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Inverse scattering for geophysical problems. III. On the velocity-inversion problems of acoustics†

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A bounded inhomogeneity $\mathscr{D}$ is immersed in an acoustic medium; the speed of sound is a function of position in $\mathscr{D}$, and is constant outside. A time-harmonic source is placed at a point y and the pressure at a point x is measured. Given such measurements at all $x \in P$ for all $y \in P$, where P is a plane that does not intersect $\mathscr{D}$, can the speed of sound (in the unknown region $\mathscr{D}$) be recovered? This is a velocity-inversion problem. The three-dimensional problem has been solved analytically by Ramm (*Phys. Lett.* **99**A, 258–260 (1983)). In the present paper, analogous one-dimensional and two-dimensional problems are solved, as well as the problem where the plane P is the interface between two different acoustic media.

1. Introduction

Consider a bounded three-dimensional inhomogeneity $\mathscr{D}$ immersed in an acoustic medium; the speed of sound is a function of position in $\mathscr{D}$, and is constant outside. A time-harmonic source is placed at $y \notin \mathscr{D}$ and the pressure at $x \notin \mathscr{D}$ is measured. Given such measurements at all positions $x \in P$, for all source positions $y \in P$, where P is a plane that does not intersect $\mathscr{D}$, we would like to recover the speed of sound (in $\mathscr{D}$).

More precisely, take cartesian coordinates (x_1, x_2, x_3) so that $x_3 = 0$ is the plane P and $\mathscr{D}$ lies in the lower half-space ($x_3 < 0$). We have

$$\nabla^2 u + k^2(1+v(x))\,u = -\delta(x-y), \tag{1.1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, k is the (constant) wavenumber outside $\mathscr{D}$, and $v(x) = 0$ outside $\mathscr{D}$. We suppose that the solution of (1.1) is known on the plane $x_3 = 0$ for all positions of the source on $y_3 = 0$ and for small values of k, and are required to find $v(x)$. Of course, we do not know the location of $\mathscr{D}$, but assume merely that, for example, $v(x) = 0$ for $|x| \geqslant R$, where R is an arbitrary (large) fixed positive number. This inverse problem was solved analytically by Ramm (1983*a*); his method is described in §2. Problems of this type ('velocity-inversion' problems) have been discussed previously; for reviews in a geophysical context, see Bleistein & Cohen (1982) and Weglein (1982).

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Ramm (1983a) lists some possible applications of the model problem to various geophysical situations, for example, the plane P could be a lode of metallic ore, or a pipeline. To improve the model, one could take P to be the interface between two different acoustic media. In §3, we show that this problem can also be solved analytically.

When $\mathscr{D}$ corresponds to a pipeline, it is natural to seek a two-dimensional analogue of the three-dimensional theory given by Ramm (1983a). It turns out that this analogue is quite different, for example, it is no longer possible to pass to the limit $k \to 0$ in the scheme of §2, and this leads to new features that are not present for three dimensions. This problem is discussed in §4. For completeness, we also discuss one-dimensional problems (in §5); again, new features arise.

Henceforth, we assume, as is customary, that $v(x)$ is compactly supported. However, most of our results hold for inhomogeneities that satisfy

$$|v(x)| \leqslant C(1+|x_1|^2+|x_2|^2)^{-a}\,(1+|x_3|)^{-a},$$

where C and a are constants ($a > 1$).

Finally, we remark that Ramm & Weglein (1984) have treated a related two-parameter (density and bulk modulus) inversion problem.

2. The basic scheme in three dimensions

The differential equation (1.1) can be recast as the integral equation

$$u(x,y) = g(x,y)+k^2 \int g(x,z)\,v(z)\,u(z,y)\,\mathrm{d}z, \tag{2.1}$$

where
$$g(x,y) = \exp(\mathrm{i}k\,|x-y|)/4\pi\,|x-y| \tag{2.2}$$

is the solution of (1.1) with $v \equiv 0$, and the integration in (2.1) is over the support of v (i.e. $\mathscr{D}$); since we assumed that $\mathscr{D}$ lies in the lower half-space, we can suppose for definiteness that the integration is over this entire half-space ($z_3 < 0$). We assume that $v(z)$ is independent of k; if $v = v(z, k)$ and is continuous in k near $k = 0$, then our method is still valid but will only yield $v(z, 0)$.

Set $w = u-g$, whence (2.1) becomes

$$w-Tw = h, \tag{2.3}$$

where
$$Tw = k^2 \int g(x,z)\,v(z)\,w(z,y)\,\mathrm{d}z$$

and $h = Tg$. Equation (2.3) can be considered in the space of continuous functions defined in the lower half-space, with the usual norm. The operator T is bounded in this space, whence (2.3) is solvable by iteration whenever $\|T\| < 1$, i.e. for sufficiently small k. Thus, we can write

$$w(x,y;k) = \sum_{n=0}^{\infty} k^n w_n(x,y). \tag{2.4}$$

Substituting into (2.3), we see that $w_0 = w_1 = 0$ and

$$w_2(x,y) = \int g_0(x,z)\, v(z)\, g_0(z,y)\, \mathrm{d}z,$$

where

$$g(x,y) = \sum_{n=0}^{\infty} k^n g_n(x,y); \tag{2.5}$$

hence, $v(z)$ satisfies

$$\int \frac{v(z)}{|x-z|\,|y-z|}\, \mathrm{d}z = 16\pi^2 \lim_{k\to 0} \left\{\frac{u(x,y)-g(x,y)}{k^2}\right\} \equiv f(x,y), \tag{2.6}$$

say. Note that, in principle, $f(x, y)$ can be measured. Thus, given f, we can consider (2.6) as an integral equation for v.

2.1. *Solution of the integral equation* (2.6)

Suppose that we know $f(x, y)$ for $x = (x_1, x_2, 0) \in P$ and $y = (y_1, y_2, 0) \in P$. Take Fourier transforms in x_1, x_2, y_1 and y_2, with transform variables λ_1, λ_2, μ_1 and μ_2, respectively, to give

$$\int v(z) \exp\{\mathrm{i}(\lambda_1+\mu_1) z_1 + \mathrm{i}(\lambda_2+\mu_2) z_2 - (|\lambda|+|\mu|)\,|z_3|\}\, \mathrm{d}z$$
$$= 4\pi^2 |\lambda|\,|\mu|\, F(\lambda_1, \lambda_2, \mu_1, \mu_2), \tag{2.7}$$

where F is the corresponding Fourier transform of f, $|\lambda|^2 = \lambda_1^2+\lambda_2^2$, $|\mu|^2 = \mu_1^2+\mu_2^2$, and we define the Fourier transform (in one variable) by

$$F(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{\mathrm{i}\lambda x}\, \mathrm{d}x.$$

We begin by proving that the homogeneous form of (2.7) has only the trivial solution $v \equiv 0$. Introduce new variables

$$p_1 = \lambda_1+\mu_1, \quad p_2 = \lambda_2+\mu_2, \quad p_3 = |\lambda| \quad \text{and} \quad p_4 = |\mu|, \tag{2.8}$$

whence (2.7) becomes (with $F \equiv 0$)

$$\int v(z) \exp\{\mathrm{i}p_1 z_1 + \mathrm{i}p_2 z_2 - (p_3+p_4)\,|z_3|\}\, \mathrm{d}z = 0. \tag{2.9}$$

Consider the left side of (2.9) as a function of four independent complex variables p_n ($n = 1, 2, 3, 4$). This function is entire in each of these variables, since $v(z)$ has compact support. In particular, (2.9) holds for

$$-\infty < p_1 < \infty, \quad -\infty < p_2 < \infty, \quad 0 < p_3 < \infty \quad \text{and} \quad 0 < p_4 < \infty.$$

Since $v(z)$ is a function of only three variables, set $q = p_3+p_4$, $0 < q < \infty$. Then the left side of (2.9) is just the double Fourier transform (in z_1 and z_2) of the Laplace transform (in $-z_3$) of $v(z_1, z_2, z_3)$. These transforms can be inverted uniquely to prove that $v \equiv 0$.

When this approach is adopted for the inhomogeneous equation (2.7), a difficulty arises, namely, the transformation (2.8) is not always (uniquely) invertible (the corresponding Jacobian vanishes whenever $\mu_1\lambda_2 = \mu_2\lambda_1$). When it is not, we may not be able to write

$$F(\lambda_1, \lambda_2, \mu_1, \mu_2) = \mathscr{F}(p_1, p_2, p_3, p_4), \tag{2.10}$$

say, for an arbitrary F. However, we know from (2.7) that (2.10) indeed holds if the data f is known exactly. In this case, we can proceed as for the homogeneous equation: set $p_3 = p_4 = \frac{1}{2}q$, whence $v(z)$ satisfies

$$\int v(z) \exp(\mathrm{i}p_1 z_1 + \mathrm{i}p_2 z_2 - q\,|z_3|)\,\mathrm{d}z = \pi^2 q^2 \mathscr{F}(p_1, p_2, \tfrac{1}{2}q, \tfrac{1}{2}q); \tag{2.11}$$

the integral transforms can be inverted, in principle, to obtain $v(z)$ from $\mathscr{F}$.

We conclude by noting that the above analysis has some similarities with that given by Lavrentiev *et al.* (1970, ch. 5) for a different class of problems.

2.2. *Practical considerations*

We can compute $F(\lambda_1, \lambda_2, \mu_1, \mu_2)$ for arbitrary real values of λ_1, λ_2, μ_1 and μ_2. Since we have set $p_3 = p_4 = \frac{1}{2}q$, let us write

$$(\lambda_1, \lambda_2, \mu_1, \mu_2) = \tfrac{1}{2}q(\cos\phi, \sin\phi, \cos\theta, \sin\theta), \tag{2.12}$$

where $0 < q < \infty$, $0 \leqslant \phi < 2\pi$ and $0 \leqslant \theta < 2\pi$. Hence

$$p_1 = q\cos\alpha\cos\beta \quad \text{and} \quad p_2 = q\cos\alpha\sin\beta, \tag{2.13}$$

where

$$2\alpha = \phi - \theta \quad \text{and} \quad 2\beta = \phi + \theta. \tag{2.14}$$

Thus (2.13) shows that the point Q with Cartesian coordinates (p_1, p_2, q) lies inside a circle of radius q, centred at $(0, 0, q)$, i.e. Q lies inside a right circular cone with a vertex angle of $\frac{1}{2}\pi$. We denote this semi-infinite conical region by $\mathscr{P}$ (see figure 1).

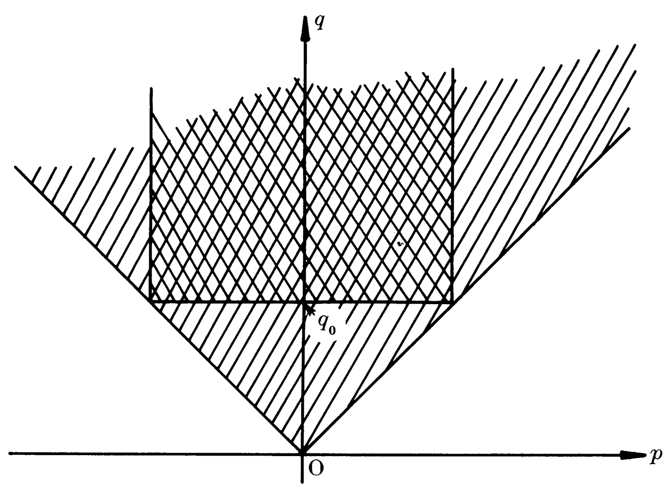

FIGURE 1. The region $\mathscr{P}$ is shown hatched. The region $\mathscr{P}_0 \subset \mathscr{P}$ is shown cross-hatched.

The Jacobian of the transformation (2.8) vanishes when $2\alpha = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, whence it is convenient to take

$$0 < \alpha < \tfrac{1}{2}\pi \quad \text{and} \quad -\pi < \beta < \pi.$$

It is a simple matter to obtain α and β within these ranges, given p_1, p_2 and q, and then to recover λ_1, λ_2, μ_1 and μ_2 from (2.12) and (2.14). Hence, for $Q \in \mathscr{P}$, we can compute $\mathscr{F}$ in (2.11).

If we invert the Laplace transform in (2.11), using the Mellin contour $\mathrm{Re}(q) = c$, then we obtain the function

$$\int v(\tilde{z}, z_3)\, \mathrm{e}^{\mathrm{i}p\cdot\tilde{z}}\, \mathrm{d}\tilde{z} = G(p, z_3), \tag{2.15}$$

say, for $z_3 < 0$ and $|p| < c$, where $p = (p_1, p_2)$ and $\tilde{z} = (z_1, z_2)$. Thus, given any particular p, we can always choose c in order to compute $G(p, z_2)$. Therefore, G can be calculated for all p, whence the double Fourier transform in (2.15) can be inverted to obtain $v(z)$.

However, it may be more convenient to fix c initially. So, choose a positive number q_0 and then consider (2.11) for $0 \leqslant |p| < q_0$, $q \geqslant q_0 > 0$. We denote this semi-infinite cylindrical region by $\mathscr{P}_0 \subset \mathscr{P}$ (see figure 1). Invert the Laplace transform in (2.11), by using an inversion contour along $\mathrm{Re}(q) = c > q_0$ (see Appendix A for some remarks on the numerical inversion of Laplace transforms). This gives us $G(p, z_3)$ for $|p| \leqslant q_0$ and $z_3 < 0$, i.e.

$$\int_{|\tilde{z}|\leqslant R} v(\tilde{z})\, \mathrm{e}^{\mathrm{i}p\cdot\tilde{z}}\, \mathrm{d}\tilde{z} = G(p), \quad |p| \leqslant q_0, \tag{2.16}$$

where we have suppressed the parametric dependence on z_3. Although (2.16) only gives us the Fourier transform of $v(\tilde{z})$ for $|p| \leqslant q_0$, we are compensated by knowing that v has compact support: $v(\tilde{z}) = 0$ for $|\tilde{z}| > R$. We cannot give an exact solution to (2.16), but we can give good closed-form approximations. To do this we can use the following theorem, which gives Fourier inversion of incomplete data.

THEOREM 1. *Suppose that $f(x)$ solves*

$$\int_{|x|\leqslant R} f(x)\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}x = F(\xi), \quad |\xi| \leqslant X, \tag{2.17}$$

where $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$. Then, given $\epsilon > 0$, there exists $n_0(\epsilon)$ such that

$$f_n(x) = \int_{|\xi|\leqslant X} h_n(\xi)\, F(\xi)\, \mathrm{e}^{-\mathrm{i}\xi\cdot x}\, \mathrm{d}\xi \tag{2.18}$$

satisfies

$$\|f - f_n\| < \epsilon \quad \textit{for all} \quad n \geqslant n_0(\epsilon), \tag{2.19}$$

where

$$h_n(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \delta_n(x)\, \mathrm{e}^{\mathrm{i}\xi\cdot x}\, \mathrm{d}x, \tag{2.20}$$

$$\delta_n(x) = \tilde{\delta}_n(x) \left(1 - \frac{|x|^2}{4R^2}\right)^n \left(\frac{n}{4\pi R^2}\right)^{N/2}, \tag{2.21}$$

$$\tilde{\delta}_n(x) = \left\{\frac{1}{|B_X|} \int_{|\xi|\leqslant X} \exp\left(-\frac{\mathrm{i}\xi\cdot x}{2n+N}\right) \mathrm{d}\xi\right\}^{2n+N}, \tag{2.22}$$

$B_X = \{\xi : |\xi| \leqslant X\}$ is the ball in $\mathbb{R}^N$ of radius X, and $|B_X|$ is the volume of B_X. The estimate (2.19) holds in $C(B_R)$ $(L^2(B_R))$ if $f \in C(B_R)$ $(L^2(B_R))$.

Proof. Multiply (2.17) by $e^{-i\xi\cdot y}h_n(\xi)$, where $|y| \leqslant R$ and $h_n(\xi)$ will be determined, and integrate over $|\xi| \leqslant X$ to give

$$\int_{|x|\leqslant R} f(x) \int_{|\xi|\leqslant X} h_n(\xi)\, e^{i\xi\cdot(x-y)}\, d\xi\, dx = \int_{|\xi|\leqslant X} h_n(\xi)\, F(\xi)\, e^{-i\xi\cdot y}\, d\xi, \quad |y| \leqslant R.$$

We choose $h_n(\xi)$ so that

$$\int_{|\xi|\leqslant X} h_n(\xi)\, e^{i\xi\cdot(x-y)}\, d\xi = \delta_n(y-x), \quad |x| \leqslant R, \quad |y| \leqslant R, \tag{2.23}$$

is a 'delta-sequence' (in $C(B_R)$ or $L^2(B_R)$), i.e. so that

$$\left\| f(x) - \int_{|y|\leqslant R} f(y)\, \delta_n(x-y)\, dy \right\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.24}$$

From (2.23), we have

$$\delta_n(x) = \int_{|\xi|\leqslant X} h_n(\xi)\, e^{-i\xi\cdot x}\, d\xi, \quad |x| \leqslant 2R; \tag{2.25}$$

(2.20) follows by inversion if $\delta_n(x)$ is defined for all x. The construction of a suitable delta-sequence (it must satisfy (2.24) and be representable as (2.25)) was first given by Ramm (1970) in a study of apodization theory for linear optical instruments; see Ramm (1980, pp. 210–215). This book also contains further references and detailed proofs, and gives error bounds on $\|f-f_n\|$, given bounds on $\|f\|$ and $\|\operatorname{grad} f\|$.

The integral in (2.22) can be evaluated analytically for all N (Ramm 1983*b*) to give

$$\tilde{\delta}_n(x) = \{2^\nu \Gamma(\nu+1)\, J_\nu(\lambda|x|)\, (\lambda|x|)^{-\nu}\}^{2n+N}, \tag{2.26}$$

where $N = 2\nu$ and $2\lambda = X/(n+\nu)$. Hence, it is clear that $\delta_n(x)$ is a function of one variable, namely $|x|$.

Returning to our two-dimensional equation (2.16), we obtain the following approximation to v:

$$v(\tilde{z}) \simeq v_n(\tilde{z}) = \int_{|p|\leqslant q_0} h_n(p)\, G(p)\, e^{-ip\cdot\tilde{z}}\, dp, \quad |\tilde{z}| < R, \tag{2.27}$$

where $$h_n(p) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \delta_n(x)\, e^{ip\cdot x}\, dx = \frac{1}{2\pi}\int_0^\infty \delta_n(r)\, J_0(|p|r)\, r\, dr, \tag{2.28}$$

$$\delta_n(r) = \left[\frac{2}{\lambda r} J_1(\lambda r)\right]^{2n+2} \left(1 - \frac{r^2}{4R^2}\right)^n \frac{n}{4\pi R^2},$$

and $2\lambda = q_0/(n+1)$. (Note that $r\delta_n(r)$ is a delta-sequence in $C(0, R)$ or in $L^2(0, R)$.)

This concludes our discussion on the practical inversion of (2.11): the dependence of v on z_3 is recovered by inverting the Laplace transform, with q as the transform variable (the parameter q_0 is at our disposal); the dependence on $\tilde{z} = (z_1, z_2)$ is recovered by approximately inverting the Fourier-type transform, with $p = (p_1, p_2)$ as the transform variable.

Finally, we note that Ramm (1983*a*) has shown how to deal with random errors on the right side of (2.7).

3. The interface problem

Let us now complicate the previous problem by supposing that the plane $P(x_3 = 0)$ is an interface between two different acoustic media, with transmission conditions across this interface. We assume that the upper half-space $(x_3 > 0)$ has density ρ_1 and wavenumber k_1, where

$$\rho_1 = \rho\delta, \quad k_1 = \tau k, \tag{3.1}$$

ρ is the density of the lower half-space $(x_3 < 0)$ and k is the corresponding wavenumber; δ and τ are constants. As before, the lower half-space contains a bounded inhomogeneity. For a point source at y, with $y_3 > 0$, we have

$$\nabla^2 u_1 + \tau^2 k^2 u_1 = -\delta(x-y), \quad x_3 > 0, \tag{3.2}$$

$$\nabla^2 u + k^2(1+v(x))\, u = 0, \quad x_3 < 0, \tag{3.3}$$

with the conditions

$$\frac{\partial u}{\partial x_3} = \frac{\partial u_1}{\partial x_3}, \quad u = \delta u_1 \quad \text{on} \quad x_3 = 0. \tag{3.4}$$

Physically, (3.4) means that both the normal velocity and the pressure are continuous across the interface P. We suppose that we know the solution of (3.2)–(3.4) on P, for small k and for all positions of the source on P, and wish to determine $v(x)$.

The solution to (3.2)–(3.4) with $v \equiv 0$ is given by Ewing *et al.* (1957, pp. 94–96): if we denote the solution by $g(x, y)$, we have, for $y_3 > 0$,

$$g(x,y) = \frac{1}{2\pi}\int_0^\infty \frac{J_0(\rho\xi)\,\xi}{\nu+\delta\nu_1} \exp(\nu x_3 - \nu_1 y_3)\,\mathrm{d}\xi \quad \text{for} \quad x_3 < 0,$$

where

$$\nu(\xi) = (\xi^2 - k^2)^{\frac{1}{2}}, \quad \nu_1(\xi) = (\xi^2 - k_1^2)^{\frac{1}{2}} \quad \text{and} \quad \rho^2 = (x_1-y_1)^2 + (x_2-y_2)^2;$$

a similar expression can be found for $x_3 > 0$, but we do not need it. Letting $k \to 0$ $(\nu \to \xi,\ \nu_1 \to \xi)$, we obtain

$$g \to g_0(x,y) = \frac{1}{2\pi}\int_0^\infty \frac{J_0(\rho\xi)}{1+\delta} \mathrm{e}^{-\xi(y_3 - x_3)}\,\mathrm{d}\xi$$

$$= \frac{2}{1+\delta}\,\frac{1}{4\pi}\,\frac{1}{|x-y|}, \quad x_3 < 0, \quad y_3 > 0.$$

(One can verify that the same limit is obtained regardless of the location of x and y, relative to P.) So, if we use the method of §2 to solve the present problem, we obtain

$$\int \frac{v(z)}{|x-z|\,|y-z|}\,\mathrm{d}z = \frac{(1+\delta)^2}{4} f(x,y), \quad x, y \in P \tag{3.5}$$

as our integral equation for $v(z)$; the function f is defined by (2.6). We note that the presence of the interface has merely introduced an additional constant factor, $\frac{1}{4}(1+\delta)^2$; the integral equation is essentially the same, and can be solved as before.

This simplification occurred because we took the limit $k \to 0$. We note that our method for solving (2.6) and (3.5) will also work on a similar equation, namely

$$\int g(x,z)\, v(z)\, g(y,z)\, \mathrm{d}z = k^{-2}\{u(x,y)-g(x,y)\}, \quad x, y \in P, k > 0,$$

which one obtains by using the Born approximation ($w = h$) to treat the problem solved in §2. The corresponding Born-approximation integral equation for the present problem is much more complicated.

4. The two-dimensional problem

Let us consider the two-dimensional analogue of the three-dimensional problem solved in §2. Take cartesian coordinates (x_1, x_2) and assume that there is a bounded inhomogeneity $\mathscr{D}$ in the lower half-plane ($x_2 < 0$). Thus, we have

$$\nabla^2 u + k^2(1+v(x))\, u = -\delta(x-y), \tag{4.1}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $v(x) = 0$ outside $\mathscr{D}$. We suppose that we know the solution of (4.1), for small values of k, everywhere on the line $x_2 = 0$ for all positions of the source on this line ($y_2 = 0$), and wish to find $v(x)$ given that $v(x) = 0$ for $|x| \geqslant R$.

We recast (4.1) as the integral equation

$$u(x,y) = g(x,y) + k^2 \int g(x,z)\, v(z)\, u(z,y)\, \mathrm{d}z, \tag{4.2}$$

where
$$g(x,y) = \tfrac{1}{4}\mathrm{i} H_0^{(1)}(k\,|x-y|) \tag{4.3}$$

and the integration is over the lower half-plane, $z_2 < 0$. The behaviour of g for small k, and fixed x and y, is given by

$$g(x,y) = \alpha(k) + g_0(x,y) + O(k^2 \ln k)$$

as $k \to 0$, where, here,

$$g_0(x,y) = \frac{-1}{2\pi} \ln|x-y|, \quad \alpha(k) = \frac{-1}{2\pi}(\ln \tfrac{1}{2}k + \gamma - \tfrac{1}{2}\pi \mathrm{i})$$

and $\gamma = 0.5572\ldots$ is Euler's constant. Thus, as $k \to 0$, $g \to \infty$. This behaviour prevents us from letting $k \to 0$ in (4.2). Similar difficulties arise in the problem of two-dimensional scattering by a sound-soft obstacle, and have been treated by several authors (see, for example, Noble 1962; MacCamy 1965; Ramm 1968). Our analysis is similar to that of the last of these references.

From (4.2), we obtain

$$u = \alpha(k) + g_0 + O(1) \quad \text{as} \quad k \to 0.$$

More precisely, we have

$$u - g = k^2\{\alpha^2 U_0 + \alpha U_1 + U_2\} + o(k^2) \quad \text{as} \quad k \to 0, \tag{4.4}$$

where
$$U_0 = \int v(z)\,\mathrm{d}z, \tag{4.5}$$

$$U_1(x,y) = \int \{g_0(x,z)+g_0(y,z)\}\,v(z)\,\mathrm{d}z \tag{4.6}$$

and
$$U_2(x,y) = \int g_0(x,z)\,v(z)\,g_0(y,z)\,\mathrm{d}z. \tag{4.7}$$

From (4.4), we obtain

$$U_0 = \lim_{k\to 0}\left\{\frac{u-g}{\alpha^2k^2}\right\} \equiv f_0(x,y), \tag{4.8}$$

$$U_1 = \lim_{k\to 0}\left\{\frac{u-g-\alpha^2k^2U_0}{\alpha k^2}\right\} \equiv f_1(x,y) \tag{4.9}$$

and
$$U_2 = \lim_{k\to 0}\left\{\frac{u-g-\alpha^2k^2U_0-\alpha k^2U_1}{k^2}\right\} \equiv f_2(x,y), \tag{4.10}$$

where f_0, f_1 and f_2 are, in principle, measurable quantities.

Equation (4.8) says that we can recover U_0 from measurements of $u(x, y)$ at one point x, with the source at one point y (x and y can coincide), and at small values of k. The number U_0 might be called the *intensity* of the inhomogeneity.

In the next two subsections, we examine (4.9) and (4.10). These are both integral equations satisfied by $v(z)$.

4.1. *The integral equation* (4.9)

From (4.6) and (4.9), we have

$$\int \{g_0(x,z)+g_0(y,z)\}\,v(z)\,\mathrm{d}z = f_1(x,y)$$

whence
$$\int g_0(x,z)\,v(z)\,\mathrm{d}z = \tfrac{1}{2}f_1(x,x). \tag{4.11}$$

This integral equation arises in various inverse problems of potential theory and, in general, it is not uniquely solvable. However, if further restrictions are placed on v, a uniquely solvable equation can be obtained, as we shall now show. Suppose that the source and receiver coincide, and that both are on the line $x_2 = 0$. Then (4.11) becomes

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty} v(z_1,z_2)\ln\{(x_1-z_1)^2+z_2^2\}\,\mathrm{d}z_1\,\mathrm{d}z_2 = \tilde{f}_1(x_1), \quad -\infty < x_1 < \infty, \tag{4.12}$$

where $\tilde{f}_1(x_1) = -2\pi f(x_1, 0; x_1, 0)$. Take the Fourier transform of (4.12) to obtain (see Appendix B)

$$\int_{-\infty}^{0}\int_{-\infty}^{\infty} v(z_1,z_2)\exp(\mathrm{i}\lambda z_1+|\lambda|z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 = -2|\lambda|\tilde{F}_1(\lambda), \tag{4.13}$$

where $\tilde{F}_1(\lambda)$ is the Fourier transform of $\tilde{f}_1(x_1)$. Now, suppose that

$$v(z_1, z_2) = v(z_2), \quad a < z_1 < b, \quad 0 > z_2 > -R,$$

and is zero otherwise. Then (4.13) becomes

$$\int_{-\infty}^{0} v(z_2)\, e^{|\lambda| z_2}\, dz_2 = \frac{-2i\lambda\,|\lambda|\,\tilde{F}_1(\lambda)}{e^{i\lambda b} - e^{i\lambda a}}.$$

Inverting this Laplace transform (for $\lambda > 0$, say) gives an explicit formula for $v(z_2)$, in terms of back-scattered data at all points on the line $z_2 = 0$.

4.2. *The integral equation* (4.10)

Suppose that x and y both lie on the line $x_2 = 0$, and consider the integral equation

$$\int \ln|x-z|\,\ln|y-z|\,v(z)\,dz = f(x_1, y_1), \tag{4.14}$$

where $x = (x_1, 0)$, $y = (y_1, 0)$ and $f(x_1, y_1) = 4\pi^2 f_2(x, y)$. Equation (4.14) is the two-dimensional analogue of (2.6). To solve it, take Fourier transforms in x_1 and y_1, with transform variables λ and μ, respectively, to obtain (see Appendix B)

$$\int v(z) \exp\{i(\lambda+\mu) z_1 + (|\lambda|+|\mu|) z_2\}\, dz = 4|\lambda||\mu| F(\lambda, \mu), \tag{4.15}$$

where F is the corresponding Fourier transform of f.

Introduce new variables

$$p = \lambda+\mu \quad \text{and} \quad q = |\lambda|+|\mu|. \tag{4.16}$$

The arguments at the end of §2.1 show that $f = 0$ implies that $v \equiv 0$, i.e. (4.15) has at most one solution.

The transformation (4.16) is clearly invertible if λ and μ have opposite signs: we take $\mu > 0$ and $\lambda < 0$. Thus, (4.15) becomes

$$\int v(z) \exp(ipz_1 + qz_2)\, dz = (q^2-p^2)\,\mathscr{F}(p, q), \tag{4.17}$$

where $\mathscr{F}(p, q) = F(\tfrac{1}{2}(p-q), \tfrac{1}{2}(p+q))$. The restrictions on λ and μ imply that the point Q with cartesian coordinates (p, q) lies inside a right-angled wedge, $0 < q < \infty$, $-q < p < q$. We denote this region by $\mathscr{P}$ (see figure 1). We can invert (4.17) to obtain $v(z)$, using the methods described in §2. So we choose a positive number q_0 and then consider (4.17) for $-q_0 < p < q_0$ and $q \geqslant q_0 > 0$; this is the region $\mathscr{P}_0$ shown in figure 1. Inverting the Laplace transform in z_2, we obtain the function

$$\int_{-R}^{R} v(z_1, z_2)\, e^{ipz_1}\, dz_1 = G(p, z_2), \tag{4.18}$$

say, for $z_2 < 0$ and $-q_0 < p < q_0$. Again, we find an approximate solution to (4.18) by using theorem 1:

$$v(z_1, z_2) \simeq v_n(z_1, z_2) = \int_{-q_0}^{q_0} h_n(p)\, G(p, z_2)\, e^{-ipz_1}\, dp, \tag{4.19}$$

where
$$h_n(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta_n(x)\,\mathrm{e}^{\mathrm{i}px}\,\mathrm{d}x,$$
$$\delta_n(x) = \left(\frac{\sin\lambda x}{\lambda x}\right)^{2n+1}\left(1-\frac{x^2}{4R^2}\right)^n\left(\frac{n}{4\pi R^2}\right)^{\frac{1}{2}},$$
$\lambda = q_0/(2n+1)$ and $\|v-v_n\| \to 0$ as $n\to\infty$.

5. The one-dimensional problem

For completeness, we now consider a one-dimensional analogue, namely, given the solution of
$$\{\mathrm{d}^2/\mathrm{d}x^2+k^2(1+v(x))\}\,u(x,y) = -\delta(x-y) \tag{5.1}$$
for $x \geqslant 0$ and $y \geqslant 0$, determine $v(x)$, where $v(x) = 0$ for $x > -\epsilon$ and $x < -R$ (ϵ and R are positive constants). As before, we replace (5.1) by the integral equation (2.1), where now
$$g(x,y) = \frac{\mathrm{i}}{2k}\,\mathrm{e}^{\mathrm{i}k|x-y|} \tag{5.2}$$
and the integration is along the half-line $z < 0$.

As in §2, set $u-g = w$, whence w satisfies
$$w(x,y) = \tfrac{1}{2}\mathrm{i}k\int v(z)\,w(z,y)\,\mathrm{e}^{\mathrm{i}k|x-z|}\,\mathrm{d}z+h(x,y), \tag{5.3}$$
where
$$h(x,y) = -\frac{1}{4}\int v(z)\,\exp\{\mathrm{i}k(|x-z|+|y-z|)\}\,\mathrm{d}z. \tag{5.4}$$

Equation (5.3) is uniquely solvable by iteration for sufficiently small k. Thus, w is an analytic function of k in a neighbourhood of $k = 0$: expanding w as in (2.4) and substituting into (5.3), we easily obtain
$$w_0 = -\tfrac{1}{4}v_0,$$
where
$$v_n \equiv \int z^n v(z)\,\mathrm{d}z \quad (n = 0,1,\ldots)$$
are the *moments of* v. Thus v_0 (the intensity of v) can be obtained by determining the limit of $u-g$ as $k\to 0$ at any fixed values of x and y.

Similarly, we have
$$w_1(x,y) = -\tfrac{1}{8}v_0^2-\frac{1}{4}\int(|x-z|+|y-z|)\,v(z)\,\mathrm{d}z.$$
Since $x > 0$, $y > 0$ and $z < 0$, this reduces to
$$w_1(x,y) = -\tfrac{1}{8}v_0^2-\tfrac{1}{4}(x+y)\,v_0+\tfrac{1}{2}v_1,$$
whence the first moment of v can be obtained from measurements of w_0 and w_1. This procedure can be repeated: given w_n for $n = 0, 1, \ldots, N$, this information determines v_n for $n = 0, 1, \ldots, N$. (Note that it is only necessary to know $w_n(x, y)$

for one value of x and one value of y; we can take $x = y = 0$, giving a closer analogue to the problems treated previously. In fact, we are unable to use the additional information obtained by allowing x and y to vary.)

The problem of determining a function v from all of its moments $\{v_n\}$ $(n = 0, 1, \ldots)$ is classical (see Appendix A). This problem is uniquely solvable, since v has compact support.

Practically, it is probably difficult to measure w_n for $n > 1$. Therefore, we conclude by outlining two other ways for treating our one-dimensional inverse problem.

First, we can make the Born approximation (i.e. $w = h$) in (5.4) to give

$$\int v(z)\,\mathrm{e}^{-2\mathrm{i}kz}\,\mathrm{d}z = -4\,\mathrm{e}^{-2\mathrm{i}kx}\,w(x, x) \tag{5.5}$$

for $x = y$. Thus, if we measure w at one point x (when the source is at the same point) and for all wavenumbers $k > 0$, then (5.5) is an integral equation for v, which can be solved by Fourier inversion. If w is only known for a finite range, $k_0 \leqslant k \leqslant k_1$, then (5.5) can be solved by using the method given in §4.2.

Secondly, if the incident field is a plane wave, and the reflection coefficient is measured for all values of $k > 0$, then the one-dimensional problem can be reduced by the Liouville transform to the quantum-mechanical problem of inverse scattering by a potential. The theory for this inverse problem is well developed; see, for example, Chadan & Sabatier (1977).

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Appendix A. Numerical inversion of Laplace transforms

Let

$$F(p) = \int_0^\infty f(t)\, \mathrm{e}^{-pt}\, \mathrm{d}t \tag{A 1}$$

denote the Laplace transform of a function $f(t)$, where p is a complex variable. Suppose that $F(p)$ is analytic in the half-plane $\mathrm{Re}(p) > c_0$. Then (A 1) can be inverted to give

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-i\infty}^{c+i\infty} F(p)\, \mathrm{e}^{pt}\, \mathrm{d}p,$$

where $t > 0$ and $c > c_0$.

There are many methods for obtaining $f(t)$ from $F(p)$ numerically (see the books by Bellman *et al.* (1966), Krylov & Skoblya (1969), and the review by Davies & Martin (1979)). Here, we shall concentrate on methods that only use $F(p)$ for real values of p.

Let

$$f_n(t) = \int_0^\infty \delta_n(t, u) f(u)\, \mathrm{d}u,$$

where δ_n is a delta-sequence (i.e. $\|f - f_n\| \to 0$ as $n \to \infty$). Choosing the functions

$$\delta_n(t, u) = \left(\frac{nu}{t}\right)^n \exp\left(-\frac{nu}{t}\right)$$

gives

$$f(t) \simeq f_n(t) = \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} F^{(n)}\left(\frac{n}{t}\right),$$

where $F^{(n)}$ is the nth derivative of F. This formula (due to D. V. Widder; refer to Davies & Martin (1979) for complete references) is probably not useful in practice (unless F is a rational function) because it requires derivatives of F. However, there are variants that only use evaluations of F, and one of these (due to D. P. Gaver and H. Stehfest) is recommended by Davies & Martin (1979); see their paper for further details.

Davies & Martin (1979) also recommend a method (due to R. Piessens) in which $F(p)$ is approximated by a series of Chebyshev polynomials as

$$F(p) \simeq p^{-\alpha-1} \sum_{n=0}^{N} a_n T_n\left(1-\frac{\beta}{p}\right),$$

where α and β are free parameters. Term-by-term inversion of this series gives an approximation to f.

Several other methods are described by Davies & Martin (1979), for example one can approximate $f(t)$ by a series of Laguerre polynomials. We conclude by mentioning two further methods.

First, one can consider (A 1) as an integral equation of the first kind for $f(t)$ (take p real and positive). This equation could be solved, numerically, by using a regularization technique.

Second, write $x = \mathrm{e}^{-t}$ in (A 1) to give

$$\int_0^1 x^{p-1} \phi(x)\,\mathrm{d}x = F(p), \tag{A 2}$$

where $\phi(x) = f(-\ln x)$. By choosing $p = 1, 2, \ldots$, (A 2) becomes

$$\int_0^1 x^m \phi(x)\,\mathrm{d}x = F_m \quad (m = 0, 1, \ldots), \tag{A 3}$$

where $F_m = F(m+1)$. The problem of finding $\phi(x)$ from (A 3) is called the *classical moment problem*. This problem is discussed in books by Shohat & Tamarkin (1943) and Akhiezer (1965). Numerical methods for its solution have also been devised (see, for example, Wimp (1979) and Greaves (1982)).

Appendix B. A Fourier transform

In §4, we used the result

$$I(x,y) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda s} \ln\{(s-x)^2+y^2\}^{\frac{1}{2}}\,\mathrm{d}s = -\frac{\pi}{|\lambda|}\exp(\mathrm{i}\lambda x - |\lambda||y|).$$

Here, we prove this result, assuming for simplicity that $\lambda \geqslant 0$ and $y \geqslant 0$.

We have $I(x,y) = \mathrm{e}^{\mathrm{i}\lambda x} J(y)$, where

$$J(y) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x} \ln(x^2+y^2)^{\frac{1}{2}}\,\mathrm{d}x.$$

In particular

$$J(0) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x} \ln|x|\,\mathrm{d}x = \frac{\mathrm{i}}{\lambda}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x}\frac{\mathrm{d}x}{x} = -\frac{\pi}{\lambda}.$$

Since $I(x, y)$ is harmonic, and we know $I(x, 0)$, Poisson's formula can be used to obtain the result. Alternatively, note that

$$J'(y) = y\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{x^2+y^2}\,\mathrm{d}x = \pi\,\mathrm{e}^{-\lambda y},$$

whence an integration, with use of $J(0)$, gives the result.