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On the Null-field Equations for Water-wave Scattering Problems

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Consider a rigid body which is partially immersed in the free surface of deep water; the body is fixed and a given time-harmonic wave is incident upon it. The corresponding linear boundary-value problem is solved using the null-field method and two systems of null-field equations are obtained; each system is always uniquely solvable—irregular frequencies do not occur. These results can be proved in two or three dimensions, for water of infinite or finite depth, and for any incident wave. Next, it is assumed that the water is deep and that the incident wave is a regular surface wave. With these assumptions, one system of null-field equations takes on a very simple form. This system is solved using the "method of projection" (Gregory & Gladwell, 1982), leading to a sequence of computable approximations to the solution of the null-field equations; this sequence is guaranteed to converge. It seems worthwhile to establish this last result, because (at present) there is a notable absence of such convergence proofs in the general theory of the null-field method. As an example of this approach, the two-dimensional problem of the interaction between a regular surface wave and a fixed, half-immersed, elliptic cylinder is solved.

1. Introduction

CONSIDER A RIGID BODY which is floating in the free surface of deep water. When the body is forced to make small time-harmonic oscillations, waves are generated which radiate energy away from the body. The corresponding linear boundary-value problem (the *radiation problem*, $\mathscr{P}$) has been studied by many authors, and is known to be well posed under certain geometrical restrictions on the shape of the wetted surface of the body. Recently, Martin (1981) has described a new method, called the *null-field method*, for solving water-wave radiation problems, which is to solve the infinite system of null-field equations. He proved that this system possesses a unique solution at all frequencies of oscillation, and that this solution can then be used to solve the original boundary-value problem, $\mathscr{P}$. These results can be proved in two and three dimensions, and for water of infinite or finite (constant) depth.

In the present paper, we shall use the null-field method to solve water-wave scattering problems. In such problems, the rigid body is held fixed, and a given time-harmonic wave is incident upon it. We are required to determine how the incident wave is modified by the presence of the body, and the resulting hydrodynamic forces acting on the body.

From the outset, we shall only consider the two-dimensional problem of scattering by a horizontal, partially immersed cylinder; moreover, we shall suppose

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that the water is of infinite depth. In Section 2, we formulate the boundary-value problem for the total potential ϕ. This problem can be solved by solving either of two systems of null-field equations; these are obtained in Section 3. Each system is uniquely solvable at all frequencies. Similar results can be proved in three dimensions and for water of constant finite depth.

In the remainder of the paper, we describe a novel procedure for solving the null-field equations, under the following two essential restrictions:

(i) the water is of infinite depth, and
(ii) the incident wave is a plane surface wave (given by (2.7) below).

With these restrictions, we obtain a sequence of approximations which converges (in the mean-square sense) to the exact solution. Our procedure is based on the "method of projection", as devised by Gregory & Gladwell (1982) for the treatment of various boundary-value problems for the semi-infinite elastic strip. After some preliminary results on the completeness of certain sets of functions in Section 4, the method of projection is described in Section 5; some computational aspects are also discussed. Finally, in Section 6, we use the method of projection to solve a particular scattering problem, namely the scattering of regular surface waves by a fixed, half-immersed, elliptic cylinder.

2. The Boundary-value Problem

Consider a rigid horizontal cylinder which is partially immersed in the free surface of a fluid. We suppose that the cylinder is fixed, and that the fluid is incompressible, inviscid and of infinite depth. We take Cartesian co-ordinates (x, y, z), with the z-axis parallel to the generators of the cylinder, the x-axis horizontal and the y-axis vertical (y increasing with depth) such that the free surface occupies a portion of the plane $y = 0$.

For simplicity, we consider beam seas (waves with crests parallel to the z-axis) whence the fluid motion can be considered to be two-dimensional, i.e. independent of z. We shall use the same notation as used by Martin (1981); henceforth, this paper will be referred to as I. Thus (in the xy-plane), we denote the fluid domain by D, the free surface by F and the wetted surface of the cylinder by ∂D, which we assume has properties J (i.e., if ∂D^* denotes the union of ∂D and its mirror image in F, then ∂D^* is required to be convex and twice-differentiable).

We assume that the fluid motion is irrotational, whence a velocity potential exists. If we further assume that the motion has a harmonic time-dependence (with radian frequency ω), then we can write the velocity potential as the real part of $\phi(P)e^{-i\omega t}$; henceforth, we shall suppress the factor $e^{-i\omega t}$. For waves of small amplitude, the *total potential* ϕ solves the following linear two-dimensional boundary-value problem:

Scattering boundary-value problem $\mathbb{S}$

Determine a function $\phi(P)$, such that ϕ satisfies Laplace's equation

$$\nabla^2\phi(P) = 0 \quad \text{in } D, \tag{2.1}$$

the free-surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on } F \tag{2.2}$$

the boundary condition

$$\frac{\partial \phi(p)}{\partial n_p} = 0 \quad \text{on } \partial D, \tag{2.3}$$

and

$$|\nabla \phi| \to 0 \quad \text{as } y \to \infty. \tag{2.4}$$

In addition, if we define ϕ_D by

$$\phi_D = \phi - \phi_I, \tag{2.5}$$

then ϕ_D must satisfy the radiation condition

$$\left(\frac{\partial}{\partial r_p} - iK\right)\phi_D \to 0 \quad \text{as } r_p \to \infty. \tag{2.6}$$

As in I, capital letters P, Q denote points of D; lower-case letters p, q denote points of ∂D; the origin O is assumed to lie in F_-, the portion of the line $y = 0$ which is inside the body; D_- denotes the interior of the body, i.e. the region with boundary $\partial D \cup F_-$; P_-, Q_- denote points of D_-; r_p is the length OP; and $\partial/\partial n_p$ denotes normal differentiation at the point p, in the direction from ∂D into D. Also, $K = \omega^2/g$, where g is the acceleration due to gravity, and ϕ_I is the velocity potential of the given incident wave. ϕ_I satisfies (2.1) *everywhere* in $y > 0$ (except possibly at a finite number of isolated points in D) and (2.2) *everywhere* on $y = 0$. For example, ϕ_I may correspond to a radiating (line) source in D, or to a train of regular surface waves. This second example is important, and will be considered in more detail in Sections 5 and 6; for a regular wave incident from $x = +\infty$, with amplitude A, we have

$$\phi_I = (Ag/\omega)e^{-Ky - iKx}. \tag{2.7}$$

If (as we assume) ∂D has properties J, the existence and uniqueness theorems of John (1950) assure us that there is precisely one function ϕ that solves $\mathbb{S}$. (Here, "ϕ solves $\mathbb{S}$" means that ϕ is a classical solution, i.e. $\phi \in C^2(D) \cap C(\bar{D})$, where $\bar{D} = D \cup \partial D \cup F$.)

Several authors have studied the scattering problem $\mathbb{S}$, for various ∂D. Much of this work is concerned with the half-immersed circular cylinder; for a detailed review of this problem, see Martin & Dixon (1983). The work on elliptic cylinders is reviewed in Section 6. For other geometries, see the review articles by Newman (1972), Mei (1978) and Yeung (1982).

3. The Null-field Equations

We shall give two systems of null-field equations, one for $\phi_D(q)$ and one for $\phi(q)$.

Setting $V(q)$ equal to $-\partial\phi_I/\partial n_q$ in I. (Section 3), we obtain the first system:

$$\int_{\partial D} \phi_D(q)\Psi_m^\sigma(q)\,ds_q = V_m^\sigma (\sigma = 1, 2; \quad m = 0, 1, \ldots). \tag{3.1}$$

Here, V_m^σ are known constants, given by

$$V_m^\sigma = -\int_{\partial D} \Phi_m^\sigma(q)\frac{\partial}{\partial n_q}\phi_I(q)\,ds_q, \tag{3.2}$$

$$\Psi_m^\sigma(q) = \frac{\partial}{\partial n_q}\Phi_m^\sigma(q), \tag{3.3}$$

and Φ_m^σ are Ursell's *multipole potentials*:

$$\Phi_0^1(P) = \int_0^\infty e^{-ky}\cos kx\,\frac{dk}{k-K}, \qquad \Phi_0^2(P) = -\frac{1}{K}\frac{\partial}{\partial x}\Phi_0^1(P) \tag{3.4}$$

$$\Phi_m^1(P) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}, \tag{3.5}$$

$$\Phi_m^2(P) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m}\frac{\sin 2m\theta}{r^{2m}}, \tag{3.6}$$

$m = 1, 2, \ldots$, and the point $P \equiv (x, y)$ has circular polar co-ordinates given by

$$x = r\sin\theta, \qquad y = r\cos\theta$$

(with $r = r_p$). (Note that the definitions of Φ_0^2 and α_0^2 (see below) differ slightly from those given in I.) Each multipole potential satisfies the free-surface and radiation conditions, and satisfies Laplace's equation except at O where it is singular.

The system (3.1) is to be solved for $\phi_D(q)$. Since (3.1) is a special case of the system of null-field equations for water-wave radiation problems, as derived in I, we have

THEOREM 3.1 *Suppose that ∂D has properties J. Then, the null-field equations* (3.1) *possess a unique solution, $\phi_D(q) \in C(\partial D)$, for all values of K. Moreover, the function*

$$\phi(P) = \phi_I(P) - \frac{1}{2\pi}\int_{\partial D}\left\{G_0(P, q)\frac{\partial}{\partial n_q}\phi_I(q) + \phi_D(q)\frac{\partial}{\partial n_q}G_0(P, q)\right\}ds_q \tag{3.7}$$

solves $\mathbb{S}$.

Here, G_0 is the potential of a simple wave source, given by

$$G_0(P, Q) \equiv G_0(x, y; \xi, \eta) = \tfrac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\,\frac{dk}{k-K}. \tag{3.8}$$

In order to obtain a system of null-field equations for $\phi(q)$, we merely add the known constants

$$\int_{\partial D}\phi_I(q)\Psi_m^\sigma(q)\,ds_q$$

to both sides of (3.1) and obtain

$$\int_{\partial D} \phi(q)\Psi_m^\sigma(q)\, ds_q = d_m^\sigma (\sigma = 1, 2; \quad m = 0, 1, \ldots) \tag{3.9}$$

where

$$d_m^\sigma = \int_{\partial D} \left\{ \phi_I(q)\Psi_m^\sigma(q) - \Phi_m^\sigma(q) \frac{\partial}{\partial n_q} \phi_I(q) \right\} ds_q. \tag{3.10}$$

The constants d_m^σ have a different interpretation: apply Green's theorem in D_- to ϕ_I and G_0, yielding

$$2\pi\phi_I(P_-) = \int_{\partial D} \left\{ \phi_I(q) \frac{\partial}{\partial n_q} G_0(P_-, q) - G_0(P_-, q) \frac{\partial}{\partial n_q} \phi_I(q) \right\} ds_q \tag{3.11}$$

when $P_- \in D_-$. Now, $G_0(P, Q)$ has the bilinear expansion (Ursell, 1981)

$$G_0(P, Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P)\Phi_m^\sigma(Q) \tag{3.12}$$

for $r_P < r_Q$, where

$$\alpha_0^1(P) = -2e^{-Ky} \cos Kx, \qquad \alpha_0^2(P) = -2e^{-Ky} \sin Kx, \tag{3.13}$$

$$\alpha_m^1(P) = -\frac{2(2m-1)!}{K^{2m}} \sum_{q=2m}^{\infty} \frac{(-Kr)^q}{q!} \cos q\theta, \tag{3.14}$$

and

$$\alpha_m^2(P) = \frac{2(2m)!}{K^{2m+1}} \sum_{q=2m+1}^{\infty} \frac{(-Kr)^q}{q!} \sin q\theta. \tag{3.15}$$

Let C_- be the inscribed circle to ∂D^* and let D_N be the semi-circular region bounded by F_- and the lower half of C_-. Then, if we restrict P_- to lie in D_N (where $r_p < r_q$), we can substitute (3.12) into (3.11) giving

$$2\pi\phi_I(P_-) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} d_m^\sigma \alpha_m^\sigma(P_-), \quad P_- \in D_N, \tag{3.16}$$

where d_m^σ are given (uniquely) by (3.10). Thus, d_m^σ are the coefficients in the expansion of ϕ_I in terms of the regular functions α_m^σ (these functions are harmonic and satisfy the free-surface condition).

We have

THEOREM 3.2 *Suppose that ∂D has properties J. Then, the null-field equations* (3.9) *possess a unique solution, $\phi(q) \in C(\partial D)$, for all values of K. Moreover, the function*

$$\phi(P) = \phi_I(P) - \frac{1}{2\pi} \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G_0(P, q)\, ds_q \tag{3.17}$$

solves $\mathbb{S}$.

The proof of this theorem is similar to that for Theorem 3.1; cf. Martin (1982, theorem 9).

In summary, we have two systems of null-field equations, namely (3.1) and (3.9). Each system is uniquely solvable at all frequencies, i.e. unlike the familiar integral-equation methods (see I, Section 2), the null-field method does not suffer from irregular frequencies. So, which system should we choose for a particular scattering problem? For an arbitrary incident potential ϕ_I, the first system (3.1) may be preferable, since the constants V_m^σ [given by (3.2)] have a slightly simpler form than the constants d_m^σ [given by (3.10)]. However, the most important scattering problem arises when ϕ_I corresponds to a train of regular surface waves; for a wave incident from $x = +\infty$, ϕ_I is given by (2.7). For this particular incident wave, the second system is preferable, because the constants d_m^σ take on a very simple form: from (2.7), (3.13) and (3.16), we find that

$$d_0^1 = -\pi A g/\omega, \qquad d_0^2 = i\pi A g/\omega \tag{3.18}$$

and $d_m^\sigma = 0$ for $m > 0$. Thus, the null-field equations, (3.9), become

$$\int_{\partial D} \phi(q)\Psi_m^\sigma(q)\, ds_q = d_0^\sigma \delta_{0m} \quad (\sigma = 1, 2; \quad m = 0, 1, \ldots), \tag{3.19}$$

where δ_{ij} is the Kronecker delta. In Section 5, we shall describe a numerical scheme for solving this particular system. This scheme is convergent. To prove this result, we shall need some properties of $\{\Psi_m^\sigma(q)\}$; these are described in the next section.

4. Complete Sets of Functions

Let H be a separable Hilbert space, with inner product $(\cdot, \cdot)$ and corresponding norm $\|\cdot\|$. We are interested in sets of functions in H; let $S = \{\phi_n\}$ $(n = 0, 1, \ldots)$ be any such set. (Note that we do *not* assume that the functions in S are orthogonal in H.) We begin by recalling some definitions.

S is called *complete* if, when $h \in H$,

$$(h, \phi_n) = 0$$

for $n = 0, 1, \ldots$, implies that $h = 0$.

S is called *total* if, for any $h \in H$, there are coefficients $a_n(N)$ such that, given $\varepsilon > 0$,

$$\left\| h - \sum_{n=0}^{N} a_n(N)\phi_n \right\| \leqslant \varepsilon \tag{4.1}$$

for N sufficiently large. In general, the coefficients $a_n(N)$ depend on N, i.e. if a new ε is given, the whole set of coefficients may have to be changed.

It is well known that S is complete if, and only if, it is total (Higgins, 1977, p. 15).

S is called *minimal* if every finite subset of S is linearly independent. This is the same as requiring that there is a unique biorthogonal sequence $\{\psi_n\}$ such that $(\phi_n, \psi_m) = \delta_{nm}$.

Finally, a complete set S is called a *basis* if the coefficients $a_n(N)$ in (4.1) are independent of N and are unique. This is equivalent to requiring that S is minimal and

$$\left\| h - \sum_{n=0}^{N} (h, \psi_n)\phi_n \right\| \to 0 \tag{4.2}$$

as $N \to \infty$ for any $h \in H$ (Higgins, 1977, p. 20).

Having established our terminology, we now consider a particular Hilbert space: let $L_2(\partial D)$ denote the Hilbert space of complex-valued, square-integrable functions, defined on ∂D, with inner product

$$(f, g) = \int_{\partial D} f(q)g^*(q)\, ds_q$$

and norm $\|f\| = \{(f,f)\}^{\frac{1}{2}}$, where $f, g \in L_2(\partial D)$ and the asterisk denotes the complex conjugate. We have

THEOREM 4.1 *In $L_2(\partial D)$, the set of functions*

$$X \equiv \{\Psi_m^\sigma(q)\} \quad (\sigma = 1, 2; \quad m = 0, 1, \ldots)$$

is complete and minimal.

Proof. (i) X is complete. Let h be an arbitrary function in $L_2(\partial D)$. We must show that

$$(h, \Psi_m^\sigma) = 0 \quad \text{for } \sigma = 1, 2 \quad \text{and} \quad m = 0, 1, 2, \ldots, \tag{4.3}$$

implies that $h = 0$. Now, from Theorem 3.1, we know that the system of null-field equations (3.1) is always uniquely solvable; when $\phi_I \equiv 0$, the only solution of (3.1) *in* $C(\partial D)$ is $\phi_D(q) = 0$. Thus, if h is continuous and satisfies (4.3), then $h \equiv 0$. At first sight, this, together with a density argument, would seem to prove completeness; see, e.g., Colton (1982) or Millar (1983). However, there is a defect in this argument (Millar, 1983, p. 37); we can only assume that $h \in L_2(\partial D)$ and then show that it is actually continuous. This is easily done: the proof of Theorem 3.1 (see I, p. 321) demonstrates that h satisfies (4.3) if, and only if, h satisfies a homogeneous Fredholm integral equation of the second kind, with a continuous kernel [I, (3.9), with $V = 0$]; it follows that h must be continuous. (In three dimensions, it is necessary to iterate the corresponding integral equation to obtain the desired continuity; cf. Müller & Kersten, 1980.) Hence, $h = 0$ in $L_2(\partial D)$, and completeness is proved.

(ii) X is minimal. We must show that there exists a unique biorthogonal sequence $\{\psi_n^\nu\}$ such that $(\Psi_m^\sigma, \psi_n^\nu) = \delta_{mn}\delta_{\sigma\nu}$. Now, consider the null-field equations (3.9) and choose $\phi_I = \alpha_n^\nu/2\pi$, whence, from (3.16), $d_m^\sigma = \delta_{mn}\delta_{\sigma\nu}$. Since the null-field equations (3.9) are always uniquely solvable (Theorem 3.2), we can assert that ψ_n^ν exists. Uniqueness follows from the completeness of X.

For symmetric cylinders, we have the following simple corollary to Theorem 4.1.

Corollary 4.1 Let ∂D be symmetric about $\theta = 0$ (the y-axis) and let $L_2^1(\partial D)$ denote the subspace of symmetric functions in $L_2(\partial D)$. Then, the set $\{\Psi_m^1(q)\}$ $(m = 0, 1, \ldots)$ is complete and minimal in $L_2^1(\partial D)$. A similar result holds for the set $\{\Psi_m^2(q)\}$ with respect to $L_2^2(\partial D)$, the subspace of antisymmetric functions in $L_2(\partial D)$.

5. Application of the Method of Projection

Consider a fixed, partially-immersed cylinder which is symmetric about $\theta = 0$ (the asymmetric problem is considered in an appendix). If we write

$$\phi(q) = \phi^1(q) + \phi^2(q), \tag{5.1}$$

where $\phi^\sigma(q) \in L_2^\sigma(\partial D)$, the null field equations for $\phi(q)$, (3.9), decouple: in the present

notation, we obtain

$$(\phi^\sigma, \Psi_m^{\sigma *}) = d_m^\sigma \quad (\sigma = 1, 2; \quad m = 0, 1, \ldots), \tag{5.2}$$

where ϕ^1 and ϕ^2 are to be determined, and we may now suppress the dependence on σ.

For a train of regular surface waves incident upon the cylinder, the system (5.2) simplifies further (see the end of Section 3): we must determine the total (symmetric or antisymmetric) potential $\phi(q)$ from

$$(\phi, \Psi_0^*) = d_0 \tag{5.3}$$

and

$$(\phi, \Psi_m) = 0 \quad (m = 1, 2, \ldots), \tag{5.4}$$

where we have noticed that Ψ_m is real for $m \geqslant 1$.

Let M be the closure of the (infinite-dimensional) space spanned by the set $\{\Psi_m\}$ $(m \geqslant 1)$; M is a proper subspace of $L_2(\partial D)$. Then, (5.4) states that the unknown function $\phi(q)$ is orthogonal to every function in M, i.e. ϕ lies in the *orthogonal complement*, $M^\perp$, of M.

We can give M a physical interpretation. Consider the water-wave radiation problem $\mathscr{P}$ (I, p. 315), for a velocity potential $\phi_R(P)$; on ∂D, $\partial\phi_R/\partial n = V$, a prescribed function. Then, M is just (the closure of) the set of all functions V such that the corresponding velocity potential ϕ_R gives no waves at infinity. It follows that M is a proper subspace of $L_2(\partial D)$, since $\Psi_0 \in L_2(\partial D)$ but $\Psi_0 \notin M$.

Now, since M is closed, it follows that $L_2(\partial D)$ can be written as the direct sum of M and $M^\perp$, i.e. any $f \in L_2(\partial D)$ has a unique decomposition

$$f = f_1 + f_2,$$

where $f_1 \in M$ and $f_2 \in M^\perp$.

We know that $\phi \in M^\perp$. We now assert that $M^\perp$ is one-dimensional. For if it had a higher dimension, it would be possible to find a non-trivial function $f \in L_2(\partial D)$ that was orthogonal to every member of the set $S \equiv \{\Psi_m^*\}$ $(m \geqslant 0)$, and this would contradict the completeness of S, as guaranteed by Corollary 4.1.

Let us choose *any* function $h(q) \in L_2(\partial D)$. Then, since $M^\perp$ is one-dimensional, h can be uniquely decomposed as

$$h = p + \lambda\phi$$

where $p \in M$ and λ is a complex number. Thus, provided $\lambda \neq 0$, we have

$$\phi = (h - p)/\lambda. \tag{5.5}$$

Now, p is the *projection* of the chosen function h onto the subspace M. So, (5.5) states that if we choose any $h \in L_2(\partial D)$ and then subtract from h its projection onto M, we are left with a scalar multiple of the required total potential, $\phi(q)$. The constant λ may then be found from (5.3), i.e.

$$(h - p, \Psi_0^*) = \lambda d_0. \tag{5.6}$$

The above method is called the *method of projection* by Gregory & Gladwell

(1982); they devised, and used, it for solving various boundary-value problems for the semi-infinite elastic strip.

In any numerical implementation of the method of projection, we must necessarily use a finite-dimensional subspace of M. However, we can still construct a convergent sequence of approximations to the projection p, by the next theorem.

THEOREM 4.2 *Let h be any fixed member of $L_2(\partial D)$. Then, for each N, choose the coefficients, $a_n (n = 1, 2, \ldots, N)$ so as to minimize*

$$\left\| h - \sum_{n=1}^{N} a_n \Psi_n \right\|. \tag{5.7}$$

Then,

$$\left\| p - \sum_{n=1}^{N} a_n \Psi_n \right\| \to 0$$

as $N \to \infty$, where p is the projection of h onto M.

This theorem can be proved using the arguments given by Gregory & Gladwell (1982, theorem 6).

It is well known that the quantity (5.7) will be minimized if the coefficients $a_n(N)$ satisfy

$$\sum_{n=1}^{N} K_{mn} a_n = (h, \Psi_m) \quad (m = 1, 2, \ldots, N), \tag{5.8}$$

where

$$K_{mn} = (\Psi_m, \Psi_n). \tag{5.9}$$

(5.8) is a system of N simultaneous linear algebraic equations for the coefficients $a_n (n = 1, 2, \ldots, N)$. The matrix K_{mn} is real, symmetric and non-singular. Hence, the system (5.8) has the unique solution $a_1(N), a_2(N), \ldots, a_N(N)$. By Theorem 4.2,

$$h - \sum_{n=1}^{N} a_n(N) \Psi_n \to \lambda \phi \tag{5.10}$$

as $N \to \infty$, where $\phi(q)$ is the unknown total potential on ∂D, λ is a complex constant, and the convergence is in the mean-square sense. λ can be found from (5.6) as

$$\lambda \simeq (h - \sum_{n=1}^{N} a_n \Psi_n, \Psi_0^*)/d_0. \tag{5.11}$$

The initial function h. In order to implement the method of projection, we must choose a function h from $L_2(\partial D)$. An arbitrary choice for h will lead to mean-square convergence in (5.10). However, it may be possible to achieve pointwise convergence by making a sufficiently judicious choice. We conclude this section with a discussion on how such a choice might be made.

If $h \in M$, then $\lambda = 0$ and the method of projection fails. From our physical interpretation for M (given earlier), we know that if this function h is used as the prescribed normal velocity in the corresponding radiation problem, then the solution of this problem will be wave-free. Conversely, if h corresponds to a wave-

like solution of the radiation problem, then $h \notin M$, as required. There are two obvious ways of meeting this criterion. One is to set

$$h^1(q) = \mathbf{n}(q).\mathbf{j} \quad \text{and} \quad h^2(q) = \mathbf{n}(q).\mathbf{i} \tag{5.12}$$

(where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions, respectively), corresponding to rigid-body motions of the cylinder. (Recall that h^1 and h^2 are required to be even and odd, respectively, about $\theta = 0$.)

The second way is to set

$$h^\sigma = \Psi_0^\sigma = \mathbf{n}.\operatorname{grad} \Phi_0^\sigma \tag{5.13}$$

leading to the approximation

$$\phi(q) \simeq \left\{\Psi_0(q) - \sum_{n=1}^{N} a_n \Psi_n(q)\right\} \Big/ \lambda. \tag{5.14}$$

The choice (5.13) is equivalent to solving the null-field equations (3.19) directly: if (5.14) is substituted into (3.19) [Corollary 4.1 allows us to represent $\phi(q)$ as (5.14)], we obtain (5.8) and (5.11) [with (5.13)]. Since Ψ_0^σ is expensive to evaluate (by comparison with Ψ_m^σ for $m \geqslant 1$), (5.13) is not an attractive choice.

Both of the choices (5.12) and (5.13) are dependent on the geometry (through the normal vector $\mathbf{n}$). Physical arguments lead us to two more choices, both of which are independent of the geometry.

From (5.10), we see that we can think of $h(q)$ as an initial approximation to $\phi(q)$. If the incident waves are long, then a familiar approximation is $\phi \simeq \phi_I$ (the Froude–Krylov approximation), leading to

$$h^\sigma = \mu^\sigma \hat{h}^\sigma, \tag{5.15}$$

where

$$\begin{gathered} \mu^1 = i\mu^2 = Ag/\omega, \\ \hat{h}^1 = e^{-Ky}\cos Kx \quad \text{and} \quad \hat{h}^2 = e^{-Ky}\sin Kx; \end{gathered} \tag{5.16}$$

this corresponds to total transmission of the incident wave. For short waves, the incident wave will be almost totally reflected at the front face of the cylinder; if the half-beam of the cylinder is a, we have

$$\phi \simeq \begin{cases} 2(Ag/\omega)e^{-iKa}e^{-Ky}\cos K(x-a), & x > 0 \\ 0, & x < 0 \end{cases}$$

(cf. Ursell, 1961, p. 641), leading to (5.15) with

$$\mu^1 = \mu^2 = (Ag/\omega)e^{-iKa}$$

and

$$\hat{h}^1 = \hat{h}^2 = e^{-Ky}\cos K(x-a). \tag{5.17}$$

6. The Scattering of Regular Surface Waves by a Fixed, Half-immersed, Elliptic Cylinder

Consider the interaction between a train of regular surface waves and a fixed, half-immersed, elliptic cylinder. As an example, we have solved the corresponding

scattering boundary-value problem, using the method of projection to solve the system of null-field equations (3.19).

The incident wave will be partially reflected and partially transmitted. Let $\phi \to \phi_\pm$ as $x \to \pm\infty$. Then, the *reflection coefficient* R and the *transmission coefficient* T are defined by

$$\phi_+ = (1 + Re^{2iKx})\phi_I$$

and

$$\phi_- = T\phi_I,$$

respectively. R and T are related by

$$RR^* + TT^* = 1$$

and

$$|\arg R - \arg T| = \tfrac{1}{2}\pi \quad \textit{modulo } \pi;$$

these relations may be used to check numerical calculations (see, e.g. Newman, 1976).

From (3.17) and the asymptotic behaviour of Φ_m^σ as $|x| \to \infty$, we find that

$$\phi_+ = \phi_I + \tfrac{1}{2}(ic_0^1 + c_0^2)e^{-Ky+iKx}$$

and

$$\phi_- = \phi_I + \tfrac{1}{2}(ic_0^1 - c_0^2)e^{-Ky-iKx}$$

where

$$c_m^\sigma = -\int_{\partial D} \phi(q)\frac{\partial}{\partial n_q}\alpha_m^\sigma(q)\,ds_q$$

and we have also used (3.12). Writing $c_0^\sigma = (Ag/\omega)\hat{c}_0^\sigma$, we obtain

$$R = \tfrac{1}{2}(i\hat{c}_0^1 + \hat{c}_0^2)$$

and

$$T = 1 + \tfrac{1}{2}(i\hat{c}_0^1 - \hat{c}_0^2).$$

In Table 1, we give values of R for various Ka and $H = a/b$, where $2a$ and b are the beam and draught, respectively, of the cylinder. (For each pair (Ka, H), the upper number is $|R|$ and the lower (in parentheses) is arg R.) Similarly, we give values of T in Table 2. All the results shown were obtained using $N \leqslant 25$. We used two different choices for h, namely (5.16) and (5.17). (5.16) (the long-wave approximation) was found to be most efficient for $Ka \leqslant 1$, whilst (5.17) (the short-wave approximation) was found to be most efficient for $Ka > 1$; these observations accord with the physical arguments leading to the relevant choice for h. As Kb increases, $|T|$ decreases rapidly and becomes increasingly more difficult to compute accurately; however, R and arg T are easily computed.

When $H = 1$ (corresponding to a circular cylinder), our results are seen to be in excellent agreement with those obtained by Martin & Dixon (1983), using Ursell's

TABLE 1

The complex reflection coefficient, R, for various Ka and H

	Wavenumber, Ka				
H	0·5	1·0	1·5	2·0	2·5
0·5	9·760(−1)	0·996(−1)	1·000	1·000	1·000
	(−1·391)	(−2·143)	(−3·087)	(2·220)	(1·233)
1·0	8·403(−1)	9·862(−1)	9·981(−1)	9·996(−1)	9·999(−1)
	(−1·855)	(−2·438)	(2·990)	(2·061)	(1·103)
1·5	7·515(−1)	9·584(−1)	9·910(−1)	9·975(−1)	9·992(−1)
	(−2·047)	(−2·648)	(2·809)	(1·903)	(0·962)
2·0	7·061(−1)	9·336(−1)	9·809(−1)	9·932(−1)	9·971(−1)
	(−2·137)	(−2·781)	(2·679)	(1·779)	(0·845)
2·5	6·805(−1)	9·148(−1)	9·752(−1)	9·876(−1)	9·941(−1)
	(−2·188)	(−2·870)	(2·562)	(1·687)	(0·758)

multipole method. Several other authors have treated this case; see Martin & Dixon (1983) for a review. However, we are only aware of three studies on scattering by a fixed, half-immersed *elliptic* cylinder; we shall now describe these.

Kim (1964) solved the source integral equation [I, (2.8)] (for $H = 0{\cdot}2$, 1, 2 and 4) and then computed dimensionless forms of $\mathbf{F}-\mathbf{F}_I$, where $\mathbf{F}$ is the hydrodynamic force exerted on the cylinder and $\mathbf{F}_I$ is the Froude–Krylov force. (By definition, $\mathbf{F} = \mathbf{F}_I$ when $\phi = \phi_I$.)

Leppington (1973), using the method of matched asymptotic expansions, has obtained the following short-wave estimate for T:

$$T \sim (2i/\pi)H^6(Ka)^{-4}e^{-2iKa} \quad \text{as } Ka \to \infty;$$

when $H = 1$, it is known that the exact solution for T approaches the asymptotic

TABLE 2

The complex transmission coefficient, T, for various Ka and H

	Wavenumber, Ka				
H	0·5	1·0	1·5	2·0	2·5
0·5	2·178(−1)	2·728(−2)	5·584(−3)	1·561(−3)	5·512(−4)
	(0·179)	(−0·573)	(−1·516)	(−2·492)	(2·804)
1·0	5·420(−1)	1·655(−1)	6·093(−2)	2·663(−2)	1·316(−2)
	(−0·284)	(−0·868)	(−1·722)	(−2·652)	(2·674)
1·5	6·597(−1)	2.853(−1)	1·340(−1)	7·091(−2)	4·110(−2)
	(−0·476)	(−1·077)	(−1·903)	(−2·810)	(2·533)
2·0	7·080(−1)	3·582(−1)	1·947(−1)	1·165(−1)	7·641(−2)
	(−0·566)	(−1·211)	(−2·033)	(−2·934)	(2·415)
2.5	7·328(−1)	4·038(−1)	2·213(−1)	1·571(−1)	1·084(−1)
	(−0·617)	(−1·300)	(−2·150)	(−3·025)	(2·329)

solution (obtained by Ursell, 1961) rather slowly (Martin & Dixon, 1983, Section 3), and so we expect that Leppington's formula (for $H \neq 1$) will be difficult to verify.

Vongvisessomjai & Hanif (1979) have used a finite-element method to solve $\mathbb{S}$ when the water is of constant finite depth ($= h$, say). They have presented graphs of $|R|$, $|T|$ and dimensionless forms of $\mathbf{F}$, for $H = \frac{2}{3}$, 1 and $\frac{4}{3}$, and $h/a = 6$. For $|R|$, we find that our values (for $h/a = \infty$) lie above their curves (for $h/a = 6$). However, this discrepancy is probably due to the different values of h/a; for $H = 1$, Naftzger & Chakrabarti (1979) have shown that $|R|$ varies significantly with h/a, for small wavenumbers.

7. Conclusions and Discussion

In this paper, we have used the null-field method to solve water-wave scattering problems. We have provided two systems of null-field equations, one for $\phi_D(q)$ and one for $\phi(q)$; each system is uniquely solvable at all frequencies—irregular frequencies do not occur.

The usual procedure for solving null-field equations is to replace the unknown function by a series of basis functions with unknown coefficients; these coefficients satisfy an infinite system of linear algebraic equations. To obtain a numerical method, this system is truncated, to an $N \times N$ system, say. Does the solution of this truncated system approach that of the infinite system as $N \to \infty$? In general, this question has not been answered, although the numerical method has been used widely.

In most applications of the null-field method, attention is focused on the potential outside C_+, the escribed circle to ∂D^*, centred on O. This leads naturally to the so-called *T-matrix*, which relates the incident wave to the diffracted wave, without explicit computation of the potential on ∂D; see Martin (1982, Section 7) and Varadan & Varadan (1980). The T-matrix formulation is probably most useful for solving multiple-scattering problems, i.e. problems where there are two, or more, scatterers; see, e.g., Peterson & Ström (1974). The corresponding water-wave problems arise when studying, e.g., the interaction between neighbouring ships or between neighbouring wave-power devices. We hope to report on the application of the null-field method to such problems in a subsequent paper.

In Section 5, we described another method (the method of projection) for solving the null-field equations for $\phi(q)$, under two essential restrictions: (i) the incident wave must be a regular surface wave and (ii) the water must be of infinite depth. With these restrictions, the method of projection was shown to yield a convergent sequence of approximations to the solution of the infinite system of null-field equations. This result is interesting, because (as yet) there is no analogue in acoustics, for example. Also, the method fails for a general incident wave and for water of finite depth. However, the method can be used to treat three-dimensional problems when the scatterer is axisymmetric about a vertical axis. As an example, we have successfully used the method of projection (in Section 6) to solve the two-dimensional problem of scattering by a half-immersed elliptic cylinder.

Finally, we have also proved that the set of functions $\{\Psi_m^\sigma(q)\}$ is complete and minimal in $L_2(\partial D)$; it remains to show whether or not this set is actually a basis.

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Appendix. Extension of the Method of Projection to Asymmetric Cylinders

When ∂D is asymmetric about $\theta = 0$, the null-field equations, (3.9), do not decouple. For a regular incident wave, we must solve (3.19), i.e.

$$(\phi, \Psi_m^{\sigma *}) = d_0^\sigma \delta_{0m} \quad (\sigma = 1, 2; \quad m = 0, 1, 2, \ldots). \tag{A.1}$$

To apply the method of projection, we let M be the closure of the space spanned

by

$$\{\Psi_m^\sigma\} \quad (\sigma = 1, 2; \quad m = 1, 2, \ldots).$$

It quickly follows, from (A.1) and Theorem 4.1, that $\phi \in M^\perp$ and $M^\perp$ is a *two-dimensional* subspace of $L_2(\partial D)$.

Choose two functions h^1 and h^2 from $L_2(\partial D)$. Let their projections onto M be p^1 and p^2, respectively; these may be computed using theorem 4.2 as before. Then, $h^1 - p^1$ and $h^2 - p^2$ are two elements of $M^\perp$; suppose they are linearly independent (this imposes a restriction on the choice of h^1 and h^2). It follows that

$$\phi = \lambda^1(h^1 - p^1) + \lambda^2(h^2 - p^2),$$

for some scalars λ^1 and λ^2. These scalars are found by solving the pair of equations,

$$\left.\begin{aligned} \lambda^1(h^1 - p^1, \Psi_0^{1*}) + \lambda^2(h^2 - p^2, \Psi_0^{1*}) &= d_0^1 \\ \lambda^1(h^1 - p^1, \Psi_0^{2*}) + \lambda^2(h^2 - p_0^2, \Psi_0^{2*}) &= d^2 \end{aligned}\right\}$$

that is obtained from (A.1), with $m = 0$.

It is now clear why the method of projection is not useful for an arbitrary incident wave. As the number of non-zero terms on the right-hand side of (3.9) is increased, so the dimension of $M^\perp$ increases too; for each dimension, we must choose a function h and then find its projection onto M. For an arbitrary incident wave, $M^\perp$ is infinite-dimensional. Also, for water of constant finite depth, $M^\perp$ is generally infinite-dimensional, even for a regular surface wave. Similar remarks apply to scattering problems in other branches of mathematical physics, e.g., in the theory of acoustics.