ORTHOGONAL POLYNOMIAL SOLUTIONS FOR PRESSURIZED ELLIPTICAL CRACKS

By P. A. MARTIN

(*Department of Mathematics, University of Manchester, Manchester* M13 9PL)

[Received 15 July 1985]

SUMMARY

Consider an infinite elastic solid containing a flat elliptical crack, which is opened symmetrically by a prescribed pressure p. We expand p and the crack-face displacement w as Fourier series in ϕ, and expand each Fourier component as a series of orthogonal polynomials in ρ, where (in Cartesian coordinates) the crack occupies the surface $\{(x, y, z): x = a\rho\cos\phi,\ \ y = b\rho\sin\phi,\ \ z = 0,\ \ 0 \leqslant \rho < 1,\ 0 \leqslant \phi < 2\pi\}$. We obtain explicit relations between the coefficients in the series for w and p, and derive a formula for the stress-intensity factor. As an example, we consider the quadratic pressure $p(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$ in detail, and compare our solution with those of other authors.

1. Introduction

It is now 40 years since Sneddon (**1**) and Sack (**2**) first studied the static loading of a penny-shaped crack in an otherwise unbounded homogeneous isotropic elastic solid. Both authors considered the crack to be opened symmetrically by a prescribed pressure p. It is instructive to look at the methods used by these authors. Let x, y, z be Cartesian coordinates, so that the crack occupies the region

$$\Omega = \{(x, y, z): 0 \leqslant r < 1,\ 0 \leqslant \theta < 2\pi,\ z = 0\},$$

where

$$x = ar\cos\theta, \qquad y = ar\sin\theta,$$

and a is the radius of the crack. As the problem is symmetric about $z = 0$, Sneddon (**1**) considered an equivalent half-space problem. Taking p to be axisymmetric, and using an integral representation for the displacement in $z \geqslant 0$ (involving Hankel transforms), he obtained a pair of dual integral equations, which he could solve. Sack's approach (**2**) was more direct: he reduced the problem to one in potential theory, which he solved using the method of separation of variables in oblate spheroidal coordinates, when p was constant over Ω.

Since the pioneering work of Sneddon and Sack on the simplest problem in three-dimensional fracture mechanics, there has been much written on

the static loading of a penny-shaped crack. Sneddon's method has been extended to arbitrary loadings by Kassir and Sih (**3**, Chapter 1) and Bell (**4**) (see also (**5**)); an alternative method, based on the Somigliana formula, has been used by Guidera and Lardner (**6**) (see also (**7**) and §3.2 below). Polynomial loadings have also been considered. Thus, Payne (**8**), England and Shail (**9**), and Shail (**10**) all, like Sack (**2**), used oblate spheroidal coordinates, and obtained polynomial solutions for polynomial loads. Similar solutions have been obtained using different methods by Gladwell and England (**11**) and Krenk (**12**). For example, Krenk showed that if a penny-shaped crack is inflated symmetrically by the pressure

$$p(r, \theta) = \mu \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} S_j^n \frac{\Gamma(n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j)!} \frac{C_{2j+1}^{n+\frac{1}{2}}((1-r^2)^{\frac{1}{2}})}{(1-r^2)^{\frac{1}{2}}} r^n \cos n\theta, \quad (1.1)$$

then the normal displacement of the upper crack face, say, will be given by

$$u_z(r, \theta, 0) = \tfrac{1}{2}a \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} W_j^n \frac{\Gamma(n+\frac{1}{2})j!}{\Gamma(n+j+\frac{3}{2})} C_{2j+1}^{n+\frac{1}{2}}((1-r^2)^{\frac{1}{2}}) r^n \cos n\theta, \quad (1.2)$$

where

$$W_j^n = -(1-\nu)S_j^n, \quad (1.3)$$

μ is the shear modulus and ν is Poisson's ratio. Here, $C_m^{\lambda}(x)$ is a Gegenbauer polynomial of degree m with index λ (**13**, §10.9); these polynomials are orthogonal and satisfy

$$\int_0^1 \frac{r^{2m+1}}{(1-r^2)^{\frac{1}{2}}} C_{2j+1}^{m+\frac{1}{2}}((1-r^2)^{\frac{1}{2}}) C_{2k+1}^{m+\frac{1}{2}}((1-r^2)^{\frac{1}{2}})\, dr = h_j^m \delta_{jk}, \quad (1.4)$$

where δ_{ij} is the Kronecker delta and h_j^m is a known constant. (An alternative derivation of (1.3) is given in (**7**).) The authors of (**8** to **11**) used associated Legendre functions $P_n^m(x)$, whilst Bell (**4**) used Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ to expand the Fourier components of $p(r, \theta)$. All these solutions are equivalent, since

$$r^n C_{2m+1}^{n+\frac{1}{2}}((1-r^2)^{\frac{1}{2}}), \quad P_{2m+n+1}^n((1-r^2)^{\frac{1}{2}}), \quad r^n(1-r^2)^{\frac{1}{2}} P_m^{(n,\frac{1}{2})}(1-2r^2)$$

are all proportional to one another. However, we prefer to use Gegenbauer polynomials, so that we can compare with Krenk's result (1.3).

1.1. *The pressurized elliptical crack*

Some of the methods used to solve the problem of a pressurized penny-shaped crack have been adapted to the corresponding problem for a flat elliptical crack. Let

$$\Omega = \{(x, y, z) : 0 \leq \rho < 1, 0 \leq \phi < 2\pi, z = 0\}, \quad (1.5)$$

where

$$x = a\rho \cos \phi, \qquad y = b\rho \sin \phi, \tag{1.6}$$

and $0 < b \leqslant a$; Ω is an elliptical region in the plane $z = 0$. Green and Sneddon (**14**) reduced the problem to the determination of a single harmonic function, $V(x, y, z)$, say (see §3.1 below). In the special case of a crack inflated by a constant pressure p, this potential problem had a known solution. This solution represents the gravitational potential of a uniform elliptical plate, and is proportional to $V^{(1)}$, where

$$V^{(\alpha)}(x, y, z) = \int_{\lambda}^{\infty} \frac{\{\omega(s)\}^{\alpha}\, ds}{\{Q(s)\}^{\frac{1}{2}}}, \tag{1.7}$$

$$\omega(s) = 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s}, \qquad Q(s) = s(s+a^2)(s+b^2),$$

λ is an ellipsoidal coordinate, defined as the positive root of $\omega(s) = 0$, and $\alpha > -\frac{1}{2}$ is a real number. The harmonic functions $V^{(\alpha)}$ were introduced by Kassir and Sih (**15**) and Segedin (**16**) in order to treat polynomial loads; Segedin (**16**) also suggested using partial derivatives of $V^{(\alpha)}$ with respect to x and y. This method has been developed by Shah and Kobayashi (**17**), Kassir and Sih (**3**, Chapter 3), Vijayakumar and Atluri (**18**) and Nishioka and Atluri (**19**); for example, if the prescribed pressure has the form (**18**)

$$p(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{m} A^n_{m-n} x^{2m-2n} y^{2n}, \tag{1.8}$$

then take

$$V(x, y, z) = \sum_{m=0}^{M} \sum_{n=0}^{m} C^n_{m-n} \frac{\partial^{2m} V^{(2m+1)}}{\partial x^{2m-2n}\, \partial y^{2n}}, \tag{1.9}$$

and determine the coefficients C^n_m by imposing the boundary condition on Ω, (3.4). This method is complicated (because of the difficulty in expanding the right-hand side of (1.9) as a polynomial), but tractable (**18**, **19**).

An alternative method for finding V has been given by Shail (**20**). He uses the method of separation of variables in ellipsoidal coordinates, leading to solutions as products of Lamé functions. This method is elegant, and is the natural generalization of the corresponding approach for the penny-shaped crack (**2**, **8** to **12**). Its main drawback lies in the Lamé functions themselves, since these functions are not easily computed and their properties are not all well understood.

Shibuya (**21**) has used a method involving dual integral equations and a conformal mapping between Ω and the unit circle. Later, Sneddon (**22**) showed that Shibuya's method is essentially based on certain properties of two-dimensional Fourier transforms (see §2 below), although he only

considered the case of constant p. One purpose of the present paper is to extend this approach to general polynomial pressures.

Somigliana's formula has been used by Eshelby (**23**), Willis (**24**), Walpole (**25**) and Gladwell (**26**). Eshelby (**23**) gave the first solution for a uniform shear stress on Ω, whilst Willis (**24**) has presented a general method which is applicable to anisotropic media.

One general result which features in much of the work on elliptical cracks is an analogue of Galin's theorem, which we state here in a form given by Shail (**20**).

THEOREM. *Suppose that an elliptical crack is inflated by equal and opposite pressures $p(x, y)$, where*

$$p(x, y) = P_1(x^2, y^2) + xP_2(x^2, y^2) + yP_3(x^2, y^2) + xyP_4(x^2, y^2), \quad (1.10)$$

and P_i $(i = 1, 2, 3, 4)$ are polynomials in x^2 and y^2. Then, the normal displacements of the crack faces are $\pm w(x, y)$, where

$$w(x, y) = (1 - \rho^2)^{\frac{1}{2}}\{Q_1(x^2, y^2) + xQ_2(x^2, y^2) + \\ + yQ_3(x^2, y^2) + xyQ_4(x^2, y^2)\}, \quad (1.11)$$

$$\rho = (x^2/a^2 + y^2/b^2)^{\frac{1}{2}} \quad (1.12)$$

and Q_i is a polynomial in x^2 and y^2 of the same degree as P_i.

Concise proofs of this theorem have been given by Willis (**24**), Walpole (**25**), Shail (**20**) and Gladwell (**26**); indeed, the theorem is true when the elastic solid is arbitrarily anisotropic (**24**).

In the present paper, we combine Sneddon's approach (**22**) with Krenk's polynomial expansions (**12**). We obtain results which generalize (1.3) to elliptical cracks; these are also implied by some results of Gladwell (**26**), who used a different method. Rather than the simple uncoupled relations (1.3), we now have certain systems of simultaneous linear algebraic equations. In section 6, we show how these systems can be properly truncated. In section 5, we derive a simple formula for the stress-intensity factor $k_1(\phi)$. Finally, in section 7, we solve five particular problems. The first four are simple, and correspond to $P_i = \delta_{ij}$ $(j = 1, 2, 3, 4)$ in (1.10). For the fifth problem, we take $p(x, y) = Ax^2 + By^2$; this problem is non-trivial and provides a better illustration of the method.

Computationally, the method presented here appears to be more attractive than Atluri's method (**18, 19**). Both yield similar systems of linear equations for the pressurized crack, although here (i) the matrix elements are much simpler, (ii) all systems are completely uncoupled from one another, and (iii) the load coefficients S_j^n are easily computed (using (1.4)). However, Atluri *et al.* can also treat polynomial shear loadings of the crack; such loadings have yet to be treated by the present method.

2. Two-dimensional Fourier transforms

Define the two-dimensional Fourier transform of $f(\mathbf{x})$ by (**27**, §7.1)

$$F(\boldsymbol{\xi}) = \mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = \frac{1}{2\pi} \int_{\mathbf{R}^2} f(\mathbf{x}) \exp\{i\boldsymbol{\xi} \cdot \mathbf{x}\}\, d\mathbf{x}; \tag{2.1}$$

its inverse is

$$f(\mathbf{x}) = \mathscr{F}_2^{-1}[F(\boldsymbol{\xi}); \mathbf{x}] = \frac{1}{2\pi} \int_{\mathbf{R}^2} F(\boldsymbol{\xi}) \exp\{-i\boldsymbol{\xi} \cdot \mathbf{x})\, d\boldsymbol{\xi}. \tag{2.2}$$

Here, $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $\boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2$ and $\boldsymbol{\xi} \cdot \mathbf{x} = \xi x + \eta y$. Make the substitutions

$$x = a\rho \cos\phi, \quad y = b\rho \sin\phi, \quad \xi = (\lambda/a) \cos\psi, \quad \eta = (\lambda/b) \sin\psi \tag{2.3}$$

in (2.1) and (2.2) to give

$$\mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = \frac{ab}{2\pi} \int_0^\infty \int_0^{2\pi} f(\mathbf{x}) \exp\{i\lambda\rho \cos(\phi - \psi)\}\, \rho\, d\phi\, d\rho \tag{2.4}$$

and

$$\mathscr{F}_2^{-1}[F(\boldsymbol{\xi}); \mathbf{x}] = \frac{1}{2\pi ab} \int_0^\infty \int_0^{2\pi} F(\boldsymbol{\xi}) \exp\{-i\lambda\rho \cos(\phi - \psi)\} \lambda\, d\psi\, d\lambda. \tag{2.5}$$

Suppose that $f(\mathbf{x})$ has the Fourier expansion

$$f(\mathbf{x}) = \sum_{m=0}^{\infty} f_m(\rho) \cos m\phi + \sum_{m=1}^{\infty} \bar{f}_m(\rho) \sin m\phi. \tag{2.6}$$

Since (**13**, Equation 7.2.4 (27))

$$\exp\{\pm ir \cos\theta\} = \sum_{n=0}^{\infty} \varepsilon_n (\pm i)^n J_n(r) \cos n\theta, \tag{2.7}$$

where

$$\varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0, \end{cases}$$

we can integrate over ϕ in (2.4) to give

$$\mathscr{F}_2[f(\mathbf{x}); \boldsymbol{\xi}] = ab \sum_{m=0}^{\infty} i^m \cos m\psi \mathscr{H}_m[f_m(\rho); \lambda] + \\ + ab \sum_{m=1}^{\infty} i^m \sin m\psi \mathscr{H}_m[\bar{f}_m(\rho); \lambda], \tag{2.8}$$

where

$$\mathscr{H}_\nu[f(\rho); \lambda] = \int_0^\infty f(\rho) J_\nu(\lambda\rho) \rho\, d\rho. \tag{2.9}$$

Sneddon (**22**) has used the 'axisymmetric' form of (2.8), in which $f(\mathbf{x}) = f_0(\rho)$; when $a = b$, $f(\mathbf{x}) = f_0(r)$ and we recover the well-known relation (**27**, §11)

$$\mathscr{F}_2[f_0(r); \xi] = a^2 \mathscr{H}_0[f_0(r); \lambda].$$

3. The pressurized elliptical crack

3.1. *Solution using one harmonic function*

As the problem is symmetric about the plane $z = 0$, it is equivalent to determining the displacement $\mathbf{u} = (u_x, u_y, u_z)$ and the corresponding stresses τ_{ij} in the half-space $z > 0$, when the boundary conditions are

$$\tau_{zz}(x, y, 0) = p(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{3.1}$$

$$u_z(x, y, 0) = 0, \qquad \mathbf{x} \in \mathbb{R}^2 \backslash \Omega, \tag{3.2}$$

$$\tau_{xz}(x, y, 0) = \tau_{yz}(x, y, 0) = 0, \qquad \mathbf{x} \in \mathbb{R}^2. \tag{3.3}$$

Because of (3.3), this problem can be reduced to the determination of a single harmonic function, $V(x, y, z)$ (**28**, §5.7), where

$$2\mu u_x = z \frac{\partial^2 V}{\partial x\, \partial z} + (1 - 2\nu) \frac{\partial V}{\partial x}, \quad 2\mu u_y = z \frac{\partial^2 V}{\partial y\, \partial z} + (1 - 2\nu) \frac{\partial V}{\partial y},$$

$$2\mu u_z = z \frac{\partial^2 V}{\partial z^2} - 2(1 - \nu) \frac{\partial V}{\partial z}.$$

This representation satisfies (3.3). Also,

$$\tau_{zz}(x, y, 0) = -\partial^2 V / \partial z^2 \tag{3.4}$$

and

$$w(\mathbf{x}) \equiv u_z(x, y, 0) = -\frac{(1 - \nu)}{\mu} \frac{\partial V}{\partial z}. \tag{3.5}$$

Suppose that, for $\mathbf{x} \in \mathbb{R}^2$,

$$w(\mathbf{x}) = \tfrac{1}{2} a \sum_{m=0}^{\infty} w_m(\rho) \cos m\phi + \tfrac{1}{2} a \sum_{m=1}^{\infty} \tilde{w}_m(\rho) \sin m\phi. \tag{3.6}$$

Following Sneddon (**22**), take the harmonic function V to be

$$V(x, y, z) = \frac{\mu}{1 - \nu} \mathscr{F}_2^{-1}[|\xi|^{-1} U(\xi) \exp(-|\xi|\, z); \mathbf{x}], \tag{3.7}$$

where

$$U(\xi) = \tfrac{1}{2} a^2 b \sum_{m=0}^{\infty} i^m \mathscr{H}_m[w_m(\rho); \lambda] \cos m\psi + \\ + \tfrac{1}{2} a^2 b \sum_{m=1}^{\infty} i^m \mathscr{H}_m[\tilde{w}_m(\rho); \lambda] \sin m\psi, \tag{3.8}$$

$|\boldsymbol{\xi}| = (\xi^2 + \eta^2)^{\frac{1}{2}}$ and $\lambda = (a^2\xi^2 + b^2\eta^2)^{\frac{1}{2}}$. Then (2.8) shows that (3.6) is consistent with (3.5); (3.2) will be satisfied if $w_m(\rho)$ and $\tilde{w}_m(\rho)$ vanish for $\rho \geqslant 1$. Equation (3.1) will be satisfied if w_m and $\tilde{w}_m$ are also chosen to satisfy

$$p(\mathbf{x}) = -\frac{\mu}{1-\nu}\,\mathscr{F}_2^{-1}[|\boldsymbol{\xi}|\,U(\boldsymbol{\xi});\mathbf{x}]. \tag{3.9}$$

Noting that $|\boldsymbol{\xi}| = (\lambda/b)(1 - k^2\cos^2\psi)^{\frac{1}{2}}$, where $k^2 = 1 - (b/a)^2$, and using (2.7) in (2.5), we can reduce (3.9) to

$$p(\mathbf{x}) = \mu \sum_{n=0}^{\infty} \tau_n(\rho)\cos n\phi + \mu \sum_{n=1}^{\infty} \tilde{\tau}_n(\rho)\sin n\phi, \tag{3.10}$$

where

$$\pi(1-\nu)k'\tau_n(\rho) = -\tfrac{1}{2}\varepsilon_n \sum_{m=0}^{\infty} I^c_{mn}(k)\mathscr{H}_n[\lambda\mathscr{H}_m\{w_m(\rho);\lambda\};\rho], \tag{3.11}$$

$$\pi(1-\nu)k'\tilde{\tau}_n(\rho) = -\sum_{m=1}^{\infty} I^s_{mn}(k)\mathscr{H}_n[\lambda\mathscr{H}_m\{\tilde{w}_m(\rho);\lambda\};\rho], \tag{3.12}$$

$$I^c_{mn}(k) = \tfrac{1}{2}i^m(-i)^n \int_0^{2\pi} (1 - k^2\cos^2\psi)^{\frac{1}{2}} \cos m\psi \cos n\psi \,\mathrm{d}\psi, \tag{3.13}$$

$$I^s_{mn}(k) = \tfrac{1}{2}i^m(-i)^n \int_0^{2\pi} (1 - k^2\cos^2\psi)^{\frac{1}{2}} \sin m\psi \sin n\psi \,d\psi, \tag{3.14}$$

$k' = b/a$ and we have noticed that if, in (3.13), we replace $\cos n\psi$ by $\sin n\psi$, then the corresponding integral is zero.

Splitting the range of integration in (3.13), we obtain

$$I^c_{2m,2n} = 2\int_0^{\frac{1}{2}\pi} \Delta \cos 2mx \cos 2nx \,dx \tag{3.15}$$

and

$$I^c_{2m+1,2n+1} = 2\int_0^{\frac{1}{2}\pi} \Delta \sin(2m+1)x \sin(2n+1)x \,dx, \tag{3.16}$$

where $\Delta(x) = (1 - k^2\sin^2 x)^{\frac{1}{2}}$; also $I^c_{2m,2n+1} = 0$. Similarly,

$$I^s_{2m,2n} = 2\int_0^{\frac{1}{2}\pi} \Delta \sin 2mx \sin 2nx \,dx, \tag{3.17}$$

$$I^s_{2m+1,2n+1} = 2\int_0^{\frac{1}{2}\pi} \Delta \cos(2m+1)\cos(2n+1)x \,dx \tag{3.18}$$

and $I^s_{2m,2n+1}=0$. Thus, (3.11) and (3.12) separate as follows:

$$\pi(1-\nu)k'\tau_{2n}(\rho) = -\tfrac{1}{2}\varepsilon_{2n}\sum_{m=0}^{\infty} I^c_{2m,2n}\mathscr{H}_{2n}[\lambda\mathscr{H}_{2m}\{w_{2m}(\rho);\lambda\};\rho], \quad (3.19a)$$

$$\pi(1-\nu)k'\tau_{2n+1}(\rho) = -\sum_{m=0}^{\infty} I^c_{2m+1,2n+1}\mathscr{H}_{2n+1}\times \times[\lambda\mathscr{H}_{2m+1}\{w_{2m+1}(\rho);\lambda\};\rho], \quad (3.19b)$$

$$\pi(1-\nu)k'\bar{\tau}_{2n}(\rho) = -\sum_{m=1}^{\infty} I^s_{2m,2n}\mathscr{H}_{2n}[\lambda\mathscr{H}_{2m}\{\bar{w}_{2m}(\rho);\lambda\};\rho], \quad (3.20a)$$

$$\pi(1-\nu)k'\bar{\tau}_{2n+1}(\rho) = -\sum_{m=0}^{\infty} I^s_{2m+1,2n+1}\mathscr{H}_{2n+1}\times \times[\lambda\mathscr{H}_{2m+1}\{\bar{w}_{2m+1}(\rho);\lambda\};\rho]. \quad (3.20b)$$

This separation corresponds to the four-term splitting of $p(\mathbf{x})$ in (1.10).

The integrals I^c_{mn} and I^s_{mn} can be simplified; we have

$$I^c_{2m,2n} \pm I^s_{2m,2n} = 2\int_0^{\frac{1}{2}\pi} \Delta\cos 2(m\mp n)x\,dx$$

and

$$I^s_{2m+1,2n+1} \pm I^c_{2m+1,2n+1} = 2\int_0^{\frac{1}{2}\pi} \Delta\cos\{2m+1\mp(2n+1)\}x\,dx,$$

whence the basic integrals are seen to be

$$E_m(k) = \int_0^{\frac{1}{2}\pi} \Delta\cos 2mx\,dx, \qquad m=0,1,2,\ldots. \quad (3.21)$$

These integrals are easily computed; we have

$$E_0 = E(k), \qquad 3k^2E_1 = (1+k'^2)E - 2k'^2K$$

and

$$(2m+3)k^2E_{m+1} = -4m(1+k'^2)E_m - (2m-3)k^2E_{m-1}$$

for $m\geqslant 1$, where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively. Alternatively, $E_m(k)$ has a power-series expansion (**29**, Equation 806.03)

$$E_m = \frac{(-1)^{m+1}}{4}\sum_{j=m}^{\infty}\frac{\Gamma(j+\frac{1}{2})\Gamma(j-\frac{1}{2})}{(j+m)!(j-m)!}k^{2j}.$$

As a preliminary check on our analysis, we can let $b\to a$ $(k\to 0)$, corresponding to a penny-shaped crack. We have

$$\varepsilon_n I^c_{mn}(0) = \pi\delta_{mn}, \quad (3.22)$$

whence (3.11) simplifies to

$$(1-\nu)\tau_n(r) = -\tfrac{1}{2}\mathcal{H}_n[\lambda\mathcal{H}_n\{w_n(r);\lambda\};r], \tag{3.23}$$

which is precisely the integral equation solved by Krenk (**12**, Equation (25)), using expansions of τ_n and w_n in series of Gegenbauer polynomials. We shall adopt a similar approach here for the elliptical crack.

3.2. *Solution using the Somigliana formula*

An alternative approach is based on the Somigliana formula; this has been used by, for example, Willis (**24**), Walpole (**25**) and Guidera and Lardner (**6**). We have (**6**, Equation (2.4))

$$\tau_{zz}(x',y',z') = \frac{-\mu}{4\pi(1-\nu)}\int_\Omega \left\{\frac{\partial}{\partial x}[u_z(\mathbf{x})]\frac{\partial}{\partial x}+\frac{\partial}{\partial y}[u_z(\mathbf{x})]\frac{\partial}{\partial y}\right\}\times \\ \times\left(\frac{1}{R}+\frac{z'^2}{R^3}\right)d\mathbf{x}, \tag{3.24}$$

where Ω is defined by (1.5), $R^2=(x-x')^2+(y-y')^2+z'^2$ and

$$[u_z(\mathbf{x})] = u_z(x,y,0+)-u_z(x,y,0-) = 2w(\mathbf{x}) \tag{3.25}$$

is the discontinuity in u_z across the crack. Following Willis (**24**), we introduce the two-dimensional Fourier transform of R^{-1},

$$R^{-1} = \mathcal{F}_2^{-1}[|\xi|^{-1}\exp\{i\xi\,.\,\mathbf{x}-|\xi|\,z'\};\mathbf{x}'] \tag{3.26}$$

where $\mathbf{x}'=(x',y')$. Substitute (3.26) into (3.24), integrate by parts (noting that $[u_z]=0$ around the crack edge), let $z'\to 0$ and use the boundary condition (3.1) to give

$$p(\mathbf{x}) = -\frac{\mu}{1-\nu}\mathcal{F}_2^{-1}[|\xi|\,U(\xi);\mathbf{x}], \tag{3.27}$$

where

$$U(\xi) = \frac{1}{4\pi}\int_\Omega [u_z(\mathbf{x})]\exp\{i\xi\,.\,\mathbf{x}\}\,d\mathbf{x} = \mathcal{F}_2[w(\mathbf{x});\xi];$$

here, we have noted that $[u_z(\mathbf{x})]=0$ for $\mathbf{x}\in\mathbb{R}^2\backslash\Omega$ and used (3.25). Using (3.6) and (2.8), we see that U is given by (3.8) and hence (3.27) is identical to (3.9).

4. Polynomial solutions

Following Krenk (**12**), take

$$w_m(\rho) = H(1-\rho)\rho^m\sum_{j=0}^{\infty} W_j^m\frac{\Gamma(m+\frac{1}{2})j!}{\Gamma(m+j+\frac{3}{2})}C_{2j+1}^{m+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}), \tag{4.1}$$

where $H(t)$ is the Heaviside unit function and W_j^m are unknown coefficients. We have (**12, 26**)

$$\int_0^1 J_\nu(\xi t)t^{\nu+1}C_{2k+1}^{\nu+\frac{1}{2}}((1-t^2)^{\frac{1}{2}})\,dt = \frac{\Gamma(\nu+k+\frac{3}{2})}{\Gamma(\nu+\frac{1}{2})k!}\frac{2}{\xi}j_{2k+\nu+1}(\xi), \tag{4.2}$$

where $j_\nu(z) = (\frac{1}{2}\pi/z)^{\frac{1}{2}}J_{\nu+\frac{1}{2}}(z)$ is a spherical Bessel function. Hence

$$\mathscr{H}_m[w_m(\rho);\lambda] = \frac{2}{\lambda}\sum_{j=0}^{\infty} W_j^m j_{2j+m+1}(\lambda) \tag{4.3}$$

and

$$\mathscr{H}_n[\lambda\mathscr{H}_m\{w_m(\rho);\lambda\};\rho] = 2\sum_{j=0}^{\infty} W_j^m L_{m,n}^{2j}(\rho), \tag{4.4}$$

where

$$L_{m,n}^{2j}(\rho) = \int_0^\infty \lambda J_n(\lambda\rho)j_{2j+m+1}(\lambda)\,d\lambda. \tag{4.5}$$

Consider $L_{\mu,\nu}^{2j}(\rho)$, where μ and ν are integers. From (3.19), we see that we only need to consider those cases where μ and ν are either both even or both odd; thus, we can define integers p and q by

$$\mu+\nu = 2p, \qquad \mu-\nu = 2q. \tag{4.6}$$

Now, (4.5) is a Weber–Schafheitlin integral (**13**, §7.7.4). For $0 \leq \rho < 1$, we have

$$L_{\mu,\nu}^{2j}(\rho) = \frac{\pi^{\frac{1}{2}}\rho^\nu\Gamma(j+p+\frac{3}{2})}{\nu!\,\Gamma(j+q+1)}F(j+p+\tfrac{3}{2}, -j-q; \nu+1; \rho^2), \tag{4.7}$$

where F is the Gauss hypergeometric function. Note that

$$L_{\mu,\nu}^{2j} = 0 \quad \text{whenever} \quad j+q = -1, -2, -3, \ldots, \tag{4.8}$$

whereas if $j+q = 0, 1, 2, \ldots$, the series terminates; in fact, in this case, we have

$$L_{\mu,\nu}^{2j}(\rho) = \rho^\nu\frac{\Gamma(\nu+\frac{1}{2})\Gamma(j+q+\frac{3}{2})}{(j+p)!\,(1-\rho^2)^{\frac{1}{2}}}\,C_{2j+2q+1}^{\nu+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}), \tag{4.9}$$

where we have used (**13**, Equation 10.9(22)) and the duplication formula for $\Gamma(2z)$. Using (4.9) in (4.4), (3.19a) becomes

$$\tau_{2n}(\rho) = \frac{-\varepsilon_{2n}\rho^{2n}\Gamma(2n+\frac{1}{2})}{\pi(1-\nu)k'}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} I_{2m,2n}^c \times$$
$$\times \frac{\Gamma(m-n+j+\frac{3}{2})}{(m+n+j)!}\,\frac{C_{2j+2m-2n+1}^{2n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}})}{(1-\rho^2)^{\frac{1}{2}}}, \tag{4.10}$$

where the summation is only over those values of j and m that satisfy $j+m \geq n$. To make this explicit, set $l=j+m$, whence

$$S \equiv \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} a_{mj} = \sum_{m=0}^{\infty}\sum_{l=m}^{\infty} a_{m,l-m} = \sum_{l=0}^{\infty}\sum_{m=0}^{l} a_{m,l-m}.$$

Using the restriction on l, we obtain

$$S = \sum_{l=n}^{\infty}\sum_{m=0}^{l} a_{m,l-m} = \sum_{j=0}^{\infty}\sum_{m=0}^{n+j} a_{m,n+j-m}.$$

Thus, (4.10) becomes

$$\tau_{2n}(\rho) = \frac{-\varepsilon_{2n}\rho^{2n}\Gamma(2n+\frac{1}{2})}{\pi(1-\nu)k'}\sum_{j=0}^{\infty}\frac{\Gamma(j+\frac{3}{2})}{(2n+j)!}\times$$
$$\times\frac{C_{2j+1}^{2n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}})}{(1-\rho^2)^{\frac{1}{2}}}\sum_{m=0}^{n+j} I_{2m,2n}^{c} W_{n+j-m}^{2m}. \quad (4.11)$$

So if, following Krenk, we suppose that $\tau_n(\rho)$ has the expansion

$$\tau_n(\rho) = \rho^n \sum_{j=0}^{\infty} S_j^n \frac{\Gamma(n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(n+j)!}\frac{C_{2j+1}^{n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}})}{(1-\rho^2)^{\frac{1}{2}}}, \quad (4.12)$$

where the coefficients S_j^n are assumed to be known, then, since the Gegenbauer polynomials are orthogonal (see (1.4)), (4.11) and (4.12) give

$$S_j^{2n} = \frac{-\varepsilon_{2n}}{\pi(1-\nu)k'}\sum_{m=0}^{n+j} I_{2m,2n}^{c}(k) W_{n+j-m}^{2m} \quad (4.13)$$

for $n=0, 1, \ldots$ and $j=0, 1, \ldots$. Similarly, (3.19b) gives

$$S_j^{2n+1} = \frac{-2}{\pi(1-\nu)k'}\sum_{m=0}^{n+j} I_{2m+1,2n+1}^{c} W_{n+j-m}^{2m+1}. \quad (4.14)$$

For a penny-shaped crack we have $k=0$; if we use (3.22) in (4.13) and (4.14), we recover Krenk's relation, (1.3). Thus, for this particular geometry, we have a simple relation between W_j^n and S_j^n.

If $k \neq 0$, then (4.13), for example, is an infinite system of linear algebraic equations for W_l^{2m} in terms of S_j^{2n}. In section 6, we give a proper truncation of (4.13); we shall use this to solve some particular problems.

The corresponding results for (3.20) are as follows:

$$\tilde{S}_j^{2n} = \frac{-2}{\pi(1-\nu)k'}\sum_{m=1}^{n+j} I_{2m,2n}^{s}\tilde{W}_{n+j-m}^{2m}, \quad (4.15)$$

$$\tilde{S}_j^{2n+1} = \frac{-2}{\pi(1-\nu)k'}\sum_{m=0}^{n+j} I_{2m+1,2n+1}^{s}\tilde{W}_{n+j-m}^{2m+1}, \quad (4.16)$$

where $\tilde{w}_m$ is given by (4.1) with W_j^n replaced by $\tilde{W}_j^m$, and $\tilde{\tau}_n$ is given by (4.12) with S_j^n replaced by $\tilde{S}_j^n$.

The formulae (4.13 to 4.16) can be used to solve the converse problem, in which it is required to determine the pressure distribution needed to maintain a prescribed crack-face displacement. We remark that Kassir and Sih (**15**) treated this problem by writing

$$w_0(\rho) = (1-\rho^2)^{\frac{1}{2}} \sum_{j=0}^{N} c_j (1-\rho^2)^j.$$

5. The stress-intensity factor

Consider two points P and P_1, where P is on the edge of the crack and has Cartesian coordinates $(a\cos\phi, b\sin\phi, 0)$, and P_1 is in the plane of the crack and has coordinates $(a\rho\cos\phi_1, b\rho\sin\phi_1, 0)$ with $\rho > 1$. Let s denote the distance between P and P_1. Let P_1 approach P along the normal at P. Then, Sneddon (**22**) has shown that the stress-intensity factor at P is given by

$$\begin{aligned} k_1(\phi) &= \lim_{s\to 0+} (2s)^{\frac{1}{2}} \tau_{zz}(a\rho\cos\phi_1, b\rho\sin\phi_1, 0) \\ &= (ab)^{\frac{1}{2}}(a^2\sin^2\phi + b^2\cos^2\phi)^{-\frac{1}{4}} \times \\ &\quad \times \lim_{\rho\to 1+} \{(\rho^2-1)^{\frac{1}{2}} \tau_{zz}(a\rho\cos\phi, b\rho\sin\phi, 0)\}. \end{aligned} \tag{5.1}$$

We have

$$\tau_{zz}(a\rho\cos\phi, b\rho\sin\phi, 0) = \mu \sum_{n=0}^{\infty} \tau_n(\rho)\cos n\phi + \mu \sum_{n=1}^{\infty} \tilde{\tau}_n(\rho)\sin n\phi, \tag{5.2}$$

where τ_n and $\tilde{\tau}_n$ are given by (3.19) and (3.20), respectively. For example,

$$\tau_{2n}(\rho) = \frac{-\varepsilon_{2n}}{\pi(1-\nu)k'} \sum_{m=0}^{\infty} I^c_{2m,2n} \sum_{j=0}^{\infty} W_j^{2m} L^{2j}_{2m,2n}(\rho), \tag{5.3}$$

where we have used (4.4), and $L^{2j}_{\mu,\nu}$ is defined by (4.5). For $\rho > 1$, we have (**13**, §7.7.4)

$$\begin{aligned} L^{2j}_{\mu,\nu}(\rho) = \frac{\pi^{\frac{1}{2}}\Gamma(j+p+\frac{3}{2})}{\rho^{2j+\mu+3}\Gamma(2j+\mu+\frac{5}{2})\Gamma(-j-q-\frac{1}{2})} \times \\ \times F(j+p+\tfrac{3}{2}, j+q+\tfrac{3}{2}; 2j+\mu+\tfrac{5}{2}; 1/\rho^2), \end{aligned}$$

where the integers p and q are defined by (4.6). Since (**30**, Equations 9.131.1, 9.122.1, 8.339.2 and 8.339.3)

$$F(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; z),$$

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

$$\Gamma(\tfrac{1}{2}+n)\Gamma(\tfrac{1}{2}-n)=(-1)^n\pi,$$

we obtain

$$\lim_{\rho\to 1+}\{(\rho^2-1)^{\frac{1}{2}}L^{2j}_{\mu,\nu}(\rho)\}=(-1)^{j+q+1}.$$

Hence from (5.3)

$$\lim_{\rho\to 1+}\{(\rho^2-1)^{\frac{1}{2}}\tau_{2n}(\rho)\}=\frac{\varepsilon_{2n}}{(1-\nu)k'}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}}{\pi}I^c_{2m,2n}\sum_{j=0}^{\infty}(-1)^jW^{2m}_j. \quad (5.4)$$

Now from (3.13), we have

$$\sum_{n=0}^{\infty}\varepsilon_{2n}\frac{(-1)^{m+n}}{\pi}I^c_{2m,2n}\cos 2n\phi=(1-k^2\cos^2\phi)^{\frac{1}{2}}\cos 2m\phi, \quad (5.5)$$

whence

$$\lim_{\rho\to 1+}\{(\rho^2-1)^{\frac{1}{2}}\sum_{n=0}^{\infty}\tau_{2n}(\rho)\cos 2n\phi\}$$
$$=\frac{(1-k^2\cos^2\phi)^{\frac{1}{2}}}{(1-\nu)k'}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}(-1)^jW^{2m}_j\cos 2m\phi. \quad (5.6)$$

Similar results obtain for τ_{2n+1}, $\tilde{\tau}_{2n}$ and $\tilde{\tau}_{2n+1}$; combining these gives

$$k_1(\phi)=\frac{\mu}{1-\nu}\left(\frac{a}{b}\right)^{\frac{1}{2}}(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{1}{4}}W(\phi), \quad (5.7)$$

where

$$W(\phi)=\sum_{j=0}^{\infty}(-1)^j\left\{\sum_{m=0}^{\infty}W^m_j\cos m\phi+\sum_{m=1}^{\infty}\tilde{W}^m_j\sin m\phi\right\}. \quad (5.8)$$

For the penny-shaped crack, this formula for k_1 reduces to that given by Krenk (**12**).

6. Truncation

Consider (4.13), which is an infinite system of linear equations for W^{2m}_j; the right-hand side involves W^{2m}_{l-m}, where $0\leqslant m\leqslant l=n+j$. So, to obtain a proper truncation of this system, we must restrict l. To fix ideas, assume that the prescribed loading is given by

$$p(\mathbf{x})=\mu\sum_{n=0}^{\infty}\tau_{2n}(\rho)\cos 2n\phi=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}a_{nj}(\rho,\phi),$$

say. Rearranging, we have

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{l=n}^{\infty} a_{n,l-n} = \sum_{l=0}^{\infty} \sum_{n=0}^{l} a_{n,l-n}.$$

Let us now suppose that the loading is such that this infinite series can be truncated at $l = N$:

$$p(\mathbf{x}) = \sum_{l=0}^{N} \sum_{n=0}^{l} a_{n,l-n} = \sum_{n=0}^{N} \sum_{l=n}^{N} a_{n,l-n} = \sum_{n=0}^{N} \sum_{j=0}^{N-n} a_{nj}.$$

Thus, we are led to consider prescribed loadings of the form

$$p(\mathbf{x}) = \tau_{zz}(x, y, 0) = \mu \sum_{n=0}^{N} \tau_{2n}(\rho) \cos 2n\phi, \tag{6.1}$$

where

$$\tau_{2n}(\rho) = \rho^{2n} \sum_{j=0}^{N-n} S_j^{2n} \frac{\Gamma(2n+\frac{1}{2})\Gamma(j+\frac{3}{2})}{(2n+j)!\,(1-\rho^2)^{\frac{1}{2}}} C_{2j+1}^{2n+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}}). \tag{6.2}$$

When (6.2) is compared with (4.11), we find that (4.13) holds, but only in the range $0 \leqslant n \leqslant N$, $0 \leqslant j \leqslant N-n$, that is,

$$S_{l-n}^{2n} = \frac{-\varepsilon_{2n}}{\pi(1-\nu)k'} \sum_{m=0}^{l} I_{2m,2n}^{c} W_{l-m}^{2m}, \qquad 0 \leqslant n \leqslant N, \quad n \leqslant l \leqslant N. \tag{6.3}$$

All other coefficients W_j^{2m} can be set to zero.

For each l in $0 \leqslant l \leqslant N$, (6.3) gives $l+1$ equations in the $l+1$ unknown coefficients W_l^0, W_{l-1}^2, $W_{l-2}^4, \ldots, W_1^{2l-2}$, W_0^{2l} involving the $l+1$ known coefficients S_l^0, S_{l-1}^2, $S_{l-2}^4, \ldots, S_1^{2l-2}$, S_0^{2l}. For example, $l = 0$ gives

$$-\pi(1-\nu)k'S_0^0 = I_{00}^{c} W_0^0 \tag{6.4}$$

whilst $l = 1$ (for $N \geqslant 1$) gives

$$\left.\begin{aligned} -\pi(1-\nu)k'S_1^0 &= I_{00}^{c} W_1^0 + I_{20}^{c} W_0^2, \\ -\tfrac{1}{2}\pi(1-\nu)k'S_0^2 &= I_{02}^{c} W_1^0 + I_{22}^{c} W_0^2. \end{aligned}\right\} \tag{6.5}$$

We note that each system of equations is uncoupled from all the others. Atluri *et al.* (**18**, **19**) obtain equations with a similar structure, except that their systems are weakly coupled, thus, in order to solve their system corresponding to $l = L$, say, they first need the solutions for the larger systems with $L < l \leqslant N$ (see (**18**, p. 92)).

The displacement of the upper crack-face is given by

$$w(\mathbf{x}) = \tfrac{1}{2}a \sum_{m=0}^{N} \sum_{j=0}^{N-m} W_j^{2m} \frac{\Gamma(2m+\frac{1}{2})j!}{\Gamma(2m+j+\frac{3}{2})} C_{2j+1}^{2m+\frac{1}{2}}((1-\rho^2)^{\frac{1}{2}})\rho^{2m} \cos 2m\phi,$$

and the corresponding stress-intensity factor is given by (5.7) with

$$W(\phi) = \sum_{m=0}^{N} \sum_{j=0}^{N-m} (-1)^j W_j^{2m} \cos 2m\phi. \tag{6.6}$$

Similar results obtain for prescribed loadings with other symmetries.

Finally, we could now give another proof of the theorem stated in section 1, although we shall not do so here.

7. Five examples

To illustrate our method, we shall consider some particular loadings of the crack. We begin with the four simplest cases, namely (i) $p(\mathbf{x}) = -p_0$, (ii) $p(\mathbf{x}) = -p_0 x/a$, (iii) $p(\mathbf{x}) = -p_0 y/b$, and (iv) $p(\mathbf{x}) = -p_0 xy/(ab)$, where p_0 is a constant. These are simple because they all involve just one non-zero load coefficient (S_0^0, S_0^1, $\bar{S}_0^1$ or $\bar{S}_0^2$), that is, they correspond to $P_i = \delta_{ij}$ ($j = 1, 2, 3, 4$) in (1.10).

(i) Since $C_1^\lambda(x) = 2\lambda x$, we obtain

$$S_0^0 = \frac{-2p_0}{\pi\mu},$$

whence (6.4) gives (using $I_{00}^c = 2E(k)$)

$$W_0^0 = \frac{p_0(1-\nu)k'}{\mu E}.$$

Hence

$$w(\mathbf{x}) = aW_0^0(1-\rho^2)^{\frac{1}{2}} = \frac{p_0(1-\nu)b}{\mu E}(1-\rho^2)^{\frac{1}{2}} \tag{7.1}$$

and

$$\begin{aligned} k_1(\phi) &= \frac{\mu}{1-\nu}\left(\frac{a}{b}\right)^{\frac{1}{2}}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}W_0^0 \\ &= \frac{p_0}{E}\left(\frac{a}{b}\right)^{\frac{1}{2}}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}. \end{aligned} \tag{7.2}$$

Equations (7.1) and (7.2) were first obtained by Green and Sneddon (**14**) and Irwin (**31**), respectively.

(ii) We have $p(\mathbf{x}) = -p_0\rho\cos\phi$, whence

$$S_0^1 = \frac{-4p_0}{3\pi\mu} \quad \text{and} \quad W_0^1 = \frac{p_0(1-\nu)k'k^2}{\mu\Omega_1},$$

where

$$\Omega_1(k) = \tfrac{3}{2}k^2 I_{11}^c = (k^2 - k'^2)E(k) + k'^2K(k).$$

Thus,

$$w(\mathbf{x}) = W_0^1 x(1-\rho^2)^{\frac{1}{2}} \tag{7.3}$$

and

$$k_1(\phi) = p_0\left(\frac{b}{a}\right)^{\frac{1}{2}} \frac{k^2}{\Omega_1}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}\cos\phi. \tag{7.4}$$

Equations (7.3) and (7.4) have both been given by Shibuya (**21**) and Shail (**20**) and, incorrectly, by Kassir and Sih (**15**). Equation (7.4) has also been given by Shah and Kobayashi (**17**).

(iii) We have $p(\mathbf{x}) = -p_0\rho\sin\phi$, whence

$$\tilde{S}_0^1 = \frac{-4p_0}{3\pi\mu} \quad \text{and} \quad \tilde{W}_0^1 = \frac{p_0(1-\nu)k'k^2}{\mu\Omega_2},$$

where

$$\Omega_2(k) = \tfrac{3}{2}k^2 I_{11}^{\mathrm{r}} = (1+k^2)E(k) - k'^2K(k).$$

Thus,

$$w(\mathbf{x}) = (a/b)\tilde{W}_0^1 y(1-\rho^2)^{\frac{1}{2}} \tag{7.5}$$

and

$$k_1(\phi) = p_0\left(\frac{b}{a}\right)^{\frac{1}{2}} \frac{k^2}{\Omega_2}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}\sin\phi. \tag{7.6}$$

Equations (7.5) and (7.6) have both been given by Shibuya (**21**).

(iv) We have $p(\mathbf{x}) = -\frac{1}{2}p_0\rho^2\sin 2\phi$, whence

$$\tilde{S}_0^2 = \frac{-8p_0}{15\pi\mu} \quad \text{and} \quad \tilde{W}_0^2 = \frac{p_0(1-\nu)k'k^4}{2\mu\Omega_3},$$

where

$$\Omega_3(k) = \tfrac{15}{8}k^4 I_{22}^{\mathrm{r}} = 2(k^2+k'^4)E(k) - k'^2(1+k'^2)K(k).$$

Thus

$$w(\mathbf{x}) = \frac{p_0(1-\nu)k^4}{\mu a\Omega_3}xy(1-\rho^2)^{\frac{1}{2}} \tag{7.7}$$

and

$$k_1(\phi) = p_0\left(\frac{b}{a}\right)^{\frac{1}{2}} \frac{k^4}{2\Omega_3}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}\sin 2\phi. \tag{7.8}$$

Equations (7.7) and (7.8) have both been given by Shail (**20**) (there is a factor of $a\mu$ missing from the denominator in (**20**, Equation (53))). Equation (7.8) has also been given by Kassir and Sih (**3**, Equation (A3.3k)).

We now consider a more complicated example:

$$p(\mathbf{x}) = p_{20}(x/a)^2 + p_{02}(y/b)^2 = \rho^2(p_0 + p_1 \cos 2\phi), \tag{7.9}$$

where $p_0 = \frac{1}{2}(p_{20} + p_{02})$ and $p_1 = \frac{1}{2}(p_{20} - p_{02})$. Taking $N = 1$ in (6.1), (6.2) gives

$$S_0^0 = \frac{4p_0}{5\pi\mu}, \quad S_1^0 = \frac{-8p_0}{15\pi\mu}, \quad S_0^2 = \frac{16p_1}{15\pi\mu};$$

W_0^0 is given by (6.4), whilst (6.5) gives

$$\Omega W_0^2 = -\tfrac{1}{2}\pi(1-\nu)k'\{S_0^2 I_{00}^c - 2S_1^0 I_{02}^c\}$$

and

$$\Omega W_1^0 = -\tfrac{1}{2}\pi(1-\nu)k'\{2S_1^0 I_{22}^c - S_0^2 I_{20}^c\},$$

where

$$\Omega(k) = I_{00}^c I_{22}^c - I_{02}^c I_{20}^c.$$

We can calculate I_{00}^c, I_{22}^c and $I_{02}^c = I_{20}^c$ using the formulae given in section 3.1; $w(\mathbf{x})$ and $k_1(\phi)$ can also be found. For example, the stress-intensity factor is

$$k_1(\phi) = \frac{-8}{15}(b/a)^{\frac{1}{2}}(a^2 \sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}\{B_0(k) + B_2(k)\cos 2\phi\}, \tag{7.10}$$

where

$$\Omega B_0 = \tfrac{1}{2}p_0(3\Omega + 2I_{00}^c I_{22}^c)/I_{00}^c + p_1 I_{02}^c$$

and

$$\Omega B_2 = p_0 I_{02}^c + p_1 I_{00}^c.$$

Kassir and Sih have also given a formula for $k_1(\phi)$ (**3**, Equation (A3.3g)). However, their formula does not reduce to the correct result for a penny-shaped crack (see the Appendix), whereas (7.10) does; this result, which can be obtained using the method of Guidera and Lardner (**6**), is

$$k_1(\phi) = \frac{-4a^{\frac{1}{2}}}{15\pi}(5p_0 + 4p_1\cos 2\phi), \quad b = a. \tag{7.11}$$

REFERENCES

1. I. N. Sneddon, *Proc. R. Soc.* A **187** (1946) 229–260.
2. R. A. Sack, *Proc. Phys. Soc.* **58** (1946) 729–736.
3. M. K. Kassir and G. C. Sih, *Three-dimensional Crack Problems* (Noordhoff International, Leyden 1975).
4. J. C. Bell, *Int. J. Fract.* **15** (1979) 85–104.
5. R. N. L. Smith, *Int. J. Num. Meth. Engng* **20** (1984) 2093–2105.

6. J. T. Guidera and R. W. Lardner, *J. Elast.* **5** (1975) 59–73.
7. P. A. Martin, *ibid.* **12** (1982) 201–218.
8. L. E. Payne, *J. SIAM* **1** (1953) 53–71.
9. A. H. England and R. Shail, *Q. Jl Mech. appl. Math.* **30** (1977) 397–414.
10. R. Shail, *ibid.* **33** (1980) 235–249.
11. G. M. L. Gladwell and A. H. England, *ibid.* **30** (1977) 175–185.
12. S. Krenk, *J. appl. Mech.* **46** (1979) 821–826.
13. A. Erdélyi *et al.*, *Higher Transcendental Functions*, Vol. 2 (McGraw–Hill, New York 1953).
14. A. E. Green and I. N. Sneddon, *Proc. Camb. phil. Soc.* **46** (1950) 159–163.
15. M. K. Kassir and G. C. Sih, *J. Math. Mech.* **16** (1967) 927–948.
16. C. M. Segedin, *Int. J. engng Sci.* **6** (1968) 309–312.
17. R. C. Shah and A. S. Kobayashi, *Eng. Fract. Mech.* **3** (1971) 71–96.
18. K. Vijayakumar and S. N. Atluri, *J. appl. Mech.* **48** (1981) 88–96.
19. T. Nishioka and S. N. Atluri, *Eng. Fract. Mech.* **17** (1983) 247–268.
20. R. Shail, *Int. J. engng Sci.* **16** (1978) 551–563.
21. T. Shibuya, *Bull. Japanese Soc. Mech. Engrs* **20** (1977) 909–914.
22. I. N. Sneddon, *Int. J. engng Sci.* **17** (1979) 185–191.
23. J. D. Eshelby, *Proc. R. Soc.* **A241** (1957) 376–396.
24. J. R. Willis, *Int. J. engng Sci.* **6** (1968) 253–263.
25. L. J. Walpole, *Proc. Camb. phil. Soc.* **67** (1970) 225–235.
26. G. M. L. Gladwell, *Q. Jl Mech. appl. Math.* **31** (1978) 251–260.
27. I. N. Sneddon, *Fourier Transforms* (McGraw–Hill, New York 1951).
28. A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd edition (Clarendon Press, Oxford 1968).
29. P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 2nd edition (Springer, Berlin 1971).
30. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Corrected and enlarged edition (Academic Press, New York 1980).
31. G. R. Irwin, *J. appl. Mech.* **29** (1962) 651–654.

APPENDIX

The stress-intensity factor for an elliptical crack opened by the quadratic pressure (7.9) has been given by Kassir and Sih (**3**, Equation (A3.3g)) as

$$k_1(\phi) = -\tfrac{2}{3}(a/b)^{\frac{3}{2}}\{A_2\cos^2\phi + A_3(a/b)^2\sin^2\phi\}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{4}}, \tag{A.1}$$

where

$$\Delta A_2 = p_{20}(J_{12} + 5J_{03}) - (a/b)^2 p_{02}(J_{21} + J_{12}), \tag{A.2}$$

$$\Delta A_3 = -p_{20}(J_{12} + J_{21}) + (a/b)^2 p_{02}(J_{21} + 5J_{30}), \tag{A.3}$$

$$\Delta(k) = (J_{21} + 5J_{30})(J_{12} + 5J_{03}) - (J_{21} + J_{12})^2, \tag{A.4}$$

and

$$J_{mn}(k) = \int_0^{K(k)} (\operatorname{sn} t)^{2m+2n}(\operatorname{nd} t)^{2n}\, dt.$$

We shall let $k \to 0$ in (A.1). Since (**29**) $\operatorname{sn} t \to \sin t$, $\operatorname{nd} t \to 1$ and $K \to \frac{1}{2}\pi$ as $k \to 0$, we obtain

$$J_{mn}(0) = \int_0^{\frac{1}{2}\pi} (\sin t)^{2m+2n}\, dt.$$

In particular

$$J_{12}(0) = J_{21}(0) = J_{30}(0) = J_{03}(0) = \tfrac{5}{32}\pi. \tag{A.5}$$

Kassir and Sih (3) give formulae for $J_{mn}(k)$ as linear combinations of E and K; letting $k \to 0$ in these, we see that their formula for J_{03} does not satisfy (A.5), and so it must be incorrect. Nevertheless, let us now use (A.5) in (A.2), (A.3) and (A.4):

$$32\Delta(0) = 25\pi^2, \quad 5\pi A_2(0) = 6p_{20} - 2p_{02} \quad \text{and} \quad 5\pi A_3(0) = 6p_{02} - 2p_{20}.$$

Hence, as $k \to 0$, (A.1) gives

$$\begin{aligned} k_1(\phi) &= \frac{-4a^{\frac{1}{2}}}{15\pi}\{(3p_{20} - p_{02})\cos^2\phi + (3p_{02} - p_{20})\sin^2\phi\} \\ &= \frac{-4a^{\frac{1}{2}}}{15\pi}(2p_0 + 4p_1\cos 2\phi), \end{aligned}$$

which should be compared with (7.11); it is seen that only the coefficients of $\cos 2\phi$ agree. Thus (A.1) must also be incorrect.