Proc. R. Soc. Lond. A **421**, 341–355 (1989)
Printed in Great Britain

On boundary integral equations for crack problems

BY P. A. MARTIN[1] AND F. J. RIZZO[2]

[1] *Department of Mathematics, University of Manchester, Manchester M*13 9*PL, U.K.*
[2] *Department of Engineering Science and Mechanics, Iowa State University, Ames, Iowa* 50011, *U.S.A.*

(*Communicated by F. Ursell, F.R.S. – Received* 16 *May* 1988)

A ubiquitous linear boundary-value problem in mathematical physics involves solving a partial differential equation exterior to a thin obstacle. One typical example is the scattering of scalar waves by a curved crack or rigid strip (Neumann boundary condition) in two dimensions. This problem can be reduced to an integrodifferential equation, which is often regularized. We adopt a more direct approach, and prove that the problem can be reduced to a hypersingular boundary integral equation. (Similar reductions will obtain in more complicated situations.) Computational schemes for solving this equation are described, with special emphasis on smoothness requirements. Extensions to three-dimensional problems involving an arbitrary smooth bounded crack in an elastic solid are discussed.

1. INTRODUCTION

Boundary integral equations are used widely to determine the effect of obstacles on otherwise uniform fields. Examples are potential flow past an aircraft, acoustic scattering by a rigid target and loading of an elastic structure with embedded cavities. In many problems of practical interest, the obstacles are *thin*. Examples are rigid plates in a flow and cracks in a solid. Conventional integral-equation methods fail for such problems.

To be specific, consider the scattering of a plane sound wave in a compressible fluid by a rigid plate, Γ, in two dimensions. (The basic ideas below can be extended to problems in three dimensions, and to other partial differential equations. Cracks are considered in §4.) Mathematically, the problem is to find the potential $\mathrm{Re}\{u(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, where

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + k^2 u = 0 \quad \text{in the fluid,} \tag{1.1}$$

$$\partial u/\partial n^{\pm} = 0 \quad \text{on} \quad \Gamma^{\pm} \tag{1.2}$$

$$u \text{ is bounded at both edges of the plate,} \tag{1.3}$$

and the scattered potential $u^{\mathrm{sc}} \equiv u - u^{\mathrm{in}}$ satisfies a radiation condition.

Here, k is the given wavenumber, assumed real and positive; $u^{\mathrm{in}}(x,y)$ is the given incident wave; $\partial/\partial n^{\pm}$ denotes normal differentiation at a point on $\Gamma^{\pm}$, in the direction from $\Gamma^{\pm}$ into the fluid; $\Gamma^{\pm}$ are the two sides of an *open* arc Γ, which is assumed to be twice continuously differentiable; and x, y are cartesian coordinates.

We shall also require the *closed* arc, $\bar{\Gamma} = \Gamma \cup \partial\Gamma$, where $\partial\Gamma$ consists of the two end points (edges) of Γ.

We introduce a fundamental solution, G, defined by

$$G(\mathrm{P}, \mathrm{Q}) \equiv G(x, y; \xi, \eta) = -\tfrac{1}{2}\mathrm{i} H_0^{(1)}(kR), \tag{1.4}$$

where R is the distance between P and Q, and $H_0^{(1)}$ is a Hankel function. A careful application of Green's theorem to u^{sc} and G gives

$$2u^{\mathrm{sc}}(\mathrm{P}) = -\int_\Gamma [u(\mathrm{q})] \frac{\partial}{\partial n_\mathrm{q}^+} G(\mathrm{P}, \mathrm{q})\, \mathrm{d}s_\mathrm{q}, \tag{1.5}$$

where

$$[u(\mathrm{q})] = u(\mathrm{q}^+) - u(\mathrm{q}^-) \tag{1.6}$$

and $\mathrm{q}^\pm$ are corresponding points on $\Gamma^\pm$. Here, and below, we use capital letters P, Q to denote points in the fluid and lower-case letters p, q to denote points on Γ. Thus, $[u(\mathrm{q})]$ is the discontinuity (jump) in u across the plate at q. Equation (1.5) states that $u^{\mathrm{sc}}(\mathrm{P})$ can be represented as a double layer, i.e. as a distribution of normal dipoles over Γ. The density is seen to be $-\tfrac{1}{2}[u]$.

Applying the boundary condition (1.2) on Γ^+ gives

$$\frac{\partial u^{\mathrm{in}}}{\partial n_\mathrm{p}^+} - \frac{1}{2}\frac{\partial}{\partial n_\mathrm{p}^+}\int_\Gamma [u(\mathrm{q})] \frac{\partial}{\partial n_\mathrm{q}^+} G(\mathrm{p}^+, \mathrm{q})\, \mathrm{d}s_\mathrm{q} = 0, \quad \mathrm{p}^+ \in \Gamma^+.$$

The same equation is obtained if we let P approach the other side of Γ, Γ^-. Thus, we can delete the superscript $+$ throughout to give

$$\frac{\partial}{\partial n_\mathrm{p}}\int_\Gamma [u(\mathrm{q})] \frac{\partial}{\partial n_\mathrm{q}} G(\mathrm{p}, \mathrm{q})\, \mathrm{d}s_\mathrm{q} = 2\frac{\partial u^{\mathrm{in}}(\mathrm{p})}{\partial n_\mathrm{p}}, \quad \mathrm{p} \in \Gamma. \tag{1.7}$$

It is this equation, and some of its variants, that we consider below; we discuss both analytical and numerical aspects.

Equation (1.7) is an integrodifferential equation for $[u(\mathrm{q})]$, $\mathrm{q} \in \Gamma$. Its solution is sought in the space

$$C_0^{1,\alpha}(\bar{\Gamma}) = \{u(\mathrm{q}) \in C^{1,\alpha}(\Gamma) : u(\mathrm{q}) = 0 \quad \text{for} \quad \mathrm{q} \in \partial\Gamma\},$$

where $C^{1,\alpha}(\Gamma)$ is the usual space of functions with one Hölder-continuous tangential derivative on Γ, and $0 < \alpha \leqslant 1$. It can be shown that if $[u(\mathrm{q})] \in C_0^{1,\alpha}(\bar{\Gamma})$, and $[u]$ solves (1.7), then $[u]$ will have the expected square-root behaviour near the edges of the plate:

$$[u(\mathrm{q})] \sim s^{\frac{1}{2}} \quad \text{as} \quad s \to 0,$$

where s is the distance (arc length) of q from $\partial\Gamma$.

Equation (1.7) is an inevitable consequence of using G, which is continuous across Γ, i.e. $[G(\mathrm{P}, \mathrm{q})] = 0$. Wickham (1982) has replaced G by another fundamental solution which is discontinuous across Γ; this leads to a Fredholm integral equation of the second kind for $[u]$, with a continuous kernel which is itself given explicitly as a certain singular integral over Γ.

It is tempting simply to take the normal derivative $\partial/\partial n_\mathrm{p}$ in (1.7) under the integral sign, but this leads to a non-integrable integrand. Instead, it is common to *regularize* (1.7) by rewriting its left-hand side as an integral involving $[u]$ and its tangential derivative;

$$\int_\Gamma \{(\boldsymbol{n}(\mathrm{q}) \times \nabla_\mathrm{q} [u(\mathrm{q})]) \cdot (\boldsymbol{n}(\mathrm{p}) \times \nabla_\mathrm{p} G(\mathrm{p}, \mathrm{q})) + k^2 \boldsymbol{n}(\mathrm{q}) \cdot \boldsymbol{n}(\mathrm{p})\, G(\mathrm{p}, \mathrm{q})\, [u(\mathrm{q})]\}\, \mathrm{d}s_\mathrm{q} = 2 \frac{\partial u^{\mathrm{in}}(\mathrm{p})}{\partial n_\mathrm{p}}. \tag{1.8}$$

This formula was first given by Maue (1949). (It is also valid in three dimensions.) For our two-dimensional problem, (1.8) reduces to

$$\oint_\Gamma \left\{ \frac{\partial}{\partial t_\mathrm{q}} [u(\mathrm{q})] \frac{\partial}{\partial t_\mathrm{p}} G(\mathrm{p}, \mathrm{q}) + k^2 \boldsymbol{n}(\mathrm{q}) \cdot \boldsymbol{n}(\mathrm{p})\, G(\mathrm{p}, \mathrm{q})\, [u(\mathrm{q})] \right\} \mathrm{d}s_\mathrm{q} = 2 \frac{\partial u^{\mathrm{in}}(\mathrm{p})}{\partial n_\mathrm{p}}, \quad \mathrm{p} \in \Gamma, \tag{1.9}$$

where $\partial/\partial t_\mathrm{q}$ denotes tangential differentiation at q. This is a regularized equation, involving both $[u]$ and its tangential derivative.

Equation (1.9) can also be written as

$$-\frac{\partial}{\partial t_\mathrm{p}} \oint_\Gamma [u(\mathrm{q})] \frac{\partial}{\partial t_\mathrm{q}} G(\mathrm{p}, \mathrm{q})\, \mathrm{d}s_\mathrm{q} + k^2 \int_\Gamma \boldsymbol{n}(\mathrm{q}) \cdot \boldsymbol{n}(\mathrm{p})\, G(\mathrm{p}, \mathrm{q})\, [u(\mathrm{q})]\, \mathrm{d}s_\mathrm{q} = 2 \frac{\partial u^{\mathrm{in}}(\mathrm{p})}{\partial n_\mathrm{p}}, \quad \mathrm{p} \in \Gamma, \tag{1.10}$$

which is a different integrodifferential equation for $[u(\mathrm{q})]$, this time involving only tangential derivatives rather than the normal derivatives occurring in (1.7).

Equations (1.9) and (1.10) are well known; see, for example, Morrison (1979), Frenkel (1983) and Zakharov & Sobyanina (1986).

If we introduce a parametrization of the curve Γ (see (3.1) below), we see that (1.10) is of the form

$$\frac{\mathrm{d}}{\mathrm{d}x} \oint_{-1}^{1} \frac{f(t)}{x-t}\, \mathrm{d}t + \int_{-1}^{1} f(t)\, K(x, t)\, \mathrm{d}t = g(x), \quad |x| < 1, \tag{1.11}$$

where g is given, K is a weakly singular kernel and $f(x)$ is to be found in $C_0^{1,\alpha}([-1, 1])$. Because

$$\frac{\mathrm{d}}{\mathrm{d}x} \oint_{-1}^{1} \frac{f(t)}{x-t}\, \mathrm{d}t = -\,⨎_{-1}^{1} \frac{f(t)}{(x-t)^2}\, \mathrm{d}t, \tag{1.12}$$

we can rewrite (1.11) as the *hypersingular integral equation*,

$$⨎_{-1}^{1} \left\{ \frac{-1}{(x-t)^2} + K(x, t) \right\} f(t)\, \mathrm{d}t = g(x), \quad |x| < 1. \tag{1.13}$$

Here, the notation ⨎ means that the integral must be interpreted as a *Hadamard finite-part integral* (Hadamard 1923); these are defined in Appendix A. (Our notation follows Mangler (1952); there does not seem to be a universal standard.) The result (1.12) is implicit in Hadamard's book (1923) – indeed, the definition of ⨎ is chosen so that (1.12) holds – and is explicitly proved by Mangler (1952).

In the present paper, we derive, directly from (1.7), a hypersingular integral equation for our scattering problem, analogous to (1.13). Then, in §3, we describe some methods for solving this equation. Finally, in §4, we consider the extensions necessary to treat arbitrary smooth cracks in three-dimensional elasticity. Since our primary goal is to develop robust computational methods for these crack problems, we focus our attentions in §3 on boundary element methods, rather than on the powerful methods (for two-dimensional problems) using orthogonal polynomials that have been developed recently by Frenkel (1983), Golberg (1983, 1985) and Kaya & Erdogan (1987). In this sense, the scalar scattering problem described above is a simple prototype (although it is equivalent to the scattering of horizontally polarized shear waves by a two-dimensional crack, Γ).

We conclude this section with some remarks. First, we note that much of the previous work on crack problems has been based on regularized equations like (1.9), presumably in an attempt to avoid apparently divergent integrals (see §4.1). We shall argue that a formulation in terms of Hadamard finite-part integrals, leading to hypersingular integral equations, is a preferred alternative to regularized equations, both conceptually and computationally. Besides, if one intends to collocate at an edge (or crack tip) with the regularized equation, one is faced with a one-sided Cauchy principal-value integral with a density (the tangential derivative of $[u]$) which is singular at the edge. This integral, although seldom identified as such, is a one-sided finite-part integral of order $\frac{3}{2}$; see Appendix A. In this paper, we hope to clarify the relations between apparently divergent integrals and various finite-part integrals. In a subsequent paper, we shall demonstrate how hypersingular integral equations can be used to obtain numerical solutions to various problems involving cracks and other thin obstacles.

2. Hypersingular integral equations

The following theorem is proved in Appendix B.

Theorem *Let* $u(\mathrm{q}) \in C_0^{1,\alpha}(\bar{\Gamma})$, $0 < \alpha \leqslant 1$. *Then*

$$\frac{\partial}{\partial n_\mathrm{p}} \int_\Gamma u(\mathrm{q}) \frac{\partial}{\partial n_\mathrm{q}} G(\mathrm{p}, \mathrm{q})\, \mathrm{d}s_\mathrm{q} = \oint\!\!\!\!\times_\Gamma u(\mathrm{q}) \frac{\partial}{\partial n_\mathrm{p}\, \partial n_\mathrm{q}} G(\mathrm{p}, \mathrm{q})\, \mathrm{d}s_\mathrm{q} \tag{2.1}$$

for any $\mathrm{p} \in \bar{\Gamma}$.

We remark that the integral on the right-hand side of (2.1) is a two-sided finite-part integral of order 2 if *not* of $\mathrm{p} \in \Gamma$, but is a one-sided finite-part integral of order $\frac{3}{2}$ if $\mathrm{p} \in \partial\Gamma$; see Appendix A. Moreover, the smoothness requirements on $u(\mathrm{q})$ are essential.

Using the theorem, (1.7) becomes

$$\oint\!\!\!\!\times_\Gamma [u(\mathrm{q})] \frac{\partial}{\partial n_\mathrm{p}\, \partial n_\mathrm{q}} G(\mathrm{p}, \mathrm{q})\, \mathrm{d}s_\mathrm{q} = 2 \frac{\partial u^{\mathrm{in}}(\mathrm{p})}{\partial n_\mathrm{p}}, \quad \mathrm{p} \in \Gamma. \tag{2.2}$$

This is a hypersingular integral equation for $[u(\mathrm{q})]$, $\mathrm{q} \in \Gamma$. It has advantages over the other equations exhibited in §1: it is not an integrodifferential equation; it does not involve tangential derivatives of $[u(\mathrm{q})]$, which are unbounded at $\partial\Gamma$; and it is conceptually attractive.

3. Numerical treatment

We begin by breaking Γ into N elements, Γ_j, $j = 1, 2, \dots, N$. To do this, we parametrize Γ as

$$\Gamma = \{(x(t), y(t)) : -1 \leqslant t \leqslant 1\}, \tag{3.1}$$

We partition $[-1, 1]$ using $N+1$ nodes t_j,

$$-1 = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = +1$$

and then set

$$\Gamma_j = \{(x(t), y(t)) : t_{j-1} \leqslant t \leqslant t_j\}.$$

Near the edges $\partial\Gamma$, i.e. near $t = \pm 1$, we know that

$$[u(\mathrm{q})] \equiv U(t) \approx A_{\pm}(1 \mp t)^{\frac{1}{2}}, \tag{3.2}$$

where the coefficients $A_{\pm}$ are essentially the stress-intensity factors. Hence

$$V(t) \equiv U(t) - A_+(1-t)^{\frac{1}{2}}\chi_+(t) - A_-(1+t)^{\frac{1}{2}}\chi_-(t) \tag{3.3}$$

satisfies

$$V(t) \in C_0^{1,\alpha}([-1,1]) \quad \text{and} \quad V'(\pm 1) = 0. \tag{3.4}$$

Here, $\chi_{\pm}(t)$ are smooth (C^{∞}) cut-off functions, with $0 \leqslant \chi_{\pm} \leqslant 1$, $\chi_{\pm}(t) = 1$ for t near ± 1 and $\chi_{\pm} = 0$ elsewhere. (Decompositions like (3.3) have been used by Stephan & Wendland (1984).)

Because of (3.4), we can readily approximate $V(t)$ by *parabolic B-splines*,

$$V(t) \approx \sum_{n=0}^{N-3} v_n B_n^{(2)}(t).$$

where $B_i^{(2)}(x) \equiv B_{i,3,\boldsymbol{t}}(x)$ is the ith B-spline of order 3 (degree 2) for the knot sequence $\boldsymbol{t} = \{t_0, t_1, t_2, \dots, t_N\}$; see de Boor (1978 ch. IX–XI). Note that $B_i^{(2)}(x)$ is non-zero only in the interval $t_i < x < t_{i+3}$. Counting up, we see that there are N unknown coefficients, namely A_+, A_- and $v_n, n = 0, 1, \dots, N-3$. To find these, we can collocate the integral equation (2.2) at N points on Γ; we choose $t = s_j$, where $t_{j-1} < s_j < t_j$, $j = 1, 2, \dots, N$. (It is known that this method, the spline-collocation method, is convergent for integral equations like (2.2) on closed curves, at least when the nodes t_j are equally spaced and $s_j = \frac{1}{2}(t_{j-1} + t_j)$; see Arnold & Wendland (1985).)

Note that we can refine the numerical method by adding terms in $(1 \pm t)^{\frac{3}{2}}$ in (3.3), and then approximating the remainder by *cubic* B-splines; this leads naturally to collocation at the nodes t_j, $j = 0, 1, \dots, N$.

To make these methods work, we must evaluate integrals over each element Γ_j. The singular integrals can be computed using a combination of analytical and numerical techniques. For an extensive list of finite-part integrals, see Kaya & Erdogan (1987); for quadrature rules, see Kutt (1975), Paget (1981), Linz (1985) and Brandão (1987); for linear variation over a flat triangular element, see Ioakimidis (1985).

Previous computational experience of solving one-dimensional hypersingular integral equations is limited. Filippi & Dumery (1969) solved (2.2) for a flat plate, using a piecewise-constant approximation to $[u]$. Macaskill & Tuck (1977) solved a related equation using a similar approximation, but with a graded partition to account for the edge behaviour (3.2).

For three-dimensional problems, two-dimensional hypersingular integral equations obtain. Cassot (1975) has solved such an equation, using a piecewise-constant approximation, for acoustic scattering by a flat rectangular rigid plate. The hypersingular operator (normal derivative of a double layer) also occurs in problems involving *closed* surfaces (see, for example, Nedelec 1978, 1982). It was used in an interesting paper by Sayhi *et al.* (1981). They consider acoustic scattering by a hard sphere, and compute the total error incurred using a piecewise-constant approximation (see their figures 8 and 9). They show that this error does not decrease as the number of elements, N, increases, even though convergence is achieved at the collocation points themselves. This latter result is in accord with the theoretical results of Zakharov & Sobyanina (1986) for the one-dimensional integrodifferential equation (1.10). Sayhi *et al.* (1981) go on to show that the total error does decrease with N if a continuous, piecewise-linear approximation is used. This is in accord with the general results of Arnold & Wendland (1985) for one-dimensional equations.

We can give a qualitative explanation for the results of Sayhi *et al.* (1981). A constant approximation over an element gives rise to a potential whose normal derivative is singular around the perimeter of the element; thus, there is a mismatch with the prescribed normal derivative, which is usually very smooth everywhere on the surface. If a continuous, piecewise-linear approximation is used, the corresponding normal derivative is still undefined at element boundaries, but the mismatch is reduced. Finally, a continuously differentiable approximation gives rise to a potential whose normal derivative is continuous everywhere on the surface.

These results suggest that, for our problems, it is probably sufficient to use a continuous, piecewise-linear approximation, together with additional square-root terms near the edges. However, such an approximation cannot satisfy the integral equation (2.2) at the nodes, and so we do not expect pointwise convergence. Nevertheless, continuous, piecewise-linear approximations are simple, and so deserve further investigation, especially for three-dimensional problems. For the present two-dimensional problem, we can write

$$V(t) \approx \sum_{n=0}^{N-2} v_n B_n^{(1)}(t)$$

where $B_i^{(1)}(x) \equiv B_{i,2,t}(x)$ is a *linear* B-spline. (This approximation satisfies $V(\pm 1) = 0$.) Thus, there are $N+1$ unknown coefficients, which we can compute by collocating at $N+1$ interior points s_j; we cannot take $s_j = t_j$, because the integral in (2.2) does not then exist. Alternatively, we could use any other continuous representation for $V(t)$, such as quadratic elements, with continuity at the nodes. Some preliminary computations of an elementary finite-part integral, using these elements, gave us good results when collocating at interior points but somewhat spurious results, as expected, when collocating at nodes t_j between elements.

4. Cracks

We conclude with some remarks on integral-equation methods for crack problems in three dimensions. Here, we are not concerned with the vast literature on the static or dynamic loadings of particular cracks, i.e. on solution strategies that make explicit use of the crack geometry.

4.1. *Regularized integral equations*

Although the Somigliana formula has been used widely for many years in crack problems, it was apparently first used to derive regularized integral equations by Guidera & Lardner (1975). They obtaned equations analogous to (1.9) for the static loading of an arbitrary flat crack; they obtaind explicit solutions for the arbitrary loading of a penny-shaped crack. Similar equations were derived by Lo (1979) for a crack in a half-space (with the crack plane parallel to the free surface), and by Budiansky & Rice (1979) for the dynamic loading of a flat crack. For arbitrary smooth cracks regularized integral equations have been derived by Sládek & Sládek (1984), Le Van & Royer (1986) and Nishimura & Kobayashi (1988). All of these equations involve tangential derivatives of $[\boldsymbol{u}]$, the jump in the displacement vector across the crack. These derivatives are unbounded at the crack edge. Indeed, Nishimura & Kobayashi (1988) introduce second derivatives of $[\boldsymbol{u}]$, which are not integrable at the edge but can be interpreted using finite-part integrals.

The first numerical solutions were achieved by Bui (1977) and by Weaver (1977). They both considered static loadings of flat cracks. Bui (1977) used a piecewise-linear representation for $[\boldsymbol{u}]$ and obtained results for elliptical and square cracks. Weaver (1977) considered rectangular cracks, and used a piecewise-quadratic, discontinuous representation for $[\boldsymbol{u}]$, with appropriate square-root behaviour near the edges. This scheme has been refined considerably by Polch *et al.* (1987); they used a complicated algorithm to guarantee that their representation for $[\boldsymbol{u}]$ was continuously differentiable everywhere on the crack surface, and gave numerical results for static loadings of penny-shaped and elliptical cracks. Nishimura & Kobayashi (1988) have used cubic B-splines (they need two tangential derivatives), and gave results for the scattering of P-waves (compressional waves) by penny-shaped and square cracks.

4.2. *Integrodifferential equations*

Some authors have given direct numerical treatments of the integrodifferential equation for $[\boldsymbol{u}]$, analogous to (1.7). The method is essentially as follows: first, break up the crack surface into elements; on each element, approximate $[\boldsymbol{u}]$ by a constant vector, except perhaps in those elements bordering the crack edge; collocate at the centroid of each element; the tractions on flat elements corresponding to a constant displacement discontinuity can be calculated analytically (for elastodynamics, Jones (1985) reduces this calculation to the evaluation of a line integral around the element's perimeter). This method (the 'displacement discontinuity method') was worked out in two-dimensional elastostatics by Crouch (1976); see also Crouch & Starfield (1983, ch. 5 and 7).

Polynomial approximations on each element were used by Crawford & Curran (1982), whereas three-dimensional problems (such as the pressurized penny-shaped crack) were solved by Wiles & Curran (1982). More recently a similar method has been used by Budreck & Achenbach (1988) to treat the diffraction of a normally-incident P-wave by a flat elliptical crack.

4.3. *Hypersingular integral equations*

Ioakimidis (1982*a*, *b*) was apparently the first to derive hypersingular integral equations analogous to (2.2) for the static loading of a flat crack. Later (Ioakimidis 1987), he showed that the character of the finite-part integral will change if one collocates at the crack edge; see §2. The same hypersingular equations were derived by Takakuda *et al.* (1985) and by Lin'kov & Mogilevskaya (1986). The former authors also gave a numerical solution for an elliptical crack, using an expansion for $[\boldsymbol{u}]$ in terms of Chebyshev polynomials and an appropriate square root. Lin & Keer (1986, 1987) have solved the corresponding equations for a P-wave normally incident upon a penny-shaped crack buried in an elastic half-space, and upon a flat elliptical crack in an unbounded solid. In each case, they correctly modelled the edge behaviour, but their representation for $[\boldsymbol{u}]$ is only piecewise continuous. The use of such a representation for solving the hypersingular integral equations is clearly equivalent to using the method described in §4.2. However, it is our contention that smoother representations should be used; these are more readily accommodated into a scheme for solving the hypersingular integral equations.

Stephan (1986) and Costabel & Stephan (1987) have derived hypersingular integral equations for static loadings of arbitrary smooth cracks. They advocate a Galerkin scheme, with singular functions near the crack edge.

We believe that the spline-collocation method using at least a continuous representation for $[\boldsymbol{u}]$, together with Stephan's singular edge-functions, will be efficient and flexible, and are currently developing such a scheme. Preliminary experiments using Overhauser splines (these give a C^1 representation), gave excellent results, as we expected. Further development is in progress.

5. Closing comment

It is well known that improper (weakly-singular) integrals arise naturally in boundary integral formulations emanating from the divergence theorem, with a fundamental singular solution of the governing differential equation as principal ingredient. For some vector problems, e.g. in linear elasticity, this process leads to a special, 'more singular' integral which exists in the sense of the Cauchy principal value. Further, whenever it is necessary or advantageous, a normal derivative of such boundary integrals is taken, as is the case here with crack problems. Then, an even higher order singular integral, interpretable as a Hadamard finite-part integral, arises. Each step upwards in singular integrals requires more and more smoothness of the density function multiplying the singular kernels for such integrals even to exist. This fact, if not the hierarchy of the ever more singular integrals, may not be generally appreciated. (Mathematically, this hierarchy corresponds to pseudo-differential operators of higher and higher order; see, for example, Wendland 1982). Indeed, the role of this smoothness requirement in algorithms for computation with these integrals is rich ground for research. In any case, the strategy of regularization, i.e. to always remove, or avoid, singular integrals before beginning any computation, seems to be a burdensome and conservative posture which we hope, with the help of this paper, to turn about.

Above all, we thank I. R. Gonsalves for many helpful conversations on finite-part integrals. We also acknowledge the contributions of D. J. Shippy and Z. Jia to this work. Partial support was provided by the U.S. office of Naval Research under Contract N00014-86-K-0551. P. A. M. is grateful to the Kentucky EPSCoR Program, under the Computational Sciences Project, for supporting his visit to the University of Kentucky in June 1987. He also acknowledges receipt of a Fulbright Travel Grant.

Appendix A. Finite-part integrals

In this Appendix, $f(t)$ is defined for $a \leqslant t \leqslant b$ and $a < x < b$.

A. 1. *Cauchy principal-value integral*

Definition: For $f \in C^{0,\alpha}$

$$-\!\!\!\!\!\!\int_a^b \frac{f(t)}{x-t}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_a^{x-\epsilon} \frac{f(t)}{x-t}\,\mathrm{d}t + \int_{x+\epsilon}^b \frac{f(t)}{x-t}\,\mathrm{d}t\right\}. \tag{A 1}$$

Regularization: For $f \in C^1$,

$$-\!\!\!\!\!\!\int_a^b \frac{f(t)}{x-t}\,\mathrm{d}t = f(a)\ln(x-a) - f(b)\log(b-x) + \int_a^b f'(t)\ln|x-t|\,\mathrm{d}t. \tag{A 2}$$

Differentiation: For $f \in C^{1,\alpha}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}-\!\!\!\!\!\!\int_a^b \frac{f(t)}{x-t}\,\mathrm{d}t = \frac{f(a)}{x-a} + \frac{f(b)}{b-x} + -\!\!\!\!\!\!\int_a^b \frac{f'(t)}{x-t}\,\mathrm{d}t. \tag{A 3}$$

A. 2. *Two-sided finite-part integrals of order* 2

Definition: For $f \in C^{1,\alpha}$,

$$=\!\!\!\!\!\!\int_a^b \frac{f(t)}{(x-t)^2}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_a^{x-\epsilon} \frac{f(t)}{(x-t)^2}\,\mathrm{d}t + \int_{x+\epsilon}^b \frac{f(t)}{(x-t)^2}\,\mathrm{d}t - \frac{2f(x)}{\epsilon}\right\}. \tag{A 4}$$

Regularization: For $f \in C^{1,\alpha}$,

$$=\!\!\!\!\!\!\int_a^b \frac{f(t)}{(x-t)^2}\,\mathrm{d}t = -\frac{f(a)}{x-a} - \frac{f(b)}{b-x} - -\!\!\!\!\!\!\int_a^b \frac{f'(t)}{x-t}\,\mathrm{d}t. \tag{A 5}$$

Comparing (A. 3) and (A. 5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}-\!\!\!\!\!\!\int_a^b \frac{f(t)}{x-t}\,\mathrm{d}t = -=\!\!\!\!\!\!\int_a^b \frac{f(t)}{(x-t)^2}\,\mathrm{d}t, \tag{A 6}$$

i.e. the differentiation can be taken under the integral. Equation (A 6) is sometimes taken as the definition of a two-sided finite-part integral of order 2. It is straightforward to extend (A 4) to higher integer powers of $(x-t)$. Note that it is common to use the notation $⨍$ specifically for two-sided finite-part integrals of order 2.

A. 3. *One-sided Cauchy principal-value integrals*

Definition:

$$⨍_a^x \frac{f(t)}{x-t}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_a^{x-\epsilon} \frac{f(t)}{x-t}\,\mathrm{d}t + f(x)\ln\epsilon\right\}, \tag{A 7}$$

$$⨍_x^b \frac{f(t)}{x-t}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_{x+\epsilon}^{b} \frac{f(t)}{x-t}\,\mathrm{d}t - f(x)\ln\epsilon\right\}. \tag{A 8}$$

It can be shown that the sum of (A 7) and (A 8) is just the corresponding Cauchy principal-value integral of f over $[a, b]$. Note that the integrals in (A 7) and (A 8) are not independent of simple changes of variable.

A. 4. *One-sided (Hadamard) finite-part integrals*

Definition: Let μ be a real number, with $1 < \mu < 2$ (in §2, we required $\mu = \frac{3}{2}$).

$$⨍_a^x \frac{f(t)}{(x-t)^\mu}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_a^{x-\epsilon} \frac{f(t)}{(x-t)^\mu}\,\mathrm{d}t - \frac{f(x)}{(\mu-1)\,\epsilon^{\mu-1}}\right\}, \tag{A 9}$$

$$⨍_x^b \frac{f(t)}{(t-x)^\mu}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_{x+\epsilon}^{b} \frac{f(t)}{(t-x)^\mu}\,\mathrm{d}t - \frac{f(x)}{(\mu-1)\,\epsilon^{\mu-1}}\right\}. \tag{A 10}$$

These are the definitions orginally introduced by Hadamard (1923, §§80–87), in his investigations into linear hyperbolic partial differential equations. Later, these equations were solved in the context of supersonic aerodynamics (see, for example, Ward 1955, §3.4).

The integrals (A 9) and (A 10) are independent of simple changes of variable. Also, the definitions are easily extended to larger non-integer values of μ.

A. 5. *Multiple integrals*

It is possible to define Cauchy principal-value integrals and finite-part integrals for functions of two variables, over smooth surfaces in three dimensions. For principal-value integrals, see, for example, Kupradze *et al.* (1979, Ch. IV). For finite-part integrals over plane regions, see Takakuda *et al.* (1985).

Appendix B. Proof of theorem

After parametrizing Γ using (3.1), the double-layer potential (1.5) becomes

$$u^{\mathrm{sc}}(\mathrm{P}) = \int_{-1}^{1} U(t)\,K(R)\,L(t,\mathrm{P})\,\mathrm{d}t,$$

where
$$K(R) = -(\tfrac{1}{4}ik/R)\,H_1^{(1)}(kR),$$
$$L(t, \mathrm{P}) = -y'(t)\,(x(t)-X)+x'(t)\,(y(t)-Y),$$
$$\mathrm{P} = (X, Y),$$
$$R^2 = (x(t)-X)^2+(y(t)-Y)^2.$$

Set
$$(X, Y) = (x(s), y(s))+(N/v)\,(-y'(s), x'(s)),$$
where $v^2 = (x'(s))^2+(y'(s))^2$ and N is the distance along the normal at $\mathrm{p} = (x(s), y(s))$ to P. We consider
$$\lim_{N\to 0}\frac{\partial}{\partial N}u^{\mathrm{sc}}(\mathrm{P}).$$

Case I. p *not at an edge*

Suppose $\mathrm{p}\notin\partial\Gamma$. Then, it is clear that we only need to consider a small neighbourhood of the singularity at $t = s$. Moreover, we can also replace $K(R)$ by its asymptotic approximation for small R, namely
$$-\tfrac{1}{2}\pi^{-1}(R^{-2}-\tfrac{1}{2}k^2\ln R),$$
since the difference leads to non-singular integrals. Straightforward computations, and properties of classical single-layer potentials, show that the term involving $\ln R$ is continuously differentiable in N at $N = 0$. So, we can go on to consider
$$\frac{-1}{2\pi}\int_{s-a}^{s+a}U(t)\,L(t, \mathrm{P})\,R^{-2}\,\mathrm{d}t \tag{B 1}$$
where $a > 0$. For small $|t-s|$ and N, we have
$$\frac{1}{R^2}\simeq\frac{1-(t-s)\,w}{N^2+(t-s)^2v^2}$$
where $w = (x'(s)\,x''(s)+y'(s)\,y''(s))/v^2$. Again, it is straightforward to show that this approximation can be used in (B 1); the difference is continuously differentiable at $N = 0$. Now, for small $|t-s|$,
$$L(t, \mathrm{P})\,(1-(t-s)\,w)\approx f(t, s)+Ng(t, s),$$
where $f\approx A(t-s)^2, g\approx -v+B(t-s)^2$ and A and B are constants. With these approximations, we readily see that the only term which gives difficulty is the leading term in g. Subtracting this, we are led to consider
$$\frac{1}{2\pi}\int_{s-a}^{s+a}U(t)\frac{Nv}{N^2+(t-s)^2v^2}\,\mathrm{d}t = \frac{D(N)}{2\pi v}, \text{ say.} \tag{B 2}$$

So far, we have only assumed that $U(t)$ is continuous. We now assume that $U(t)\in C^{1,\alpha}$, i.e. that
$$|U'(t)-U'(s)|\leqslant A\,|t-s|^\alpha$$
for some A and α, with $0 < \alpha\leqslant 1$. This condition cannot be weakened.

Example $U(t) = |t-s| \in C^{0,1}$. Then,

$$D(N) = 2Nv^2 \int_0^a x(N^2+x^2v^2)^{-1}\,dx = N\ln(1+a^2v^2/N^2),$$

$$D'(N) = \ln(1+a^2v^2/N^2) - 2a^2v^2(N^2+a^2v^2)^{-1}$$

and the first term is unbounded as $N \to 0$.

Integrating (B 2) by parts gives, for $N > 0$,

$$D(N) = v\{U(s+a)+U(s-a)\}\arctan(av/N) + D_1(N),$$

where $$D_1(N) = -v\int_{s-a}^{s+a}(U'(t)-U'(s))\arctan((t-s)\,v/N)\,dt.$$

Hence, $$D'(N) = -\{U(s+a)+U(s-a)\}\,av^2(N^2+a^2v^2)^{-1} + D_1'(N), \tag{B 3}$$

where $$D_1'(N) = v^2\int_{s-a}^{s+a}(U'(t)-U'(s))\,(t-s)\,(N^2+(t-s)^2v^2)^{-1}\,dt.$$

The first term in (B 3) is continuous at $N = 0$, whereas

$$D_1'(0) = \int_{s-a}^{s+a}(U'(t)-U'(s))\,(t-s)^{-1}\,dt.$$

An application of Hölder's inequality shows that

$$|D_1'(N) - D_1'(0)| \leqslant BN^\beta$$

where $0 < \beta < \alpha$. Thus, $D'(N)$ is continuous at $N = 0$, whence

$$D'(0) = -\frac{1}{a}\{U(s+a)+U(s-a)\} + \int_{s-a}^{s+a}(U'(t)-U'(s))\,(t-s)^{-1}\,dt. \tag{B 4}$$

Return now to (B 2); differentiate with respect to N and formally set $N = 0$ to give

$$D'(0) = \int_{s-a}^{s+a}\frac{U(t)}{(t-s)^2}\,dt.$$

Of course, this integral does not exist. However, from (A 5) we have

$$\int_{s-a}^{s+a}\frac{U(t)}{(t-s)^2}\,dt = -\frac{U(s-a)}{a} - \frac{U(s+a)}{a} + \int_{s-a}^{s+a}\frac{U'(t)}{t-s}\,dt. \tag{B 5}$$

Because the range of integration is symmetric about $t = s$, we see that the right-hand sides of (B 4) and (B 5) are equal. Thus, the theorem is proved for $\mathrm{p} \in \Gamma$ but $\mathrm{p} \notin \partial\Gamma$.

Case II. p *at an edge*

Consider the edge at $s = -1$. Extend Γ smoothly, so that $\boldsymbol{n}$ is defined at the edge. Near $s = -1$, we have

$$U(t) \approx A\{v(1+t)\}^{\frac{1}{2}} + B\{v(1+t)\}^{\frac{3}{2}} \tag{B 6}$$

(cf. (3.2)), where v is evaluated at $s = -1$. The analysis proceeds as in case I, except the range of integration is $-1 \leqslant t \leqslant -1+a$; the point of departure is (cf. (B 2))

$$D(N) = \int_{-1}^{-1+a} U(t)\, N v^2 (N^2 + (t+1)^2 v^2)^{-1}\, \mathrm{d}t. \tag{B 7}$$

We replace $u(t)$ by the approximation (B 6); this gives

$$D(N) = 2vN^{\frac{1}{2}}\{AI_{\mathrm{R}} + BN(X - I_{\mathrm{I}})\}$$

where

$$I_{\mathrm{R}} + \mathrm{i}I_{\mathrm{I}} = \int_0^X \frac{x^2+\mathrm{i}}{x^4+1}\,\mathrm{d}x = -\tfrac{1}{2}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi} \ln\left(\frac{1 - X^2 - \mathrm{i}2^{\frac{1}{2}}X}{1 + X^2 - 2^{\frac{1}{2}}X}\right)$$

and $X = (av/N)^{\frac{1}{2}}$. So, for small N, we find that

$$D(N) = -2Nv\{A(av)^{-\frac{1}{2}} - B(av)^{\frac{1}{2}}\} + O(N^2)$$

and

$$D'(0) = -2v\{A(av)^{-\frac{1}{2}} - B(av)^{\frac{1}{2}}\}. \tag{B 8}$$

Returning to (B 7) (with (B 6)), differentiate with respect to N and formally set $N = 0$ to give

$$D'(0) = \int_{-1}^{-1+a} \{A(v(1+t))^{\frac{1}{2}} + B(v(1+t))^{\frac{3}{2}}\}(1+t)^{-2}\,\mathrm{d}t$$

$$= Av^{\frac{1}{2}} \int_{-1}^{-1+a} (1+t)^{-\frac{3}{2}}\,\mathrm{d}t + 2Bv(av)^{\frac{1}{2}}. \tag{B 9}$$

The integral does not exist. However, from (A 10),

$$⨎_{-1}^{-1+a} (1+t)^{-\frac{3}{2}}\,\mathrm{d}t = -2a^{-\frac{1}{2}},$$

Thus, if the integral in (B 9) is interpreted as a one-sided finite-part integral of order $\frac{3}{2}$, we obtain agreement between (B 8) and (B 9), and the theorem is proved.

References

Arnold, D. N. & Wendland, W. L. 1985 The convergence of spline collocation for strongly elliptic equations on curves. *Numer. Math.* **47**, 317–341.

Brandão, M. P. 1987 Improper integrals in theoretical aerodynamics: the problem revisited. *AIAA Jl* **25**, 1258–1260.

Budiansky, B. & Rice, J. R. 1979 An integral equation for dynamic elastic response of an isolated 3-D crack. *Wave Motion* **1**, 187–192.

Budreck, D. E. & Achenbach, J. D. 1988 Scattering from three-dimensional planar cracks by the boundary integral equation method. *J. appl. Mech.* **55**, 405–412.

Bui, H. D. 1977 An integral equations method for solving the problem of a plane crack of arbitrary shape. *J. Mech. Phys. Solids* **25**, 29–39.

Cassot, F. 1975 Contribution à l'étude de la diffraction par un écran mince. *Acustica* **34**, 64–71.

Costabel, M. & Stephan, E. 1987 An improved boundary element Galerkin method for three-dimensional crack problems. *Integ. Equats oper. Theory* **10**, 467–504.

Crawford, A. M. & Curran, J. H. 1982 Higher-order functional variation displacement discontinuity elements. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **19**, 143–148.

Crouch, S. L. 1976 Solution of plane elasticity problems by the displacement discontinuity method. *Int. J. numer. Meth. Engng* **10**, 301–343.

Crouch, S. L. & Starfield, A. M. 1983 *Boundary element methods in solid mechanics*. London: George Allen & Unwin.

de Boor, C. 1978 *A practical guide to splines*. New York: Springer-Verlag.

Filippi, P. & Dumery, G. 1969 Étude théorique et numérique de la diffraction par un écran mince. *Acustica* **21**, 343–350.

Frenkel, A. 1983 A Chebyshev expansion of singular integrodifferential equations with a $\partial^2 \ln|s-t|/\partial s \partial t$ kernel. *J. comput. Phys.* **51**, 335–342.

Golberg, M. A. 1983 The convergence of several algorithms for solving integral equations with finite-part integrals. *J. integ. Equats* **5**, 329–340.

Golberg, M. A. 1985 The convergence of several algorithms for solving integral equations with finite-part integrals. II. *J. integ. Equats* **9**, 267–275.

Guidera, J. T. & Lardner, R. W. 1975 Penny-shaped cracks. *J. Elast.* **5**, 59–73.

Hadamard, J. 1923 *Lectures on Cauchy's problem in linear partial differential equations*. New Haven: Yale University Press.

Ioakimidis, N. I. 1982*a* Application of finite-part integrals to the singular integral equations of crack problems in plane and three-dimensional elasticity. *Acta mech.* **45**, 31–47.

Ioakimidis, N. I. 1982*b* A natural approach to the introduction of finite-part integrals into crack problems of three-dimensional elasticity. *Engng. Fract. Mech.* **16**, 669–673.

Ioakimidis, N. I. 1985 Exact expression for a two-dimensional finite-part integral appearing during the numerical solution of crack problems in three-dimensional elasticity. *Communs appl. numer. Meth.* **1**, 183–189.

Ioakimidis, N. I. 1987 Validity of the hypersingular integral equation of crack problems in three-dimensional elasticity along the crack boundaries. *Engng. Fract. Mech.* **26**, 783–788.

Jones, D. S. 1985 Boundary integrals in elastodynamics. *IMA J. appl. Math.* **34**, 83–87.

Kaya, A. C. & Erdogan, F. 1987 On the solution of integral equations with strongly singular kernels. *Q. appl. Math.* **45**, 105–122.

Kupradze, V. D., Gegelia, T. G., Basheleishvili, M. O. & Burchuladze, T. V. 1979 *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*. Amsterdam: North-Holland.

Kutt, H. R. 1975 The numerical evaluation of principal value integrals by finite-part integration. *Numer. Math.* **24**, 205–210.

Le Van, A. & Royer, J. 1986 Integral equations for three-dimensional problems. *Int. J. Fract.* **31**, 125–142.

Lin, W. & Keer, L. M. 1986 Scattering by a horizontal subsurface penny-shaped crack. *Proc. R. Soc. Lond.* A **408**, 277–294.

Lin, W. & Keer, L. M. 1987 Scattering by a planar three-dimensional crack. *J. acoust. Soc. Am.* **82**, 1442–1448.

Lin'kov, A. M. & Mogilevskaya, S. G. 1986 Finite-part integrals in problems of three-dimensional cracks. *Appl. Math. Mech. (PMM)* **50**, 652–658.

Linz, P. 1985 On the approximate computation of certain strongly singular integrals. *Computing* **35**, 345–353.

Lo, K. K. 1979 Three-dimensional crack in the interior of a half-space. *J. Elast.* **9**, 435–439.

Macaskill, C. & Tuck, E. O. 1977 Evaluation of the acoustic impedance of a screen. *J. Aust. math. Soc.* B **20**, 46–61.

Mangler, K. W. 1952 Improper integrals in theoretical aerodynamics. *Aeronaut. Res. Coun. Current Pap.* no. 94.

Maue, A. W. 1949 Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung. *Z. Phys.* **126**, 601–618.

Morrison, J. A. 1979 Integral equations for electromagnetic scattering by perfect conductors with two-dimensional geometry. *Bell Syst. tech. J.* **58**, 409–425.

Nedelec, J. C. 1978 Approximation par potential de double couche du problème due Neumann extérieur. *C.r. Acad. Sci. Paris* A **286**, 103–106.

Nedelec, J. C. 1982 Integral equations with non-integrable kernels. *Integ. Equats oper. Theory* **5**, 562–572.

Nishimura, N. & Kobayashi, S. 1988 An improved boundary integral equation method for crack problems. In *Advanced boundary element methods* (ed. T. A. Cruse), pp. 279–286. Berlin: Springer-Verlag.

Paget, D. F. 1981 The numerical evaluation of Hadamard finite-part integrals. *Numer. Math.* **36**, 447–453.

Polch, E. Z., Cruse, T. A. & Huang, C. J. 1987 Traction BIE solutions for flat cracks. *Comput. Mech.* **2**, 253–267.

Sayhi, M. N., Ousset, Y. & Verchery, G. 1981 Solution of radiation problems by collocation of integral formulations in terms of single and double layer potentials. *J. Sound Vib.* **74**, 187–204.

Sládek, V. & Sládek, J. 1984 Transient elastodynamic three-dimensional problems in cracked bodies. *Appl. math. Modelling* **8**, 2–10.

Stephan, E. P. 1986 A boundary integral equation method for three-dimensional crack problems in elasticity. *Math. Meth. appl. Sci.* **8**, 609–623.

Stephan, E. P. & Wendland, W. L. 1984 An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. *Applicable Analysis* **18**, 183–219.

Takakuda, K., Koizumi, T. & Shibuya, T. 1985 On integral equation methods for crack problems. *Bull. Jap. Soc. mech. Engrs* **28**, 217–224.

Ward, G. N. 1955 *Linearized theory of steady high-speed flow.* Cambridge University Press.

Weaver, J. 1977 Three-dimensional crack analysis. *Int. J. Solids Struct.* **13**, 321–330.

Wendland, W. L. 1982 On applications and the convergence of boundary integral methods. In *Treatment of integral equations by numerical methods* (ed. C. T. H. Baker & G. F. Miller), pp. 463–476. London: Academic Press.

Wickham, G. R. 1982 Integral equations for boundary value problems exterior to open arcs and surfaces. In *Treatment of integral equations by numerical methods* (ed. C. T. H. Baker & G. F. Miller), pp. 421–431. London: Academic Press.

Wiles, T. D. & Curran, J. H. 1982 A general 3-D displacement discontinuity method. *Proc. 4th int. Conf. numer. Meth. Geomech.* **1**, 103–111.

Zakharov, E. V. & Sobyanina, I. V. 1986 On one-dimensional integrodifferential equations for problems of diffraction by screens. *U.S.S.R. comput. Maths. math. Phys.* **26**, 2, 194–198.