be determined (at any point exterior to the scatterer) from the solution of the null-field equations. We prove these results in two dimensions, for two radiation problems (the exterior Neumann problem and the exterior Dirichlet problem) and two scattering problems (scattering by a sound-hard body and scattering by a sound-soft body); similar results can be proved in three dimensions. We also prove some subsidiary results, concerning the solvability of certain boundary integral equations and the completeness of certain sets of radiating wave-functions, and give a discussion of related numerical techniques.

1. Introduction

The null-field method (or *T*-matrix method) is used widely for obtaining numerical solutions to various scattering and radiation problems. This method was first devised by Waterman [1] for electromagnetic scattering problems. Later, it was developed for treating problems in acoustics [2-5], elastodynamics [6, 7] and hydrodynamics [8]; for a collection of papers on the null-field method, see the conference proceedings edited by Varadan and Varadan [9].

In this paper, we shall use the null-field method to solve four basic boundary-value problems in acoustics. For each problem, we derive the corresponding infinite system of null-field equations. We then prove the following theoretical results for each problem:

(a) The null-field equations always have precisely one solution.

(b) The solution of the boundary-value problem can always be constructed from the solution of the null-field equations.

These results are in marked contrast to the usual integral-equation methods, which are known to suffer from difficulties at the unphysical irregular frequencies, corresponding to eigenvalues of certain interior problems; see, e.g. [10, 11], and Sections 3 and 4 of the present paper.

In the next Section, we formulate our four boundary-value problems; two of these are radiation problems (the exterior Neumann problem and the exterior Dirichlet problem) and two are scattering problems (scattering by a sound-hard obstacle and by a sound-soft obstacle). Since our proofs of (a) and (b) are based on integral-equation methods, we give a description of these in Sections 3 and 4. In Section 4, we also prove some new results on the solvability of certain integral equations which arise in the study of scattering problems.

Section 5 includes our proof of (a) and (b) for the two radiation problems (some of these results have been stated previously in [5]). In this section, we also prove certain completeness results which are useful when solving the null-field equations (see Corollaries 7.2 and 8.2). Section 6 includes our proofs of (a) and (b) for the two scattering problems.

In Section 7, we discuss the (numerical) solution of the null-field equations. We describe two simple methods, derive the well-known T-matrix (for scattering problems), and examine the computational difficulties that can arise.

In summary, the purpose of this paper is to place the null-field method on a firm, mathematical foundation. In particular, the absence of irregular frequencies is an important feature of the method, since the location of these frequencies in the spectrum is unknown *a priori* for a given geometry. However one difficult question still remains unanswered: can it be proved that any given numerical implementation of the null-field method will yield a sequence of approximations which converges to the unique solution of the infinite system of null-field equations?

2. Statement of the four problems

In this paper, we shall be concerned with finding solutions of the two-dimensional Helmholtz equation,

$$(\nabla^2 + k^2)u = 0 \qquad (2.1)$$

in the infinite region D, exterior to a simple closed Lyapunov curve ∂D, such that u also satisfies certain other conditions. If u satisfies (2.1), we shall call u a *wave-function*. If u also satisfies the radiation condition

$$r_P^{1/2}\left(\frac{\partial u}{\partial r_P} - iku\right) \to 0 \quad \text{as } r_P \to \infty,$$

we shall call u a *radiating wave-function*. We can now state our four boundary-value problems:

Exterior Neumann problem $\mathcal{N}$. Find a radiating wave-function $u(P)$ which satisfies

$$\frac{\partial u(q)}{\partial n_q} = f(q) \quad \text{on } \partial D. \qquad (2.2)$$

Exterior Dirichlet problem $\mathcal{D}$. Find a radiating wave-function $u(P)$ which satisfies

$$u(q) = g(q) \quad \text{on } \partial D. \qquad (2.3)$$

Scattering problem $\mathcal{S}_1$. Find a wave-function $u(P)$ which satisfies

$$\frac{\partial u(q)}{\partial n_q} = 0 \quad \text{on } \partial D, \qquad (2.4)$$

and is such that $u - u^{(i)}$ is a radiating wave-function.

Scattering problem $\mathcal{S}_2$. Find a wave-function $u(P)$ which satisfies

$$u(q) = 0 \quad \text{on } \partial D, \qquad (2.5)$$

and is such that $u - u^{(i)}$ is a radiating wave-function.

We shall use the following notation: capital letters P, Q denote points of D; lower-case letters p, q denote points of ∂D; the origin O is taken at an arbitrary point inside D_-, the complement of $D \cup \partial D$ (i.e. D_- denotes the interior of the body); P_-, Q_- denote points of D_-; r_P is the length OP; and $\partial/\partial n_q$ denotes normal differentiation at the point q, in the direction from D towards ∂D (if no confusion can arise, we shall omit the subscript q).

The functions $f(p)$ and $g(p)$ are prescribed on ∂D. Thus, $\mathcal{N}$ and $\mathcal{D}$ are radiation problems, where the normal velocity and pressure, respectively, are prescribed on the body.

$\mathcal{S}_1$ and $\mathcal{S}_2$ correspond to the scattering of an incident wave $u^{(i)}(P)$ by a sound-hard body and a sound-soft body, respectively; $u^{(i)}$ is a prescribed function, which satisfies (2.1) everywhere, except possibly at a finite number of isolated points in D, e.g. $u^{(i)}$ may correspond to an incident plane wave, or a radiating line source.

The usual approach for solving these four problems is to derive an integral equation of the second kind, over the boundary ∂D. In the next two sections, we shall describe various ways of doing this.

3. Boundary integral equations for radiation problems

Let $G(P, Q)$ be any fundamental solution, i.e. $G(P, Q)$ is a radiating wave-function in D, with a logarithmic singularity at $P = Q$. The simplest choice for $G(P, Q)$ is the free-space wave source,

$$G_0(P, Q) = \tfrac{1}{2}i\pi H_0^{(1)}(k|r_P - r_Q|), \qquad (3.1)$$

where $H_n^{(1)}(z)$ denotes the Hankel function of the first kind and order n.

In order to obtain boundary integral equations, we begin by applying Green's theorem to G_0 and another radiating wave-function, v say, in the region between ∂D and S_∞, where S_∞ is a large circle, of radius r_∞ and centre O. Since G_0 and v both satisfy a radiation condition, the contribution from integrating over S_∞ vanishes as $r_\infty \to \infty$. Hence, we obtain the well-known Helmholtz formulae [5]:

$$2\pi v(P) = \int_{\partial D} \left\{ G_0(P, q) \frac{\partial}{\partial n_q} v(q) \right.$$

$$\left. - v(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} \mathrm{d}s_q, \quad (3.2)$$

$$\pi v(p) = \int_{\partial D} \left\{ G_0(p, q) \frac{\partial}{\partial n_q} v(q) \right.$$

$$\left. - v(q) \frac{\partial}{\partial n_q} G_0(p, q) \right\} \mathrm{d}s_q, \quad (3.3)$$

$$0 = \int_{\partial D} \left\{ G_0(P_-, q) \frac{\partial}{\partial n_q} v(q) \right.$$

$$\left. - v(q) \frac{\partial}{\partial n_q} G_0(P_-, q) \right\} \mathrm{d}s_q. \quad (3.4)$$

Similar formulae may be obtained when $G_0(P, Q)$ is replaced by any other fundamental solution, $G(P, Q)$.

Suppose that $u(P)$ solves $\mathcal{N}$. Then, (3.2) becomes

$$2\pi u(P) = \int_{\partial D} \left\{ G_0(P, q) f(q) \right.$$

$$\left. - u(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} \mathrm{d}s_q. \quad (3.5)$$

Similarly, if $u(P)$ solves $\mathcal{D}$, we obtain

$$2\pi u(P) = \int_{\partial D} \left\{ G_0(P, q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - g(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} \mathrm{d}s_q. \quad (3.6)$$

(3.5) ((3.6)) is a representation for the solution of $\mathcal{N}(\mathcal{D})$ in terms of a distribution of sources and dipoles over ∂D. (Other representations are possible; see Section 4.) Such distributions have well-known properties, e.g. a source distribution is continuous across ∂D, but its normal derivative is discontinuous across ∂D; see, e.g. [11]. We shall make use of these properties in deriving boundary integral equations.

Suppose that $u(P)$ solves $\mathcal{N}$. From (3.3), we have

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q)\, \mathrm{d}s_q$$

$$= \int_{\partial D} G_0(p, q) f(q)\, \mathrm{d}s_q, \qquad (3.7)$$

which is a Fredholm integral equation of the second kind for the unknown boundary values of u. This integral equation possesses a unique solution unless the corresponding homogeneous integral equation,

$$\pi v(p) + \int_{\partial D} v(q) \frac{\partial}{\partial n_q} G_0(p, q)\, \mathrm{d}s_q = 0 \qquad (3.8)$$

has a non-trivial solution. It is known that (3.8) does have non-trivial solutions if and only if k^2 coincides with an eigenvalue of the corresponding interior Dirichlet problem [10, 11]. We denote the infinite discrete set of these eigenvalues by I_N. Thus, if $k^2 \in I_N$, there exists a non-trivial wave-function in D_- which vanishes on ∂D. At these

values of k^2 (called the *irregular* values), the integral equation (3.7) does not have a unique solution for general $f(p)$.

So, if k^2 is *not* an irregular value, we can find $u(q)$, the unique solution of (3.7), and then define a function $U(P)$ by

$$U(P) = \frac{1}{2\pi} \int_{\partial D} \left\{ G_0(P, q)f(q) \right.$$

$$\left. - u(q)\frac{\partial}{\partial n_q} G_0(P, q) \right\} \, ds_q. \quad (3.9)$$

Does $U(P)$ solve $\mathcal{N}$? This (non-trivial) question is answered affirmatively by

Theorem 1. *If $k^2 \notin I_N$ and $u(q)$ is the unique solution of* (3.7), *the function $U(P)$, defined by* (3.9), *solves the boundary-value problem $\mathcal{N}$.*

Proof. (A proof of this theorem has been given by Kleinman and Roach [11]. We shall give their proof here, since we shall wish to modify it in the sequel.) It is clear that $U(P)$ is a radiating wave-function. It only remains to show that $U(P)$ also satisfies the boundary condition on ∂D, namely

$$\frac{\partial}{\partial n_p} U(p) = f(p). \quad (3.10)$$

If we differentiate (3.9) and let P approach ∂D, we see that U will satisfy (3.10) if $u(q)$ also satisfies the following *compatibility condition*:

$$\frac{\partial}{\partial n_p} \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q$$

$$= -\pi f(p) + \int_{\partial D} f(q) \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q. \quad (3.11)$$

(Note that a sufficient condition for the existence of the left-hand side of (3.11) is that u be differentiable on the Lyapunov curve ∂D; see, e.g. [11].)

Let us define a function in D_- by

$$U_0(P_-) = \int_{\partial D} \left\{ G_0(P_-, q)f(q) \right.$$

$$\left. - u(q)\frac{\partial}{\partial n_q} G_0(P_-, q) \right\} \, ds_q. \quad (3.12)$$

U_0 is a wave-function in D_- and, if we let P_- approach ∂D, we find that

$$U_0(p) = -\pi u(p) + \int_{\partial D} \left\{ G_0(p, q)f(q) \right.$$

$$\left. - u(q)\frac{\partial}{\partial n_q} G_0(p, q) \right\} \, ds_q$$

$$= 0, \quad \text{by (3.7)}.$$

Since $k^2 \notin I_N$, it follows (by the definition of I_N) that U_0 vanishes identically in D_- and, in particular, $\partial U_0(p)/\partial n_p^- = 0$, where $\partial/\partial n_p^-$ denotes the normal derivative at p when p is approached from inside D_-. Differentiating (3.12), we find that

$$0 = \frac{\partial}{\partial n_p^-} U_0(p)$$

$$= -\pi f(p) + \int_{\partial D} f(q) \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q$$

$$- \frac{\partial}{\partial n_p} \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q.$$

Thus, the compatibility condition (3.11) is satisfied, and so (3.9) solves $\mathcal{N}$. This completes the proof of Theorem 1.

Let us now consider the exterior Dirichlet problem, i.e. suppose that $u(P)$ solves $\mathcal{D}$. If we differentiate (3.2), and then let P approach ∂D, we obtain

$$\pi \frac{\partial u(p)}{\partial n_p} - \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q$$

$$= -\frac{\partial}{\partial n_p} \int_{\partial D} g(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q, \quad (3.13)$$

where we have used (2.3). This is a Fredholm integral equation of the second kind for the

unknown boundary values of $\partial u/\partial n$; it possesses a unique solution unless the corresponding homogeneous equation,

$$\pi v(p) - \int_{\partial D} v(q) \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q = 0, \quad (3.14)$$

has a non-trivial solution. It is known that (3.14) does have non-trivial solutions if and only if k^2 coincides with an eigenvalue of the corresponding interior Neumann problem [10, 11]. We denote the infinite set of these eigenvalues by I_D. Thus, if $k^2 \in I_D$, there exists a nontrivial wave-function in D_- whose normal derivative vanishes on ∂D. At these values of k^2, the integral equation (3.13) does not have a unique solution for general $g(p)$. When k^2 is not an irregular value, we can solve $\mathcal{D}$ by solving (3.13):

Theorem 2. *If* $k^2 \notin I_D$, *and* $\partial u/\partial n$ *is the unique solution of* (3.13), *the function* $V(P)$, *defined by*

$$V(P) = \frac{1}{2\pi} \int_{\partial D} \left\{ G_0(P, q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - g(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} \, ds_q, \quad (3.15)$$

solves the boundary-value problem $\mathcal{D}$.

Proof. (This proof has also been given by Kleinman and Roach [11].) Since V is clearly a radiating wave-function in D, we only need to show that $V(p) = g(p)$, i.e. $\partial u/\partial n$ must satisfy

$$\int_{\partial D} G_0(p, q) \frac{\partial u(q)}{\partial n_q} \, ds_q$$

$$= \pi g(p) + \int_{\partial D} g(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q.$$

$$(3.16)$$

We define a wave-function in D_- by

$$V_0(P_-) = \int_{\partial D} \left\{ G_0(P_-, q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - g(q) \frac{\partial}{\partial n_q} G_0(P_-, q) \right\} \, ds_q. \quad (3.17)$$

Differentiating, and letting P_- approach ∂D, we find that

$$\frac{\partial}{\partial n_p^-} V_0(p) = -\pi \frac{\partial u(p)}{\partial n_p}$$

$$+ \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q$$

$$- \frac{\partial}{\partial n_p} \int_{\partial D} g(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q$$

$$= 0, \quad \text{by (3.13)}.$$

Since $k^2 \notin I_D$, it follows that V_0 vanishes identically in D_- and, in particular, $V_0(p) = 0$. Hence, from (3.17),

$$0 = V_0(p)$$

$$= -\pi g(p) + \int_{\partial D} \left\{ G_0(p, q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - g(q) \frac{\partial}{\partial n_q} G_0(p, q) \right\} \, ds_q,$$

i.e. the compatibility condition (3.16) is satisfied, and $V(P)$ solves $\mathcal{D}$.

4. Boundary integral equations for scattering problems

Consider the boundary-value problem $\mathcal{S}_1$ (scattering by a sound-hard body). Write

$$u^{(s)}(P) = u(P) - u^{(i)}(P), \quad (4.1)$$

where $u(P)$ solves $\mathcal{S}_1$. Thus $u^{(s)}(P)$ is a radiating wave-function which satisfies

$$\frac{\partial}{\partial n_p} u^{(s)}(p) = -\frac{\partial}{\partial n_p} u^{(i)}(p) \quad (4.2)$$

on ∂D. Hence, $u^{(s)}(P)$ solves $\mathcal{N}$, with boundary data (i.e. f) given by (4.2). It follows that we can solve $\mathcal{S}_1$ by using the method described in Section 3 for solving $\mathcal{N}$, i.e. if $k^2 \notin I_N$, we determine $u^{(s)}(q)$,

the unique solution of

$$\pi u^{(s)}(p) + \int_{\partial D} u^{(s)}(q) \frac{\partial}{\partial n_q} G_0(p,q)\, ds_q$$

$$= -\int_{\partial D} G_0(p,q) \frac{\partial}{\partial n_q} u^{(i)}(q)\, ds_q, \qquad (4.3)$$

and then, by Theorem 1,

$$u^{(s)}(P) = \frac{-1}{2\pi} \int_{\partial D} \left\{ G_0(P,q) \frac{\partial}{\partial n_q} u^{(i)}(q) \right.$$

$$\left. + u^{(s)}(q) \frac{\partial}{\partial n_q} G_0(P,q) \right\} ds_q. \qquad (4.4)$$

(4.3) is an integral equation of the second kind for $u^{(s)}(p)$. We shall now derive an integral equation of the second kind for $u(p) = u^{(s)}(p) + u^{(i)}(p)$; this equation is of the same form as (4.3), but has a much simpler right-hand side.

Apply Green's theorem in D_- to G_0 and $u^{(i)}$. Since $u^{(i)}$ is a non-singular wave-function in D_-, we obtain

$$0 = \int_{\partial D} \left\{ G_0(P,q) \frac{\partial}{\partial n_q} u^{(i)}(q) \right.$$

$$\left. - u^{(i)}(q) \frac{\partial}{\partial n_q} G_0(P,q) \right\} ds_q, \qquad (4.5)$$

$$-\pi u^{(i)}(p) = \int_{\partial D} \left\{ G_0(p,q) \frac{\partial}{\partial n_q} u^{(i)}(q) \right.$$

$$\left. - u^{(i)}(q) \frac{\partial}{\partial n_q} G_0(p,q) \right\} ds_q, \qquad (4.6)$$

and

$$-2\pi u^{(i)}(P_-) = \int_{\partial D} \left\{ G_0(P_-,q) \frac{\partial}{\partial n_q} u^{(i)}(q) \right.$$

$$\left. - u^{(i)}(q) \frac{\partial}{\partial n_q} G_0(P_-,q) \right\} ds_q. \qquad (4.7)$$

Also, since $u^{(s)}(P)$ is a radiating wave-function, it satisfies (3.2-4); adding these to (4.5-7), and using the boundary condition (4.2), we obtain

$$2\pi u^{(s)}(P) = -\int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(P,q)\, ds_q,$$

$$(4.8)$$

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p,q)\, ds_q$$

$$= 2\pi u^{(i)}(p), \qquad (4.9)$$

and

$$\int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(P_-,q)\, ds_q = 2\pi u^{(i)}(P_-).$$

$$(4.10)$$

(4.8) is an integral representation for $u^{(s)}(P)$ as a distribution of *dipoles* over ∂D, whilst (4.9) is a Fredholm integral equation of the second kind for $u(q)$. This equation is of the same form as (4.3) (and (3.7)), and hence has the same irregular values. (Although (4.9) has a very simple right-hand side, it does not seem to have been used widely for solving $\mathscr{S}_1$.) Moreover, in Theorem 3, we shall prove that if k^2 is not an irregular value, then (4.8) gives the solution of $\mathscr{S}_1$ when $u(q)$ solves (4.9).

It is interesting to compare our integral representations (4.4) and (4.8) with another standard form, obtained by writing

$$u^{(s)}(P) = \int_{\partial D} \mu(q) G_0(P,q)\, ds_q. \qquad (4.11)$$

This is an integral representation for $u^{(s)}(P)$ as a distribution of *sources* over ∂D; if we use the boundary condition (4.2), we find that the unknown source strength $\mu(q)$ satisfies

$$\pi \mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p,q)\, ds_q$$

$$= -\frac{\partial}{\partial n_p} u^{(i)}(p). \qquad (4.12)$$

(This method may also be used for solving $\mathcal{N}$.) This integral equation has a kernel which is the transpose of the kernel in (4.9), and hence has the same irregular values. (Here, we have used the symmetry of the fundamental solution (3.1).) Also, when k^2 is *not* an irregular value, we may substitute the unique solution of (4.12) into (4.11), and this will solve $\mathscr{S}_1$, since the boundary condition (4.2) is automatically satisfied if μ satisfies (4.12).

In summary, we have three different integral representations for $u^{(s)}(P)$, where $u = u^{(s)} + u^{(i)}$ is the solution of $\mathscr{S}_1$; these representations are listed in Table 1, together with the corresponding integral equations of the second kind.

Table 1

Representations and integral equations for scattering problem $\mathscr{S}_1$

Representation for $u^{(s)}(P)$		Integral equation
Sources + dipoles,	(4.4)	(4.3)
Dipoles only,	(4.8)	(4.9)
Sources only,	(4.11)	(4.12)

Let us now prove

Theorem 3. *If $k^2 \notin I_N$ and $u(q)$ is the unique solution of* (4.9), *the function $U(P)$, defined by*

$$U(P) = u^{(i)}(P)$$

$$-\frac{1}{2\pi} \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(P, q) \, ds_q,$$

(4.13)

solves the boundary-value problem $\mathscr{S}_1$.

Proof. Clearly, $U - u^{(i)}$ is a radiating wave-function in D. U will also satisfy the boundary condition (2.4) if u satisfies

$$\frac{\partial}{\partial n_p} u^{(i)}(p)$$

$$= \frac{1}{2\pi} \frac{\partial}{\partial n_p} \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q. \quad (4.14)$$

If we now define a wave-function in D_- by

$$U_0(P_-) = \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_0(P_-, q) \, ds_q$$

$$-2\pi u^{(i)}(P_-),$$

we can use exactly the same arguments as used in the proof of Theorem 1 to prove that $u(q)$ satisfies (4.14), i.e. $U(P)$ solves $\mathscr{S}_1$.

Let us now consider $\mathscr{S}_2$ (scattering by a sound-soft body). We can solve $\mathscr{S}_2$ for $u(P)$ by solving $\mathscr{D}$ for $u^{(s)}(P)$, where (4.1) is satisfied in D and

$$u^{(s)}(p) = -u^{(i)}(p) \quad (4.15)$$

on ∂D. Thus (from Section 3), if $k^2 \notin I_D$, we determine $\partial u^{(s)} / \partial n$, the unique solution of

$$\pi \frac{\partial}{\partial n_p} u^{(s)}(p)$$

$$- \int_{\partial D} \frac{\partial}{\partial n_q} u^{(s)}(q) \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q$$

$$= \frac{\partial}{\partial n_p} \int_{\partial D} u^{(i)}(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q \quad (4.16)$$

and then, by Theorem 2,

$$u^{(s)}(P) = \frac{1}{2\pi} \int_{\partial D} \left\{ G_0(P, q) \frac{\partial}{\partial n_q} u^{(s)}(q) \right.$$

$$\left. + u^{(i)}(q) \frac{\partial}{\partial n_q} G_0(P, q) \right\} \, ds_q. \quad (4.17)$$

Alternatively, we could represent $u^{(s)}(P)$ as a distribution of dipoles over ∂D,

$$u^{(s)}(P) = \int_{\partial D} \nu(q) \frac{\partial}{\partial n_q} G_0(P, q) \, ds_q; \quad (4.18)$$

if we use the boundary condition (4.15), we find that $\nu(q)$ satisfies

$$\pi \nu(p) - \int_{\partial D} \nu(q) \frac{\partial}{\partial n_q} G_0(p, q) \, ds_q$$

$$= u^{(i)}(p). \quad (4.19)$$

Finally, we can obtain a representation as a distribution of sources over ∂D, as follows. $u^{(s)}$ satisfies (3.2–4) and $u^{(i)}$ satisfies (4.5–7). If we add (3.2) and (4.5), differentiate, let P approach ∂D, and then use (4.15), we obtain

$$\pi \frac{\partial u(p)}{\partial n_p} - \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_0(p, q) \, ds_q$$

$$= 2\pi \frac{\partial}{\partial n_p} u^{(i)}(p). \quad (4.20)$$

Similarly, if we add (3.2) and (4.5), we obtain

$$u^{(s)}(P) = \frac{1}{2\pi} \int_{\partial D} G_0(P, q) \frac{\partial u(q)}{\partial n_q} ds_q, \qquad (4.21)$$

whilst from (3.4) and (4.7),

$$\int_{\partial D} G_0(P_-, q) \frac{\partial u(q)}{\partial n_q} ds_q = -2\pi u^{(i)}(P_-). \qquad (4.22)$$

We can prove

Theorem 4. *If* $k^2 \notin I_D$ *and* $\partial u/\partial n$ *is the unique solution of* (4.20), *the function* $V(P)$, *defined by*

$$V(P) = u^{(i)}(P)$$

$$+ \frac{1}{2\pi} \int_{\partial D} G_0(P, q) \frac{\partial u(q)}{\partial n_q} ds_q, \qquad (4.23)$$

solves the boundary-value problem $\mathscr{S}_2$.

Proof. Clearly, $V - u^{(i)}$ is a radiating wave-function in D. V will also satisfy the boundary condition (2.5) if $\partial u/\partial n$ satisfies

$$u^{(i)}(p) = -\frac{1}{2\pi} \int_{\partial D} G_0(p, q) \frac{\partial u(q)}{\partial n_q} ds_q. \qquad (4.24)$$

If we now define a wave-function in D_- by

$$V_0(P_-) = \int_{\partial D} G_0(P_-, q) \frac{\partial u(q)}{\partial n_q} ds_q,$$

we can use exactly the same arguments as used in the proof of Theorem 2 to prove that $\partial u/\partial n$ satisfies (4.24), i.e. $V(P)$ solves $\mathscr{S}_2$.

Thus, we have three different integral representations for $u^{(s)}(P)$, where $u = u^{(s)} + u^{(i)}$ is the solution of $\mathscr{S}_2$; these representations are listed in table 2, together with the corresponding integral equations of the second kind.

When $k^2 \in I_N(I_D)$, all of the integral equations listed in Table 1 (2) are not uniquely solvable. This difficulty may be overcome by using a different fundamental solution in place of $G_0(P, Q)$ (see the next section and [10, 12, 13]), by taking a suitable linear combination of the

Table 2

Representations and integral equations for scattering problem $\mathscr{S}_2$

Representation for $u^{(s)}(P)$		Integral equation
Sources + dipoles,	(4.17)	(4.16)
Dipoles only,	(4.18)	(4.19)
Sources only,	(4.21)	(4.20)

Helmholtz formula (3.2) and its normal-derivative form (cf. (3.13)) (see, e.g. [14]), or by using a different integral representation for $u^{(s)}(P)$ (see [11] for references). However, the irregular frequencies are unphysical (i.e. they are a consequence of the method of solution), since it is well known that the original boundary-value problems always possess unique solutions; see [10] for references. In the remainder of this paper, we shall describe an alternative method (the null-field method) which always yields (theoretically) the unique solution of the boundary-value problem, i.e. irregular values do not occur with this method.

5. The null-field equations for radiation problems

Consider the interior integral relation (3.4). This asserts that the field induced at any point in D_- by the sources on ∂D is exactly cancelled by the field induced by the dipoles on ∂D. Waterman [1, 3] calls this the 'extended boundary condition', and (3.4) the 'extended integral equation'. We shall use (3.4) to derive the null-field equations for radiation problems.

We begin by noting that the free-space wave source (3.1) may be written as [3–5, 10]

$$G_0(P, Q) = \tfrac{1}{2}i\pi \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \psi_m^\sigma(Q)\hat{\psi}_m^\sigma(P) \qquad (5.1)$$

for $r_P < r_Q$, where

$$\psi_m^\sigma(Q) = H_m^{(1)}(kr_Q)E_m^\sigma(\theta_Q),$$

$$\hat{\psi}_m^\sigma(Q) = J_m(kr_Q)E_m^\sigma(\theta_Q),$$

$$E_m^1(\theta) = 2^{1/2} \cos m\theta \quad \text{for } m > 0,$$

$$E_m^2(\theta) = 2^{1/2} \sin m\theta \quad \text{for } m \geq 0,$$

$E_0^1 = 1$ and (r, θ) are cylindrical polar coordinates centred on O. (Note that our definition of E_m^σ differs from that used in [5].)

Let C_- be the inscribed circle to ∂D, which is centred on O. Similarly, let C_+ be the escribed circle to ∂D. Denote the interior of C_- by D_N. When P_- lies in D_N (where $r_p < r_q$), we may substitute (5.1) into (3.4) to give

$$\sum_{m=0}^{\infty} \sum_{\sigma=1}^{2}{}' \hat{\psi}_m^\sigma(P_-) \int_{\partial D} \left\{ u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) \right.$$

$$\left. - \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q) \right\} \, \mathrm{d}s_q = 0. \quad (5.2)$$

Since the regular wave-functions $\hat{\psi}_m^\sigma$ are orthogonal over any circle centred on O, it follows that each term in (5.2) must vanish, i.e.

$$\int_{\partial D} \left\{ u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) - \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q) \right\} \, \mathrm{d}s_q = 0$$

$$(\sigma = 1, 2; \, m = 0, 1, \ldots). \quad (5.3)$$

These are the null-field equations of acoustics [3–5].

For the exterior Neumann problem $(\mathcal{N})$, use of the boundary condition (2.2) leads to

$$\int_{\partial D} u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) \, \mathrm{d}s_q = f_m^\sigma$$

$$(\sigma = 1, 2; \, m = 0, 1, \ldots), \quad (5.4a)$$

where

$$f_m^\sigma = \int_{\partial D} f(q) \psi_m^\sigma(q) \, \mathrm{d}s_q \quad (5.4b)$$

are known constants. If X denotes the infinite set of functions $\{\partial \psi_m^\sigma / \partial n\}$, then the null-field equations simply give the 'moment' of $u(q)$ with respect to each function in X. Thus, $u(q)$ is to be determined from the infinite set of moment-like equations, (5.4). (Note that these are *not* integral equations.)

For the exterior Dirichlet problem $(\mathcal{D})$, use of the boundary condition (2.3) in (5.3) leads to

$$\int_{\partial D} \frac{\partial u(q)}{\partial n_q} \psi_m^\sigma(q) \, \mathrm{d}s_q = g_m^\sigma$$

$$(\sigma = 1, 2; \, m = 0, 1, \ldots), \quad (5.5a)$$

where

$$g_m^\sigma = \int_{\partial D} g(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) \, \mathrm{d}s_q \quad (5.5b)$$

are known constants, and $\partial u / \partial n$ is to be determined.

If we know both u and $\partial u / \partial n$ on ∂D, then we can find $u(P)$ everywhere in D from the integral representation (3.2). In particular, if P lies outside C_+, we can use (5.1) in (3.2) to give

$$u(P) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} c_m^\sigma \psi_m^\sigma(P) \quad (5.6a)$$

where

$$c_m^\sigma = \frac{\mathrm{i}}{4} \int_{\partial D} \left\{ \hat{\psi}_m^\sigma(q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - u(q) \frac{\partial}{\partial n_q} \hat{\psi}_m^\sigma(q) \right\} \, \mathrm{d}s_q. \quad (5.6b)$$

Since $u(P)$ is a radiating wave-function, we can *assume* that (5.6a) holds exterior to some large circle S_∞, centred on O, and then proceed to derive the null-field equations more simply; see [5, 15] for details.

Let us now prove that the null-field equations are always uniquely solvable. To do this, we show that $v(q)$ solves the null-field equations if and only if $v(q)$ solves an integral equation of the second kind which is known to possess a unique solution.

Consider $\mathcal{N}$. Initially, we multiply each of (5.4) by $\hat{\psi}_m^\sigma(P_-)$, where $P_- \in D_N$, and then sum, using (5.1), to give

$$U_0(P_-) = 0, \qquad P_- \in D_N, \quad (5.7)$$

where U_0 is defined by (3.12). However, since U_0 is a wave-function which vanishes in D_N, we can use continuation arguments to assert that U_0

vanishes everywhere in D_-. In particular, if we let P_- approach ∂D, we obtain (3.7), which is an integral equation of the second kind for $u(q)$. As we have already remarked (Section 3), (3.7) has a unique solution, except when $k^2 \in I_N$. Conversely, if $k^2 \notin I_N$, it follows that the unique solution of (3.7) also solves the null-field equations, (5.4). For, if $u(q)$ solves (3.7), we can define a function $U_0(P_-)$ by (3.12), which, by the arguments used in the proof of Theorem 1, vanishes everywhere in D_-, and so $u(q)$ satisfies (5.4).

At the irregular values of k^2, this argument must be modified. Multiply each of (5.4) by $a_m^\sigma \psi_m^\sigma (P_-)$, where $P - \epsilon D_N' = D_N \backslash \{O\}$ and the constants a_m^σ are unspecified at present. Adding the resulting equations to (5.7) gives

$$U_1(P_-) \equiv \int_{\partial D} \left\{ G_1(P_-, q) f(q) \right.$$

$$\left. - u(q) \frac{\partial}{\partial n_q} G_1(P_-, q) \right\} ds_q = 0, \quad (5.8)$$

where $G_1(P, Q)$ is a new (symmetric) fundamental solution, defined by

$$G_1(P, Q) = G_0(P, Q)$$

$$+ \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} a_m^\sigma \psi_m^\sigma (P) \psi_m^\sigma (Q). \quad (5.9)$$

Proceeding as before, we let P_- approach ∂D and obtain

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_1(p, q) ds_q$$

$$= \int_{\partial D} G_1(p, q) f(q) ds_q, \quad (5.10)$$

which is another integral equation of the second kind for $u(q)$.

Ursell [10] has considered a fundamental solution of the form (5.9). He found certain (complex) values for a_m^σ which ensured that $G_1(P, Q)$ satisfied a dissipative boundary condition on a circle lying inside D_N; he proved the following theorem.

Theorem 5. (Ursell [10].) *Let the function $F(p)$ be prescribed on ∂D. Then, the integral equation of the second kind*

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_1(p, q) ds_q$$

$$= F(p) \quad (5.11)$$

is uniquely solvable for all real values of k^2. Similarly, the equation

$$\pi u(p) - \int_{\partial D} u(q) \frac{\partial}{\partial n_p} G_1(p, q) ds_q$$

$$= F(p) \quad (5.12)$$

is also uniquely solvable for all real values of k^2.

Jones [12] and Ursell [13] have considered fundamental solutions of the form (5.9), but with only a finite series added to G_0. The following theorem is typical:

Theorem 6. *Let $F(p)$ be prescribed on ∂D and let*

$$G_1^{(N)}(P, Q) = G_0(P, Q)$$

$$+ \sum_{m=0}^{N} \sum_{\sigma=1}^{2} a_m^\sigma \psi_m^\sigma (P) \psi_m^\sigma (Q).$$

$$(5.13)$$

Then, the integral equation

$$\pi u(p) + \int_{\partial D} u(q) \frac{\partial}{\partial n_q} G_1^{(N)}(p, q) ds_q$$

$$= F(p) \quad (5.14)$$

is uniquely solvable at any given real value of k^2, provided that (i) N is sufficiently large, and (ii) the constants a_m^σ satisfy

$$|2a_m^\sigma + \tfrac{1}{2}i\pi| > \tfrac{1}{2}\pi, \quad (5.15)$$

for $\sigma = 1, 2$ and $m = 0, 1, \ldots N$. (Actually, the theorem is true if the inequality in (ii) is replaced by

$$|2a_m^\sigma + \tfrac{1}{2}i\pi| < \tfrac{1}{2}\pi, \quad (5.16)$$

for $\sigma = 1, 2$ and $m = 0, 1, \ldots N$.)

Notice that, although Theorem 6 only applies to the integral equation (5.14) (which arises in the solution of $\mathcal{N}$), it is straightforward to modify Ursell's proof [13] in order to prove this theorem when (5.14) is replaced by the equation

$$\pi u(p) - \int_{\partial D} u(q) \frac{\partial}{\partial n_p} G_1^{(N)}(p,q) \, ds_q = F(p)$$

(which arises in the solution of $\mathcal{D}$). However, in the sequel, we shall only use Theorem 5, i.e. we shall suppose that the coefficients a_m^σ take on the particular values found by Ursell [10]; these values are given in an appendix.

Returning to our problem, we see that if $u(q)$ satisfies the null-field equations (5.4), then, by taking a suitable linear combination of these equations, we see that $u(q)$ also satisfies an integral equation of the second kind (5.10), which is always uniquely solvable, by Theorem 5. Let us now prove the converse.

Suppose that $u(q)$ is the unique solution of (5.10). Then, we can use (5.8) to define a wave-function $U_1(P_-)$ which, by (5.10), vanishes on ∂D. We cannot immediately assert that U_1 vanishes everywhere in D_-, since G_1 is singular at O. However, if $P_- \in D_N'$, we can use (5.1) and rewrite (5.8) as

$$U_1(P_-) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} A_m^\sigma \chi_m^\sigma(P_-) \qquad (5.17)$$

where

$$A_m^\sigma = \int_{\partial D} \left\{ f(q) \psi_m^\sigma(q) \right.$$

$$\left. - u(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) \right\} ds_q \quad (5.18)$$

and

$$\chi_m^\sigma(P_-) = \tfrac{1}{2} i\pi \hat{\psi}_m^\sigma(P_-) + a_m^\sigma \psi_m^\sigma(P_-). \qquad (5.19)$$

Now, we wish to show that $u(q)$ satisfies (5.4), i.e. that $A_m^\sigma = 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots$. This can be proved by using an argument given by Ursell [13]. Consider the integral

$$I \equiv \int_{C_-} \left\{ U_1 \frac{\partial}{\partial n} U_1^* - U_1^* \frac{\partial}{\partial n} U_1 \right\} ds, \qquad (5.20)$$

where the asterisk denotes the complex conjugate. Since U_1 and U_1^* vanish on ∂D, an application of Green's theorem in $D_- \backslash D_N$ shows that $I = 0$. Ursell [13] then proved that if (5.15) (or (5.16)) holds for $\sigma = 1, 2$ and $m = 0, 1, \ldots$, then I can only vanish if

$$A_m^\sigma = 0 \quad \text{for } \sigma = 1, 2 \text{ and } m = 0, 1, \ldots.$$

In the appendix, it is shown that the inequality (5.16) is satisfied by the particular coefficients obtained by Ursell [10]. It follows from (5.17) that $U_1(P_-)$ vanishes everywhere in D_- and that $u(q)$ satisfies the null-field equations. We have thus proved the following theorem.

Theorem 7. *The null-field equations for the exterior Neumann problem of acoustics (in two dimensions)*, (5.4), *possess a unique solution for all real values of* k^2.

Corollary 7.1. *If* $u(q)$ *satisfies the null-field equations* (5.4) *(or the integral equation* (5.10)*), the solution of* $\mathcal{N}$ *is* $U(P)$, *defined by* (3.9).

Proof. We simply replace U_0 and G_0 by U_1 and G_1, respectively, in the proof of Theorem 1. In that proof, the restriction to regular values of k^2 (i.e. $k^2 \notin I_N$) was needed to ensure that $U_0(P_-) \equiv 0$. Here, we have already shown that $U_1(P_-) \equiv 0$ for any real value of k^2, and so the solution of $\mathcal{N}$ is seen to be

$$W(P) = \frac{1}{2\pi} \int_{\partial D} \left\{ G_1(P,q) f(q) \right.$$

$$\left. - u(q) \frac{\partial}{\partial n_q} G_1(P,q) \right\} ds_q.$$

We now substitute for G_1 from (5.9) to give

$$W(P) = U(P) + \frac{1}{2\pi} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} a_m^\sigma A_m^\sigma \psi_m^\sigma(P),$$

where A_m^σ is defined by (5.18). But we also know that $A_m^\sigma = 0$ (these are just the null-field equations) and so $W(P) = U(P)$, as required.

We shall now state and prove another corollary to Theorem 7. Let $L_2(\partial D)$ denote the class of complex-valued, square-integrable functions, defined on ∂D. Then, we have

Corollary 7.2. *The set of functions*

$$X \equiv \{\partial\psi_m^\sigma/\partial n\} \quad (\sigma = 1, 2, ; m = 0, 1, \ldots)$$

is complete in $L_2(\partial D)$.

In this corollary, X is complete in the mean-square sense. Thus, if we are given a function $F(q)$ in $L_2(\partial D)$, then Corollary 7.2 states that there exist coefficients α_m^σ such that

$$\int_{\partial D} |F(q) - F_M(q)|^2 \, ds_q \to 0$$

as $M \to \infty$, where

$$F_M(q) = \sum_{m=0}^{M} \sum_{\sigma=1}^{2} \alpha_m^\sigma(M) \frac{\partial}{\partial n_q} \psi_m^\sigma(q)$$

and the coefficients $\alpha_m^\sigma(M)$ depend on M. Note also that the functions in X are *not* orthogonal over ∂D.

Proof of Corollary 7.2. Let $h(q)$ be an arbitrary function in $L_2(\partial D)$. If the orthogonality of h to every member of X,

$$\int_{\partial D} h^*(q) \frac{\partial}{\partial n_q} \psi_m^\sigma(q) \, ds_q = 0$$

$$(\sigma = 1, 2; m = 0, 1, \ldots),$$

implies that $h(q) = 0$ almost everywhere on ∂D, then X is said to be *closed* in $L_2(\partial D)$, and is hence complete in $L_2(\partial D)$ (see, e.g. [16] or pp. 90-95 of [17]). But, from Theorem 7, we know that the null-field equations (5.4) are uniquely solvable for any given $f(q)$. In particular, when $f(q) = 0$ (i.e. $f_m^\sigma = 0$), the only solution is $u(q) = 0$, and the result follows.

Corollary 7.2 has been proved (in three dimensions) by Müller and Kersten [18], using a different method; Millar [16] has also given a proof (in two dimensions), but his arguments fail when $k^2 \in I_N$.

Corollary 7.2 has been *assumed* by many authors, since it has obvious applications to the numerical solution of the null-field equations; we shall discuss this procedure in Section 7.

Let us now consider the exterior Dirichlet problem, $\mathcal{D}$, and prove

Theorem 8. *The null-field equations for the exterior Dirichlet problem of acoustics (in two dimensions), (5.5), possess a unique solution for all real values of* k^2.

Proof. Multiply each of (5.5) by $\chi_m^\sigma(P_-)$, where $P_- \in D_N'$, and sum to give

$$V_1(P_-) \equiv \int_{\partial D} \left\{ G_1(P_-, q) \frac{\partial u(q)}{\partial n_q} \right.$$

$$\left. - g(q) \frac{\partial}{\partial n_q} G_1(P_-, q) \right\} ds_q = 0. \quad (5.21)$$

V_1 is a wave-function which vanishes in D_N', and hence vanishes in D_-. In particular, $\partial V_1/\partial n = 0$ on ∂D. Differentiating, and letting P_- approach ∂D, we obtain

$$\pi \frac{\partial u(p)}{\partial n_p} - \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_1(p, q) \, ds_q$$

$$= -\frac{\partial}{\partial n_p} \int_{\partial D} g(q) \frac{\partial}{\partial n_q} G_1(p, q) \, ds_q. \quad (5.22)$$

This is an integral equation for $\partial u/\partial n$ which, by Theorem 5, possess a unique solution for all real values of k^2. Conversely, suppose that $\partial u/\partial n$ is the unique solution of (5.22). Then, we can use (5.21) to define a wave-function $V_1(P_-)$ which, by (5.22), has a vanishing normal derivative on ∂D. Now, if $P_- \in D_N'$, we can use (5.1) and rewrite (5.21) as

$$V_1(P_-) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} B_m^\sigma \chi_m^\sigma(P_-)$$

where

$$B^\sigma_m = \int_{\partial D} \left\{ \psi^\sigma_m(q)\, \frac{\partial u(q)}{\partial n_q} - g(q)\, \frac{\partial}{\partial n_q} \psi^\sigma_m(q) \right\}\, ds_q.$$

By considering the integral (5.20), with U_1 replaced by V_1, we can show that $B^\sigma_m = 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots$. Hence, V_1 vanishes everywhere in D_-, and $\partial u / \partial n$ satisfies the null-field equations, (5.5). This completes the proof of Theorem 8.

The following two corollaries can be proved by modifying the proofs of Corollaries 7.1 and 7.2 in a straightforward way.

Corollary 8.1. *If* $\partial u / \partial n$ *satisfies the null-field equations* (5.5) *(or the integral equation* (5.22)*), the solution of $\mathcal{D}$ is* $V(P)$, *defined by* (3.15).

Corollary 8.2. *The set of functions*

$$\{\psi^\sigma_m\} \quad (\sigma = 1, 2; m = 0, 1, \ldots)$$

is complete in $L_2(\partial D)$.

This result has been proved by several authors; see, e.g. [16, 18, 19].

6. The null-field equations for scattering problems

Consider the boundary-value problem $\mathcal{S}_1$, corresponding to scattering by a sound-hard body. There are two ways of obtaining null-field equations for $\mathcal{S}_1$. We could specialise the analysis of the previous section (put $f = -\partial u^{(i)} / \partial n$, as prescribed by the boundary condition (4.2)), leading to an infinite set of null-field equations for $u^{(s)}(q)$, namely (5.4). Alternatively, we can obtain null-field equations for $u(q)$ as follows.

Suppose $P_- \in D_N$. Then, using (5.1) in the interior integral relation (4.10), we obtain

$$\tfrac{1}{2}i\pi \sum_{m=0}^\infty \sum_{\sigma=1}^2 \hat{\psi}^\sigma_m(P_-) \int_{\partial D} u(q)\, \frac{\partial}{\partial n_q} \psi^\sigma_m(q)\, ds_q$$

$$= 2\pi u^{(i)}(P_-).$$

But, since $u^{(i)}(P_-)$ is assumed to be regular in D_N, there exist coefficients d^σ_m such that

$$u^{(i)}(P_-) = \sum_{m=0}^\infty \sum_{\sigma=1}^2 d^\sigma_m \hat{\psi}^\sigma_m(P_-) \tag{6.1}$$

for $P_- \in D_N$; we have

$$d^\sigma_m = \frac{1}{2\pi} \int_{C_-} u^{(i)} \hat{\psi}^\sigma_m\, ds, \tag{6.2}$$

since $\hat{\psi}^\sigma_m$ are orthogonal over C_-. (In (6.2), C_- can be replaced by any smaller, concentric circle.) Equating coefficients, we obtain

$$\int_{\partial D} u(q)\, \frac{\partial}{\partial n_q} \psi^\sigma_m(q)\, ds_q = -4i d^\sigma_m$$

$$(\sigma = 1, 2; m = 0, 1, \ldots). \tag{6.3}$$

These are the null-field equations for a sound-hard body; they were first obtained by Waterman [3]. We can prove

Theorem 9. *The null-field equations for a sound-hard body,* (6.3), *possess a unique solution for all real values of* k^2.

Proof. Multiply each of (6.3) by $\chi^\sigma_m(P_-)$, where $P_- \in D'_N$ and χ^σ_m is given by (5.19), and sum to give

$$U_1(P_-) \equiv \int_{\partial D} u(q)\, \frac{\partial}{\partial n_q} G_1(P_-, q)\, ds_q$$

$$- 2\pi \tilde{u}^{(i)}(P_-) = 0, \tag{6.4}$$

where

$$\tilde{u}^{(i)}(P_-) = u^{(i)}(P_-) - \frac{2i}{\pi} \sum_{m=0}^\infty \sum_{\sigma=1}^2 a^\sigma_m d^\sigma_m \psi^\sigma_m(P_-). \tag{6.5}$$

Proceeding as in the proof of Theorem 7, we see that U_1 must vanish on ∂D, whence, from (6.4), $u(q)$ must satisfy

$$\pi u(p) + \int_{\partial D} u(q)\, \frac{\partial}{\partial n_q} G_1(p, q)\, ds_q$$

$$= 2\pi \tilde{u}^{(i)}(p), \tag{6.6}$$

which is uniquely solvable, by Theorem 5. Conversely, suppose that $u(q)$ is the unique solution

of (6.6). Then, we can define a wave-function U_1 by (6.4), which vanishes on ∂D, by (6.6). For $P_- \in D'_N$, we may use (5.9), (6.1) and (6.5) in (6.4), i.e. $U_1(P_-)$ can be expanded as (5.17), with

$$A^\sigma_m = \int_{\partial D} u(q) \frac{\partial}{\partial n_q} \psi^\sigma_m(q) \, ds_q + 4id^\sigma_m.$$

It can then be shown that $A^\sigma_m = 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots$, and the result follows.

Corollary 9.1. *If $u(q)$ satisfies the null-field equations (6.3) (or the integral equation (6.6)), the solution of $\mathcal{S}_1$ is $U(P)$, defined by (4.13).*

Proof. As for Corollary 7.1.

Let us now discuss the null-field method for $\mathcal{S}_2$. From (4.22), we can show that

$$\int_{\partial D} \frac{\partial u(q)}{\partial n_q} \psi^\sigma_m(q) \, ds_q = 4id^\sigma_m$$

$$(\sigma = 1, 2; m = 0, 1, \ldots); \tag{6.7}$$

these are the null-field equations for a sound-soft body. We have

Theorem 10. *The null-field equations for a sound-soft body, (6.7), possess a unique solution for all real values of k^2.*

Proof. In the proof of Theorem 8, replace V_1 by

$$V_1(P_-) \equiv \int_{\partial D} \frac{\partial u(q)}{\partial n_q} G_1(P_-, q) \, ds_q + 2\pi \tilde{u}^{(i)}(P_-);$$

it follows that $\partial u/\partial n$ is the unique solution of

$$\pi \frac{\partial u(p)}{\partial n_p} - \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \frac{\partial}{\partial n_p} G_1(p, q) \, ds_q$$

$$= 2\pi \frac{\partial}{\partial n_p} \tilde{u}^{(i)}(p). \tag{6.8}$$

To prove the converse, we again follow the proof of Theorem 8; here, B^σ_m is defined by

$$B^\sigma_m = \int_{\partial D} \frac{\partial u(q)}{\partial n_q} \psi^\sigma_m(q) \, ds_q - 4id^\sigma_m.$$

Finally, we can modify the proof of Corollary 8.1 to prove

Corollary 10.1. *If $\partial u/\partial n$ satisfies the null-field equations (6.7) (or the integral equation (6.8)), the solution of $\mathcal{S}_2$ is $V(P)$, defined by (4.23).*

7. Solution of the null-field equations

In this section, we shall discuss methods for solving the null-field equations. As a representative example, we shall consider the null-field equations for the scattering by a sound-hard body (6.3). Suppressing the dependence on σ, we may write these as

$$\int_{\partial D} u(q) \frac{\partial}{\partial n_q} \psi_m(q) \, ds_q = -4id_m$$

$$(m = 0, 1, \ldots). \tag{7.1}$$

These are an infinite set of equations from which $u(q)$ is to be determined; the constants d_m are known (they are given by (6.2)).

If ∂D is a circle, centred on O, the system (7.1) decouples, yielding the Fourier components of $u(q)$. This suggests a generalization of the null-field method, which has been developed by Bates and Wall [4]: replace ψ^σ_m (which is a radiating wave-function in circular polar coordinates) by the corresponding radiating wave-function in elliptic coordinates; the resulting system of null-field equations will then decouple if ∂D is an ellipse.

For any other geometry, the null-field equations (7.1) must be solved numerically. One approach (which is suggested by the 'quadrature method' for solving integral equations) is to replace the integral by a finite series. Thus, choose a quadrature rule of the form

$$\sum_{n=0}^{N} w_n F(q_n) \tag{7.2}$$

for approximating the integral

$$\int_{\partial D} F(q) \, ds_q;$$

here, w_n and $q_n \in \partial D$ are the weights and abscissae, respectively, associated with the quadrature rule (7.2). If we use (7.2) to approximate the integral in the first $N + 1$ of (7.1), we obtain

$$\sum_{n=0}^{N} K_{mn} v_n = -4\mathrm{i} d_m$$

$$(m = 0, 1, \ldots N), \qquad (7.3)$$

where

$$K_{mn} = w_n \frac{\partial}{\partial n_q} \psi_m(q_n)$$

and v_n may be regarded as an approximation to $u(q_n)$. (7.3) is a system of $N + 1$ simultaneous linear algebraic equations for v_n. However, there is no proof that this system is non-singular. Indeed, there do not seem to be any reported attempts at using the quadrature method to solve the null-field equations.

A second approach is to choose a complete set of functions $\{\phi_n(q)\}$, and then to write $u(q)$ as

$$u(q) = \sum_{n=0}^{\infty} u_n \phi_n(q), \qquad (7.4)$$

where u_n are unknown coefficients. Substituting this representation into (7.1) yields

$$\sum_{n=0}^{\infty} Q_{mn} u_n = -4\mathrm{i} d_m$$

$$(m = 0, 1, \ldots), \qquad (7.5)$$

where

$$Q_{mn} = \int_{\partial D} \phi_n(q) \frac{\partial}{\partial n_q} \psi_m(q) \, \mathrm{d}s_q. \qquad (7.6)$$

(7.5) is an infinite system of linear algebraic equations for u_n; truncating this system leads to a numerical method for solving the null-field equations. To proceed further, we must choose a set $\{\phi_n\}$. In theory, we can choose *any* set, provided it is complete over ∂D. However, in practice, the choice may be crucial, if the truncated system of equations is to yield a good approximation to $u(q)$.

The ideal choice would be $\phi_n = \Psi_n$, where Ψ_n satisfies

$$\int_{\partial D} \Psi_n(q) \frac{\partial}{\partial n_q} \psi_m(q) \, \mathrm{d}s_q = \delta_{mn}, \qquad (7.7)$$

for then the system (7.1) will decouple, yielding $u_m = -4\mathrm{i} d_m$. However, we do not know *a priori* which functions $\{\Psi_n\}$ satisfy the orthogonality relation (7.7), for a given boundary ∂D; for each value of n, the determination of Ψ_n is equivalent to solving the null-field equations (with $-4\mathrm{i} d_m$ replaced by δ_{mn}). Computationally, it is probably not worth while to determine Ψ_n; see [4], p. 57. However, it may be possible to choose $\{\phi_n\}$ so that (7.7) is almost satisfied, i.e. so that the system of equations (7.5) is diagonally-dominant.

Several authors have advocated simple choices for $\{\phi_n\}$, e.g. trigonometric functions or orthogonal polynomials. However, most authors use wave-functions; there are six obvious choices for ϕ_m:

$$\psi_m, \quad \hat{\psi}_m, \quad \psi_m^*,$$

$$\qquad\qquad\qquad\qquad (7.8)$$

$$\partial \psi_m / \partial n, \quad \partial \hat{\psi}_m / \partial n, \quad \partial \psi_m^* / \partial n.$$

Note that $\{\partial \psi_m / \partial n\}$ and $\{\psi_m\}$ are complete, by Corollaries 7.2 and 8.2, respectively. Similarly, $\{\partial \psi_m^* / \partial n\}$ and $\{\psi_m^*\}$ are also complete. But, $\{\partial \hat{\psi}_m / \partial n\}$ and $\{\hat{\psi}_m\}$ are *not* complete when $k^2 \in I_D$ and $k^2 \in I_N$, respectively [3, 19]; thus, use of one of these two sets will reintroduce the difficulties at irregular frequencies. A good discussion of the relative merits of the six sets (7.8) (for the corresponding electromagnetic scattering problems) has been given by Waterman [20].

Several authors have solved scattering problems for polygonal cylinders, using the null-field method and different choices for $\{\phi_n\}$. For example, Wall *et al.* [21] used piecewise-smooth functions which had the correct analytical form near each corner, whilst Bates and Wall [4] used $\phi_n(q) = w(q) E_n(\Theta)$, where $w = 1$ for sound-hard bodies, $w = 1/h$ for sound-soft bodies, and h and Θ are related to the conformal mapping between ∂D and the unit circle; for square and triangular

cylinders, Bates and Wall [4] found good agreement between their numerical results and some experimental results of Iizuka and Yen [22]. However, we should remark that our theory is not immediately applicable to cylinders with corners, since ∂D is not a Lyapunov curve for such cylinders.

In many scattering problems, the quantity of interest is the far-field scattered wave, $u^{(s)}(P)$. From Corollary 9.1, (4.1) and (5.1), we have (cf. (5.6))

$$u^{(s)}(P) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} c_m^\sigma \psi_m^\sigma(P) \qquad (7.9)$$

with

$$c_m^\sigma = \frac{-i}{4} \int_{\partial D} u(q) \frac{\partial}{\partial n_q} \hat{\psi}_m^\sigma(q) \, ds_q. \qquad (7.10)$$

(In the far field, we can replace $\psi_m^\sigma(P)$ by its asymptotic form for large $|kr_p|$.) Suppressing the dependence on σ, and using (7.4), we have

$$c_m = \frac{-i}{4} \sum_{n=0}^{\infty} \hat{Q}_{mn} u_n \qquad (7.11)$$

where

$$\hat{Q}_{mn} = \int_{\partial D} \phi_n(q) \frac{\partial}{\partial n_q} \hat{\psi}_m(q) \, ds_q. \qquad (7.12)$$

Let us now use an obvious matrix notation, and rewrite (7.5) and (7.11) as

$$Qu = -4id \qquad (7.13)$$

and

$$c = -\tfrac{1}{4}i\hat{Q}u, \qquad (7.14)$$

respectively. (7.13) is equivalent to the system of null-field equations, and is therefore uniquely solvable, i.e. Q^{-1}, the inverse of (the infinite matrix) Q exists. Thus,

$$u = -4iQ^{-1}d.$$

Substituting this expression into (7.14), we obtain

$$c = Td, \qquad (7.15)$$

where

$$T = -\hat{Q}Q^{-1} \qquad (7.16)$$

is called the *transition matrix* or *T-matrix*. If T can be computed, we can then determine the scattered field outside C_+ (i.e. c) for any given incident wave (i.e. d), without computing the values of u on ∂D. Note that T depends on $\{\phi_n\}$; if we make one of the choices (7.8), then Q (or $\hat{Q}$) may have certain desirable properties, e.g. symmetry [20].

The T-matrix formulation has been used widely for solving acoustic scattering problems [9]. The numerical procedure consists of

    (i) choosing the set $\{\phi_n\}$, usually one of (7.8);

    (ii) truncating the infinite system of equations (7.15);

    (iii) computing the elements of the (finite) matrices Q and $\hat{Q}$ (the integrals may be evaluated using any suitable quadrature rule, since the integrands are non-singular);

    (iv) computing Q^{-1}; and

    (v) computing the (finite) T-matrix, from (7.16).

It is hoped that this procedure will yield a good approximation to c. However, convergence as the number of equations becomes larger has not been proved, even though the infinite system of null-field equations has been shown to be uniquely solvable.

8. Conclusions

In this paper, we have analysed the null-field method, as it is used to solve radiation and scattering problems in acoustics. This method, which was first proposed by Waterman [1, 3], consists of solving an infinite system of null-field equations. We have shown that this system always has precisely one solution. Moreover, this solution may then be used to determine the solution of the original boundary-value problem, at *any* point in the exterior, D. (It may be worth remarking here that the so-called Rayleigh hypothesis [16] plays no part in the null-field method.) Thus, irregular frequencies do not occur with the null-field method. This contrasts strongly with the more

familiar integral-equation methods (as described in Sections 3 and 4).

We have proved our results in two dimensions, for four standard boundary-value problems in acoustics; the extension to three dimensions is straightforward. Similar results can also be proved for water-wave problems [8].

In Section 7, we discussed several methods for solving the null-field equations. One method (the quadrature method) consisted of replacing the necessary integrations by finite series, leading to a (finite) system of linear algebraic equations, which may be solved numerically. We then described an exact procedure for reducing the null-field equations to an infinite system of linear algebraic equations. When this system is truncated, we obtain another method for solving the null-field equations, numerically. Suppose we choose N equations in N unknowns. Then, it has *not* been proved that the solution of these equations approaches the solution of the infinite system, as $N \to \infty$. Indeed, it is well known that numerical instabilities can occur, especially when the body is elongated [4]. However, several methods have been devised for overcoming these difficulties [4, 21, 23].

Many scattering and radiation problems have been solved successfully using simple numerical implementations of the null-field method. In the future, it is hoped that the development of more sophisticated numerical techniques will enhance the versatility and usefulness of the null-field method.

Appendix. Ursell's fundamental solution [10]

Ursell [10] has considered a fundamental solution of the form (5.9), with

$$a_m^1 = a_m^2$$
$$= -\tfrac{1}{2}i\pi\{kJ_m'(ka) + KJ_m(ka)\}/\Delta, \qquad (A.1)$$

where

$$\Delta = kH_m^{(1)'}(ka) + KH_m^{(1)}(ka),$$

$K = |K| e^{i\delta}$ is a constant such that $0 < \delta < \pi$, and a is the radius of a circle, which is smaller than, but concentric with C_-; on this smaller circle ($r_p = a$), G_1 satisfies the dissipative boundary condition

$$\left(\frac{\partial}{\partial r_p} + K\right) G_1(P, Q) = 0.$$

Henceforth, we shall write H_m and J_m for $H_m^{(1)}(ka)$ and $J_m(ka)$, respectively.

We shall prove that the coefficients (A.1) satisfy the inequality (5.16). We have

$$2a_m^\sigma + \tfrac{1}{2}i\pi = -\tfrac{1}{2}i\pi\{2(kJ_m' + KJ_m)$$
$$- (kH_m' + KH_m)\}/\Delta$$
$$= -\tfrac{1}{2}i\pi(kH_m'^* + KH_m^*)/\Delta,$$

whence

$$|2a_m^\sigma + \tfrac{1}{2}i\pi|^2 = \tfrac{1}{4}\pi^2(kH_m'^* + KH_m^*)$$
$$\cdot (kH_m' + K^*H_m)/|\Delta|^2$$
$$= \tfrac{1}{4}\pi^2\{\Delta^* + (K - K^*)H_m^*\}$$
$$\cdot \{\Delta + (K^* - K)H_m\}/|\Delta|^2$$
$$= \tfrac{1}{4}\pi^2\{|\Delta|^2 + k(K - K^*)$$
$$\cdot (H_m'H_m^* - H_mH_m'^*)\}/|\Delta|^2$$
$$= \tfrac{1}{4}\pi^2\{1 - 8|K| \sin \delta/(\pi a|\Delta|^2)\}$$
$$< \tfrac{1}{4}\pi^2,$$

as required, if $0 < \delta < \pi$.

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