End-point behaviour of solutions to hypersingular integral equations

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We consider one-dimensional hypersingular integral equations over finite intervals; the integral must be interpreted as a finite-part integral. Such equations arise naturally in various physical situations, involving thin rigid bodies or cracks; examples are given. A method is developed for determining the behaviour of the solution to a hypersingular integral equation near the end-points of the interval of integration. The method uses the Mellin transform. Several examples are worked out in detail.

1. Introduction

Many two-dimensional boundary-value problems involving thin obstacles can be reduced to hypersingular integral equations of the general form

$$(H+K)f = v(x), \quad 0 < x < a, \tag{1.1}$$

where H and K are linear operators, v is a known function and f is to be determined. H is the hypersingular integral operator defined by

$$(Hf)(x) = \frac{1}{2\pi} \times\!\!\!\!\!\!\int_0^a \frac{f(t)}{(x-t)^2}\,\mathrm{d}t, \tag{1.2}$$

where the integral must be interpreted as a finite-part integral,

$$\times\!\!\!\!\!\!\int_0^a \frac{f(t)}{(x-t)^2}\,\mathrm{d}t = \lim_{\epsilon\to 0}\left\{\int_0^{x-\epsilon} \frac{f(t)}{(x-t)^2}\,\mathrm{d}t + \int_{x+\epsilon}^a \frac{f(t)}{(x-t)^2}\,\mathrm{d}t - \frac{2f(x)}{\epsilon}\right\}; \tag{1.3}$$

here, $0 < x < a$ and $f(x)$ is required to have a Hölder-continuous derivative, $f \in C^{1,\alpha}$. Equivalently, the finite-part integral (1.3) can be defined in terms of a Cauchy principal-value integral by

$$\times\!\!\!\!\!\!\int_0^a \frac{f(t)}{(x-t)^2}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x} -\!\!\!\!\!\!\int_0^a \frac{f(t)}{x-t}\,\mathrm{d}t, \tag{1.4}$$

subject to the same smoothness restrictions on f. For further properties of finite-part integrals, and a discussion on numerical methods for treating (1.1) (see Martin & Rizzo 1989).

We assume that K is given by

$$(Kf)(x) = \lambda(x)f(x) + \frac{1}{2\pi}\int_0^a L(x,t)f(t)\,\mathrm{d}t, \tag{1.5}$$

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where $\lambda(x)$ is a given function and $L(x,t)$ is a given kernel. Let

$$\Omega = \{(x,t): 0 \leqslant x \leqslant a, 0 \leqslant t \leqslant a\}.$$

We distinguish two cases, according to properties of λ and L over Ω.

Case I. $L(x,t)$ is integrable over the whole square Ω with (at worst) a Cauchy singularity along the diagonal $x = t$. Often, L has only a logarithmic singularity at $x = t$, or is even continuous everywhere in Ω.

Case II. $\lambda \equiv 0$ and $L(x,t)$ is continuous everywhere in Ω, except for a non-integrable singularity at one corner, which we always take to be $x = t = 0$: $L(x,x) \sim x^{-2}$ as $x \to 0$. Often, L is very smooth away from this corner.

Case I arises whenever a thin obstacle Γ is embedded in an unbounded medium. Typically, (1.1) is obtained by parametrizing a boundary integral equation, which is itself obtained using a fundamental solution (Green's function) for the governing partial differential equation in an unbounded region. Case I also arises if the host medium has boundaries whose effects can be incorporated by introducing a different fundamental solution, provided that Γ does not meet these boundaries. For example, if the medium occupies a semi-infinite region $x > 0$ with a plane boundary at $x = 0$, then the effects of this boundary can often be incorporated by using an appropriate system of images, leading to very smooth terms in $L(x,t)$. Then, we have Case I if Γ lies completely in $x > 0$ but we have Case II if Γ meets the boundary at $x = 0$. In the latter situation, the image system usually gives rise to the strong singularity at the corner of Ω.

In §§2 and 3, we give many examples of hypersingular integral equations, all of which fall into Case I (§2) or Case II (§3).

We remark here that singular integral equations (i.e. equations with Cauchy principal-value integrals) are also classified in a similar manner. Thus, Case I corresponds to the simplest situation, as described by Muskhelishvili (1953). Case II corresponds to 'generalized Cauchy kernels', as described by Erdogan *et al.* (1973) and Duduchava (1982).

Our motivation for distinguishing the two cases is twofold. First, we know that to have a unique solution of (1.1) we must impose two supplementary conditions on the solution f. For Case I, these are invariably

$$f(0) = f(a) = 0. \tag{1.6}$$

Other conditions are possible, but usually the physics of the original problem requires (1.6). For Case II, the supplementary conditions are usually taken to be

$$f(a) = 0 \quad \text{and} \quad f(0) \text{ is bounded}. \tag{1.7}$$

Secondly, we are interested in the 'edge behaviour' of f, i.e. the asymptotic behaviour of $f(x)$ near the end-points $x = 0$ and $x = a$. For Case I, with (1.6), we expect (for smooth, bounded v, at least)

$$f(x) \sim f_0 \sqrt{x} \quad \text{as} \quad x \to 0+, \tag{1.8}$$

and

$$f(x) \sim g_0 \sqrt{(a-x)} \quad \text{as} \quad x \to a-, \tag{1.9}$$

where f_0 and g_0 are constants. In fact, for square-integrable v, the leading asymptotic edge behaviour is independent of the operator K and is always given by (1.8) and (1.9). For Case II, the situation is different. With (1.7) we still expect (1.9), but intuition does not readily provide the behaviour near $x = 0$; it is governed by the strong singularity in L. The situation for singular integral equations with generalized Cauchy kernels is similar.

Actually, the edge behaviour can be extracted from the governing integral equation itself. To do this, our principal tool is the Mellin transform. These are often used to find asymptotic expansions of integrals. For example, Bleistein & Handelsman (1975, ch. 4) show how to obtain asymptotic approximations of

$$I(\lambda) = \int_0^\infty h(\lambda t) f(t)\, \mathrm{d}t \tag{1.10}$$

for small or large values of $|\lambda|$, where h and f are known functions. However, we can view (1.10) as an integral equation for f: given I and h, we can use Mellin transforms to find the asymptotic behaviour of $f(t)$ near $t = 0$. We propose to use this method on (1.1).

Of course, Mellin transforms have long been used to obtain exact solutions to certain simple integral equations (see, for example, Fox 1935). More recently, they have been used by Rose (1982) to obtain the behaviour of $f(x;\lambda)$ as $\lambda \to 0$, where f solves a particular Fredholm integral equation of the second kind over $0 \leqslant x \leqslant 1$, with λ occurring as a parameter in the kernel. Costabel & Stephan (1983*a*, *b*) have used Mellin transforms to obtain the corner behaviour of solutions to boundary integral equations over piecewise smooth closed curves. The integral equations arise from the interior Neumann problem for Laplace's equation, using a double-layer representation for the solution (cf. §2.1 below).

In the present paper, we develop the Mellin-transform method. We give some general results applicable to a wide class of operators K, and obtain more precise results for integral operators of Mellin convolution type. Most of the physical applications of hypersingular integral equations described in §§2 and 3 are analysed in §§6 and 7 respectively. Section 4 is devoted to some general results on Mellin transforms, whilst §5 is concerned with the dominant equation, $Hf = v$.

Most of the result obtained are not new. However, they have been derived here in a systematic manner which should find application elsewhere. Moreover, it is hoped to develop the method further so as to treat three-dimensional problems, where one has an integral equation over a two-dimensional region with a piecewise-smooth boundary, and the behaviour of the solution near corners is sought.

2. Some examples of hypersingular integral equations: Case I

In this section, we give several examples of Case I, taken from potential theory, acoustics, hydrodynamics and elastostatics. In all the examples, f satisfies (1.6); in all but the last (§2.5), the function $\lambda(x)$ in (1.5) is identically zero. Representative references to the literature are also given.

2.1. *Potential flow past curved plates*

Consider the potential flow of an ideal fluid past a rigid plate Γ. Suppose that the corresponding velocity potential is

$$\phi_0 + \phi,$$

where ϕ_0 is the (known) potential of the flow in the absence of Γ, ϕ is due to its presence and both are harmonic. Introduce the fundamental solution

$$G_0(P;Q) = \tfrac{1}{2}\ln\{(x-\xi)^2 + (y-\eta)^2\},$$

where P and Q have cartesian coordinates (x, y) and (ξ, η), respectively. Assuming that ϕ decays at large distances and is bounded near the edges of the plate, Green's theorem gives

$$\phi(P) = \frac{1}{2\pi}\int_{\Gamma}[\phi(q)]\frac{\partial}{\partial n_q}G_0(P,q)\,\mathrm{d}s_q, \tag{2.1}$$

where P is a point in the fluid, $\partial/\partial n_q$ denotes normal differentiation at $q \in \Gamma$ and $[\phi(q)]$ is the discontinuity in ϕ across Γ at q. The boundary condition on Γ is

$$\partial\phi/\partial n_p = -\partial\phi_0/\partial n_p,$$

whence (2.1) gives

$$\frac{1}{2\pi}\frac{\partial}{\partial n_p}\int_{\Gamma}[\phi(q)]\frac{\partial}{\partial n_q}G_0(p,q)\,\mathrm{d}s_q = -\frac{\partial\phi_0}{\partial n_p}, \quad p\in\Gamma.$$

We can interchange the integration with the normal differentiation at p, provided we interpret the integral as a finite-part integral (this was proved by Martin & Rizzo (1989) for the Helmholtz equation; see §2.2 below). Thus,

$$\frac{1}{2\pi}⨎_{\Gamma}[\phi(q)]\frac{\partial^2}{\partial n_p\,\partial n_q}G_0(p,q)\,\mathrm{d}s_q = -\frac{\partial\phi_0}{\partial n_p}, \quad p\in\Gamma. \tag{2.2}$$

This hypersingular integral equation is to be solved subject to

$$[\phi] = 0 \quad \text{at the two edges of } \Gamma. \tag{2.3}$$

The kernel in (2.2) is given by

$$\partial^2 G_0/\partial n_p\,\partial n_q = -\mathcal{N}/R^2 + 2\Theta/R^4, \tag{2.4}$$

where

$$\mathcal{N} = \boldsymbol{n}(p)\cdot\boldsymbol{n}(q), \quad \Theta = (\boldsymbol{n}(p)\cdot\boldsymbol{R})\,(\boldsymbol{n}(q)\cdot\boldsymbol{R}), \quad \boldsymbol{R} = (x-\xi, y-\eta), \quad R = |\boldsymbol{R}|$$

and $\boldsymbol{n}(p) = (n_1^p, n_2^p)$ is the unit normal vector at $p \in \Gamma$.

Next, we parametrize Γ as

$$\Gamma = \{(x(t), y(t)) : 0 \leqslant t \leqslant a\}.$$

Let p and q correspond to parameters s and t respectively. Then

$$\boldsymbol{n}(q) = (n_1^q, n_2^q) = (-y'(t), x'(t))/w(t),$$

with $w(t) = \sqrt{((x'(t))^2 + (y'(t))^2)}$. If we expand the kernel (2.4) for small $|s-t|$, we find that

$$\frac{\partial^2 G_0}{\partial n_p\,\partial n_q} = \frac{-1}{w(s)\,w(t)}\left\{\frac{1}{(s-t)^2} + \mathscr{A}(s) + O(|s-t|)\right\} \tag{2.5}$$

where the singular term in (2.5) comes solely from the first term in (2.4),

$$12w^4\mathscr{A} = 6(x'y''-x''y')^2 - 3w^2(x''^2+y''^2) + 2w^2(x'x'''+y'y''')$$

and we have assumed that Γ is a C^3 curve. It follows that (2.2) can be written as (1.1), wherein $f(t) = [\phi(q)]$ and

$$v(s) = -y'(\partial\phi_0/\partial x) + x'(\partial\phi_0/\partial y).$$

Moreover the edge conditions (2.3) imply that f must satisfy (1.6). This provides an example of Case I, with a continuous kernel L.

In particular, suppose that Γ is flat, occupying the segment $0 \leqslant x \leqslant a$ of the x-axis. Then $\mathscr{N} \equiv 1$, $\Theta \equiv 0$ and we obtain

$$\frac{1}{2\pi} ⨎_0^a \frac{f(t)}{(x-t)^2} \mathrm{d}t = v(x), \quad 0 < x < a, \tag{2.6}$$

where $$v = \partial\phi_0/\partial y \quad \text{evaluated on} \quad y = 0. \tag{2.7}$$

Equation (2.6) is the simplest hypersingular integral equation over a finite interval; we call it the *dominant equation*.

Ioakimidis (1982) has derived the dominant equation (2.6) in plane elastostatics, for a straight pressurized crack in an unbounded solid. In this context, (2.6) is also equivalent (using (1.4)) to *Bueckner's equation* (Bueckner 1973, p. 268).

2.2. *Acoustic scattering by a hard plate*

Consider a plate Γ immersed in a compressible fluid. A time-harmonic incident sound wave (with velocity potential ϕ_0) is scattered by the plate; the corresponding scattered wave (with potential ϕ) must satisfy a radiation condition at large distances. The governing differential equation is the Helmholtz equation. All other conditions are as in §2.1. The appropriate fundamental solution is

$$G_1(P;Q) = -\tfrac{1}{2}\mathrm{i}\pi H_0^{(1)}(kR),$$

where $H_n^{(1)}$ is a Hankel function and k is the given wavenumber, assumed real and positive. Since

$$G_1 \sim \ln R = G_0 \quad \text{as} \quad R \to 0,$$

we obtain the integral equation (2.2), with G_0 replaced by G_1. The kernel is (cf. (2.4))

$$\frac{\partial^2 G_1}{\partial n_p \, \partial n_q} = -\frac{\mathscr{N}}{R^2}\{\mathrm{i}\pi(\tfrac{1}{2}kR)\, H_1^{(1)}(kR)\} + \frac{2\Theta}{R^4}\{\mathrm{i}\pi(\tfrac{1}{2}kR)^2 H_2^{(1)}(kR)\};$$

both expressions in braces approach unity as $kR \to 0$.

After parametrization of Γ, we obtain another example of Case I, with a logarithmic singularity in $L(x,t)$. In particular, for a straight hard strip along $0 \leqslant x \leqslant a$, we obtain

$$\frac{1}{2\pi} ⨎_0^a f(t) \left\{ \frac{1}{(s-t)^2} + L(s,t) \right\} \mathrm{d}t = v(s), \quad 0 < s < a, \tag{2.8}$$

where v is given by (2.7) and

$$L(s,t) = \frac{\mathrm{i}\pi k}{2} \frac{H_1^{(1)}(k|s-t|)}{|s-t|} - \frac{1}{(s-t)^2} \sim -\tfrac{1}{2}k^2 \ln|s-t| \quad \text{as} \quad |s-t| \to 0.$$

This problem for curved Γ is discussed further by Martin & Rizzo (1989). Numerical results are given by Frenkel (1983).

2.3. *Diffraction of water waves by a submerged barrier*

Consider a thin impermeable plate Γ submerged beneath the free surface of deep water. The plate is oscillated in calm water, or it is held fixed while a given surface wave is incident upon it. This leads to a boundary-value problem for a radiation potential ϕ in the half-plane $y > 0$, where ϕ is harmonic and satisfies the boundary condition

$$K\phi + \partial\phi/\partial y = 0 \quad \text{on} \quad y = 0,$$

the mean free surface. Here, $K = \omega^2/g$, ω is the frequency of oscillation and g is the acceleration due to gravity. The appropriate fundamental solution is

$$G_2(P;Q) = \ln R - \tfrac{1}{2}\ln(X^2+Y^2) - 2\Phi_0(X,Y), \tag{2.9}$$

where
$$\Phi_0(X,Y) = \int_0^\infty \mathrm{e}^{-kY}\cos kX \frac{\mathrm{d}k}{k-K},$$

$X = x-\xi$, $Y = y+\eta$ and the contour of integration is indented below the pole of the integrand so that G_2 satisfies the radiation condition. Since

$$G_2 \sim \ln R = G_0 \quad \text{as} \quad R \to 0, \quad \text{provided} \quad Y > 0,$$

we again obtain the integral equation (2.2), with G_0 replaced by G_2 (the right-hand side is equal to $\partial\phi/\partial n_p$ and is prescribed on Γ). The kernel is given by (cf. (2.4))

$$\frac{\partial^2 G_2}{\partial n_p\,\partial n_q} = -\frac{\mathcal{N}}{R^2} + \frac{2\Theta}{R^4} + 2K(n_1^p n_2^q - n_2^p n_1^q)\frac{\partial\Phi_0}{\partial X} - \mathcal{N}\left\{\frac{Y^2-X^2}{(X^2+Y^2)^2} + \frac{2KY}{X^2+Y^2} + 2K^2\Phi_0(X,Y)\right\}.$$

After parametrization of Γ, we obtain another example of Case I, with a continuous kernel. In particular, consider a flat plate, so that Γ is parametrized as

$$x(t) = t\sin\alpha, \quad y(t) = d + t\cos\alpha, \quad 0 \leqslant t \leqslant a,$$

where the plate is inclined at an angle α to the vertical ($|\alpha| \leqslant \frac{1}{2}\pi$) and d is the distance between the top edge and the mean free surface. The integral equation (2.2) can be written as (2.8), where

$$L(s,t) = \frac{Y^2-X^2}{(X^2+Y^2)^2} + \frac{2KY}{X^2+Y^2} + 2K^2\ {}_0(X,Y), \tag{2.10}$$

$X = (s-t)\sin\alpha$ and $Y = (s+t)\cos\alpha + 2d$. Numerical solutions have been obtained with this formulation by Higson (1988), using a boundary element method.

Three special cases are of interest. First, if the plate is deeply submerged, we can let $d \to \infty$ whence $L \to 0$ and we recover (2.6). Second, if the free surface at $y = 0$ is replaced by a rigid wall, we can set $K = 0$, whence

$$L(s,t) = (Y^2 - X^2)/(X^2+Y^2)^2. \tag{2.11}$$

Third, if the plate is vertical, we have $X = 0$ and $Y = s+t+2d$, whence

$$L(s,t) = (1/Y^2) + (2K/Y) + 2K^2\Phi_0(0,Y). \tag{2.12}$$

We note that the problems of wave radiation and scattering by submerged vertical barriers can be solved exactly (see Evans 1970). Goswami (1982) has also given a hypersingular integral equation for the scattering problem, although he does not comment on the strong singularity in his kernel.

2.4. *A pressurized crack in an elastic half-space*

Consider a homogeneous isotropic elastic half-plane, $x > 0$, with a stress-free boundary at $x = 0$. The solid contains a flat crack perpendicular to $x = 0$, with its two edges at $(d, 0)$ and $(a+d, 0)$; it is opened by a prescribed pressure, $p(x)$. The corresponding elastostatic boundary-value problem can be reduced to (2.8), where $v(s) = -(1-\nu)\,p(s+d)/\mu$, ν is Poisson's ratio, μ is the shear modulus, $f(t)$ is the discontinuity in the normal component of the displacement across the crack at $x = d+t$ and

$$L(s,t) = -(1/Y^2) + 12(s+d)(t+d)/Y^4 \tag{2.13}$$

with $Y = s+t+2d$, as in (2.12). The hypersingular integral equation for this problem was given by Kaya & Ergodan (1987). It can be derived using Melan's solution (1932; see Telles & Brebbia 1981) for a point force in a half-plane. Further crack problems, leading to similar integral equations, are treated by Kaya & Erdogan (1987) and by Bueckner (1973, §5.4).

2.5. *Reinforced cracks*

Consider an unbounded elastic solid containing a flat crack, whose faces are connected by distributed linear springs. The crack is opened by a prescribed tension perpendicular to the crack. The corresponding boundary-value problem was analysed by Rose (1987), who reduced it to a singular integral equation; subsequently, Hori & Nemat-Nasser (1990) reduced it to

$$(Hf)(x)+\lambda(x)f(x) = v(x), \quad 0 < x < a. \tag{2.14}$$

Here, f is as in §2.4, v is proportional to the prescribed tension and λ is known in terms of the springs. A physical constraint is that both λ and v should be non-positive.

Equation (2.14) is also equivalent to *Prandtl's equation*, which arises in aerodynamic theory (see, for example, Muskhelishvili 1953, §121). It also arises in the scattering of low-frequency sound waves by a thin elastic plate (Cuminato *et al.* 1990, eqn (6.3)).

3. Some examples of hypersingular integral equations: Case II

In this section, we give several examples of Case II, obtained as limiting cases of some examples in §2. In all the examples, f satisfies (2.8) and (1.7).

3.1. *Rigid plate meeting a rigid wall*

Consider potential flow past a finite flat rigid plate attached to an infinite rigid wall. This is a special case of the problem discussed in §2.3. Set $d = 0$ in (2.11), whence

$$L(s,t) = \frac{(s^2+t^2)\cos 2\alpha+2st}{(s^2+t^2+2st\cos 2\alpha)^2}, \tag{3.1}$$

where α is the angle between the plate and the normal to the wall ($|\alpha| < \frac{1}{2}\pi$). If the plate is perpendicular to the wall, we obtain

$$L(s,t) = 1/(s+t)^2. \tag{3.2}$$

3.2. *Diffraction by a vertical surface-piercing barrier*

Set $d = 0$ in (2.12) to give

$$L(s,t) = \frac{1}{(s+t)^2}+\frac{2K}{s+t}+2K^2\Phi_0(0,s+t). \tag{3.3}$$

The corresponding boundary-value problem was solved exactly by Ursell (1947).

3.3. *Pressurized edge-crack in a half-plane*

Set $d = 0$ in (2.12) to give

$$L(s,t) = -1/(s+t)^2+12st/(s+t)^4. \tag{3.4}$$

The corresponding boundary-value problem can be solved exactly; see, for example, Stallybrass (1970) who gives a solution for $v(s) = s^\beta$ with $\beta > -1$.

4. Mellin transforms

In this section, we quote some results on the Mellin transform, defined by

$$\mathscr{M}f \equiv \tilde{f}(z) = \int_0^\infty f(x)\, x^{z-1}\, \mathrm{d}x. \tag{4.1}$$

In the sequel, we always use the notation

$$z = \sigma + \mathrm{i}\tau$$

for the transform variable z.

If $f(x)$ solves (1.1), it is only defined for $0 < x < a$. We extend $f(x)$ by zero for $x > a$, whence $\tilde{f}(z)$ exists and is analytic in a right-hand plane; within this plane,

$$|\tilde{f}(\sigma + \mathrm{i}\tau)| \to 0 \quad \text{as} \quad |\tau| \to \infty. \tag{4.2}$$

If we know the behaviour of $f(x)$ for small x, more precise results are available.

Theorem 1. (Bleistein & Handelsman 1975, Lemma 4.3.6) *Suppose that $f(x) = 0$ for $x > a$ and*

$$f(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} A_{mn}\, x^{a_m} (\ln x)^n \quad \text{as} \quad x \to 0+, \tag{4.3}$$

where $\mathrm{Re}\,(a_0) \leqslant \mathrm{Re}\,(a_1) \leqslant \ldots$ and $0 \leqslant N(m)$, finite. Then $\tilde{f}(z)$ is analytic in $\sigma > -\mathrm{Re}\,(a_0)$ and can be analytically continued into $\sigma \leqslant -\mathrm{Re}\,(a_0)$, with poles at $z = -a_m$; the principal part of the Laurent expansion of $\tilde{f}(z)$ about $z = -a_m$ is

$$\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z+a_m)^{n+1}}. \tag{4.4}$$

Moreover, (4.2) *holds for all values of σ.*

So, the (asymptotic) expansion of $f(x)$ for small x determines precisely the poles of $\tilde{f}(z)$. For our application, we also need a converse result: given the poles of $\tilde{f}(z)$, deduce the expansion (4.3).

The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tilde{f}(z)\, x^{-z}\, \mathrm{d}z, \tag{4.5}$$

where $c > -\mathrm{Re}\,(a_0)$. This inversion formula holds for all $x \geqslant 0$ (the extension of $f(x)$ by zero is continuous, since $f(a) = 0$). Formally, we obtain the expansion (4.3) by moving the inversion contour to the left; each term arises as a residue contribution from an appropriate pole in the analytic continuation of $\tilde{f}(z)$, which we shall obtain from (1.1). This process can be justified.

Theorem 2. (Oberhettinger 1974, p. 7) *Suppose that $\tilde{f}(z)$ is analytic in a left-hand plane, $\sigma \leqslant c$, apart from poles at $z = -a_m$, $m = 0, 1, 2, \ldots$; let the principal part of the Laurent expansion of $\tilde{f}(z)$ about $z = -a_m$ be given by* (4.4). *Assume that* (4.2) *holds for $c' \leqslant \sigma \leqslant c$. Then, if c' can be chosen so that*

$$-\mathrm{Re}\,(a_{M+1}) < c' < \mathrm{Re}\,(a_M)$$

for some M, we have

$$f(x) = \sum_{m=0}^{M} \sum_{n=0}^{N(m)} A_{mn}\, x^{a_m} (\ln x)^n + R_M(x), \tag{4.6}$$

where $$R_M(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \tilde{f}(z)\, x^{-z}\, dz = \frac{x^{-c'}}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(c'+i\tau)\, x^{-i\tau}\, d\tau.$$

The remainder $R_M(x)$ *is* $o(x^{\mathrm{Re}(a_m)})$ *if, for example,*

$$\int_{-\infty}^{\infty} |\tilde{f}(c'+i\tau)|\, d\tau < \infty, \tag{4.7}$$

whence (4.6) *is an asymptotic approximation.*

There are similar theorems relating the behaviour of a function $v(x)$ for large x with the properties of its Mellin transform $\tilde{v}(z)$ in a right-hand plane. The following is sufficient for our purposes.

Theorem 3. (Bleistein & Handelsman 1975, Lemmas 4.3.2 and 4.3.3) *Suppose that* $v(x)$ *is bounded as* $x \to 0$ *and satisfies*

$$v(x) \sim Cx^{-\alpha}\, e^{i\omega x} \quad \textit{as} \quad x \to \infty,$$

where C, α *and* ω *are constants, with* ω *real. Then,* $\tilde{v}(z)$ *is analytic for* $0 < \sigma < \mathrm{Re}\,(\alpha)$, *and can be continued into* $\sigma \geqslant \mathrm{Re}\,(\alpha)$ *as follows.*

(i) *If* $\omega = 0$, *the continuation of* $\tilde{v}(z)$ *has a simple pole at* $z = \alpha$ *with residue* $-C$.

(ii) *If* $\omega \neq 0$, $\tilde{v}(z)$ *can be continued analytically into the whole right-hand plane* $\sigma \geqslant \mathrm{Re}\,(\alpha)$.

Bleistein & Handelsman (1975, ch. 4) give a detailed analysis of the use of Mellin transforms for finding asymptotic expansions of integrals such as (1.10). Davies (1985, §§12–14) describes other applications.

5. The dominant equation

Consider the dominant equation,

$$\frac{1}{2\pi}\,⨎_0^a \frac{f(t)}{(x-t)^2}\, dt = v(x), \quad 0 < x < a. \tag{5.1}$$

Suppose that $v(x) = g'(x)$, where $g(x) \in C^{0,\alpha}$ for $0 \leqslant x \leqslant a$; thus v can have integrable end-point singularities. Then, the general solution of (5.1) is (Martin 1990)

$$f(x) = \frac{2}{\pi}\int_0^a v(t) \ln\left(\frac{a|x-t|}{a(x+t)-2xt+2\sqrt{(xt(a-x)\,(a-t))}}\right) dt + \frac{A+Bx}{\sqrt{(x(a-x))}}, \tag{5.2}$$

where A and B are arbitrary constants. If we impose the edge conditions (1.6) on f, we must take $A = B = 0$ in (5.2). An integration by parts then gives

$$f(x) = \frac{2}{\pi}\,⨍_0^a \sqrt{\left(\frac{x(a-x)}{t(a-t)}\right)}\frac{g(t)}{x-t}\, dt. \tag{5.3}$$

If we restrict v to be square-integrable (so that $g(t) = O(t^{\mu+\frac{1}{2}})$ as $t \to 0$, with $\mu > 0$), rather than merely integrable, we can extract the behaviour of $f(x)$ as $x \to 0$; it is given by (1.8), with

$$\begin{aligned} f_0 &= -\frac{2}{\pi}\int_0^a \frac{\sqrt{(a)}\, g(t)}{t^{\frac{3}{2}}\sqrt{(a-t)}}\, dt \\ &= -\frac{4}{\pi}\int_0^a \frac{\sqrt{(a-t)}}{\sqrt{(at)}}\, v(t)\, dt, \end{aligned} \tag{5.4}$$

after another integration by parts. The expression (5.4) is well known in fracture mechanics (see Barenblatt 1962, p. 90). Note that the edge behaviour is not given by (1.8) if v is not square-integrable (see Muskhelishvili 1953, §29).

Let us now obtain similar results using Mellin transforms. For simplicity, assume that

$$v(x) = \sum_{n=0} v_n x^n \quad \text{for small } x; \tag{5.5}$$

$v(x)$ is given for $0 < x < a$. Define $v(x)$ for $x > a$ by the left-hand side of (5.1), whence $v(x) \sim x^{-2}$ as $x \to \infty$. Thus, $\tilde{v}(z)$ is analytic for $0 < \sigma < 2$ and can be analytically continued into the whole plane apart from poles. In particular, $\tilde{v}(z)$ has simple poles at $z = -N$ with residue v_N; here, and below, N is always assumed to take on integer values, starting from zero: $N = 0, 1, 2, \ldots$. (If the expansion of v is more complicated than (5.5), the corresponding poles of $\tilde{v}(z)$ in a left-hand plane are given by Theorem 1.) Moreover,

$$|\tilde{v}(\sigma + \mathrm{i}\tau)| \to 0 \quad \text{as} \quad |\tau| \to \infty \tag{5.6}$$

for all values of σ.

We also know that $f(0) = f(a) = 0$, whence

$$\tilde{f}(z) \quad \text{is analytic for} \quad \sigma \geqslant 0 \tag{5.7}$$

and has poles in $\sigma < 0$. We locate these poles using the integral equation (5.1). Taking its Mellin transform, using the result

$$⨍_0^\infty \frac{x^z}{(x-t)^2}\,\mathrm{d}x = -\pi t^{z-1} z \cot(\pi z) \quad \text{for} \quad -1 < \sigma < 1,$$

(which is obtained by combining (1.4) with (6.8) below) we obtain

$$z \cot(\pi z) \tilde{f}(z) = -2\tilde{v}(z+1) \quad \text{for} \quad -1 < \sigma < 1.$$

Hence,
$$\cos(\pi z) \tilde{f}(z) = -2 \sin(\pi z)\, \tilde{v}(z+1)/z \quad \text{for} \quad -1 < \sigma < 1.$$

Since $\cos \pi z$ has zeros, we deduce that

$$\tilde{f}(z) = -2 \sin(\pi z)\, \tilde{v}(z+1)/z \cos(\pi z) \quad \text{for} \quad -1 < \sigma < 1, \quad z \neq \pm\tfrac{1}{2}. \tag{5.8}$$

Note that this is not a formula for $\tilde{f}(z)$, since $\tilde{v}$ depends on $\tilde{f}$. However, we can use it to obtain information on f. In particular, when combined with (5.6), we have

$$\tilde{f}(\sigma + \mathrm{i}\tau) = o(\tau^{-1}) \quad \text{as} \quad |\tau| \to \infty, \tag{5.9}$$

whence (4.2) and (4.7) hold.

It follows from (5.7) that $\tilde{v}(z+1)$ must have simple zeros at the simple zeros of $\cos(\pi z)$ in $\sigma \geqslant 0$, whence

$$\tilde{v}(N + \tfrac{3}{2}) = 0.$$

We then see from (5.8) that $\tilde{f}(z)$ is analytic for $\sigma > -\frac{1}{2}$. Take the inversion contour along $\sigma = c$, with $c > -\frac{1}{2}$. By Theorem 2, we can move this contour to the left, crossing the first pole at $z = -\frac{1}{2}$; this is a simple pole, whence the edge behaviour of f is given by (1.8), as expected. The coefficient f_0 is given by the residue of $\tilde{f}(z)$ at $z = -\frac{1}{2}$,

$$f_0 = -(4/\pi)\, \tilde{v}(\tfrac{1}{2}). \tag{5.10}$$

It can be shown that the two formulae for f_0, namely (5.4) and (5.10), are in agreement; see Appendix A.

We can continue moving the inversion contour to the left. Noting that the simple poles of $\tilde{v}(z+1)$ at $z = -(N+1)$ are removed by the simple zeros of $\sin(\pi z)$, we see that $\tilde{f}(z)$ has simple poles at $z = -(N+\frac{1}{2})$, whence

$$f(x) \sim \sum_{n=0} f_n x^{n+\frac{1}{2}} \quad \text{as} \quad x \to 0. \tag{5.11}$$

6. Development of the method for Case I

We wish to use the Mellin-transform method described in §5 on more complicated equations. Consider (1.1), namely

$$(Hf)(x) + (Kf)(x) = v(x), \quad 0 < x < a; \tag{6.1}$$

as before, we use (6.1) to define $v(x)$ for $x > a$. In general, we cannot expect to be able to calculate the Mellin transform of Kf explicitly. However, we can obtain some results using asymptotic expansions of Kf: more terms in these expansions will yield more terms in the edge behaviour of f. In §6.1, we give some general results along these lines, with applications to some of the examples in §2.

In some applications, $(Kf)(x)$ can be written as a Mellin convolution (or a linear combination of several). This is useful, for we know that

$$\mathscr{M}\left\{x^{\lambda} \int_0^{\infty} t^{\mu} k\left(\frac{x}{t}\right) f(t)\,\mathrm{d}t\right\} = \tilde{k}(z+\lambda)\tilde{f}(z+\lambda+\mu+1). \tag{6.2}$$

One application of this result is described in §6.2. Further applications are made in §7, where Case II is considered.

6.1. *Use of asymptotic properties of Kf*

Suppose that there are constants α, β, C_0 and C_∞ such that

$$(Kf)(x) \sim \begin{cases} C_\infty x^{-\alpha} & \text{as} \quad x \to \infty, \\ C_0 x^{\beta} & \text{as} \quad x \to 0, \end{cases} \tag{6.3}$$

where $\alpha \geqslant 2$ and $\beta > -\frac{1}{2}$ (we could also have included logarithmic terms). Then, $(\widetilde{Kf})(z)$ is analytic for $-\beta < \sigma < \alpha$; its analytic continuation has simple poles at $z = -\beta$ and at $z = \alpha$ (Theorems 1 and 3(i)). Note that the restriction on α implies that $v(x) \sim x^{-2}$ for large x, whence $\tilde{v}(z)$ is analytic for $0 < \sigma < 2$, as before. On the other hand, the restriction on β implies that the leading edge-behaviour is determined by H.

Taking the Mellin transform of (6.1) yields

$$z\cot(\pi z)\tilde{f}(z) - 2(\widetilde{Kf})(z+1) = -2\tilde{v}(z+1)$$

for $-\sigma_1 < \sigma < 1$, where $\qquad \sigma_1 = \min(1, \beta+1). \qquad (6.4)$

Hence, $$\tilde{f}(z) = -2\sin(\pi z)\{\tilde{v}(z+1) - (\widetilde{Kf})(z+1)\}/z\cos(\pi z) \tag{6.5}$$

for $-\sigma_1 < \sigma < 1$, $z \neq -\frac{1}{2}$, using (5.7). Thus, the leading term in the asymptotic expansion of $f(x)$ near $x = 0$ is again given by (1.8). The next term depends on the magnitude of β, as the next pole to the left of $z = -\frac{1}{2}$ is at $z = -\min(\frac{3}{2}, \beta+1)$. Note that if $(\widetilde{Kf})(z)$ has a simple pole at $z = -N$, this will be removed by a corresponding zero of $\sin(\pi z)$.

An example of this situation is potential flow past a curved plate (§2.1). It can be shown that (6.3) is satisfied, with $\alpha = 2$ and $\beta = 0$; it is convenient to define $v(x)$ for $x > a$ by setting

$$x(t) = x(a) + (t-a)\,x'(a) \quad \text{and} \quad y(t) = y(a) + (t-a)\,y'(a)$$

for $t \geqslant a$. Indeed, $(Kf)(x)$ has a Taylor series expansion for small x, provided Γ is sufficiently smooth. Subject to this proviso, the edge behaviour is given by (5.11).

Suppose now that K satisfies (6.3) with $\beta > -\frac{1}{2}$ but $\max(0, -\beta) < \alpha < 2$. One consequence is that $v(x) \sim C_\infty x^{-\alpha}$ as $x \to \infty$. Nevertheless, (6.5) still holds for $-\sigma_1 < \sigma < 1$, $z \neq -\frac{1}{2}$; the expression in braces in (6.5) does not have any singularities for $\alpha - 1 \leqslant \sigma < 1$. The edge behaviour of f can now be deduced. As an example, consider

$$(Hf)(x) + Ax^\gamma f(x) + \frac{Bx^\delta}{2\pi} \int_0^a \frac{f(t)}{x-t}\,\mathrm{d}t = v(x), \quad 0 < x < a \tag{6.6}$$

where A, B, γ and δ are constants, with $\gamma > -\frac{1}{2}$ and $\delta > -\frac{1}{2}$. This equation coincides with (2.14) when $\lambda(x) = Ax^\gamma$ and $B = 0$. A comparison with (6.3) shows that $\alpha = 1 - \delta$ and $\beta = \min(\gamma, \delta)$, whence (1.8) holds.

The same method works if (6.3) is replaced by

$$(Kf)(x) \sim \begin{cases} C_\infty x^{-\alpha}\,\mathrm{e}^{\mathrm{i}\omega x} & \text{as} \quad x \to \infty, \\ C_0 x^\beta & \text{as} \quad x \to 0, \end{cases} \tag{6.7}$$

with $\alpha > \beta > -\frac{1}{2}$ and ω real. Now, (the continuation of) $(\widetilde{Kf})(z)$ is analytic throughout the right-hand plane $\sigma > -\beta$ (Theorem 3 (ii)). An example of this situation is acoustic scattering by a plate (§2.2), for which the parameters in (6.7) are $\alpha = \frac{3}{2}$, $\beta = 0$ and $\omega = k$.

6.2. *Mellin convolutions*

The singular integral in (6.6) is a Mellin convolution: its Mellin transform is given by (6.2), wherein $\lambda = \delta$, $\mu = -1$ and $k(x) = (x-1)^{-1}$. Since

$$\int_0^\infty \frac{x^z}{x-t}\,\mathrm{d}x = -\pi t^z \cot(\pi z) \tag{6.8}$$

for $-1 < \sigma < 0$, (6.6) gives

$$\tilde{f}(z) = -\frac{\tan(\pi z)}{z}\{2\tilde{v}(z+1) - 2A\tilde{f}(z+\gamma+1) + B\cot(\pi(z+\delta))\tilde{f}(z+\delta+1)\} \tag{6.9}$$

for $-\sigma_1 < \sigma < 1$, $z \neq -\frac{1}{2}$, where, from (6.4), $\sigma_1 = 1 + \min(0, \gamma, \delta)$. This is an explicit form of (6.5). It shows that $\tilde{f}(z)$ is analytic for $\sigma > -\frac{1}{2}$, with a simple pole at $z = -\frac{1}{2}$, whence (1.8) holds. The coefficient f_0 is given by

$$f_0 = -(4/\pi)\{\tilde{v}(\tfrac{1}{2}) - A\tilde{f}(\gamma+\tfrac{1}{2}) - \tfrac{1}{2}B\tan(\pi\delta)\tilde{f}(\delta+\tfrac{1}{2})\}.$$

We now move leftwards, using a 'bootstrap' argument. Thus, we know that $\tilde{f}(z)$ has a simple pole at $z = -\frac{1}{2}$. Hence, from (6.9), we deduce that the next pole is at $z = -\sigma_1 - \frac{1}{2}$ or at $z = -n - \delta$, where n is a positive integer, the actual pole depending on the values of γ and δ. Note that, in general, all these poles are simple (they are absent if they coincide with a negative integer). The result is an algebraic term in the asymptotic expansion of f,

$$f(x) \sim f_0 x^{\frac{1}{2}} + f_1 x^\nu$$

where $\nu > \frac{1}{2}$ is known. Logarithmic terms can arise for certain choices of γ and δ. For example, if $\gamma = \delta = 0$, (6.9) reduces to

$$\tilde{f}(z) = -2\tan(\pi z)\{\tilde{v}(z+1) - A\tilde{f}(z+1)\}/z - B\tilde{f}(z+1)/z,$$

and this shows that $\tilde{f}(z)$ has a double pole at $z = -\frac{3}{2}$ (if $A \neq 0$), whence

$$f(z) \sim f_0 x^{\frac{1}{2}} + x^{\frac{3}{2}}(f_1 \ln x + f_2),$$

where $f_1 = -4Af_0/(3\pi)$.

7. Development of the method for Case II

In this section, we consider Case II, i.e. we consider equations of the form

$$\frac{1}{2\pi}\fint_0^a f(t)\left\{\frac{1}{(x-t)^2} + L(x,t)\right\}\mathrm{d}t = v(x), \quad 0 < x < a \tag{7.1}$$

where
$$L(x,x) \sim x^{-2} \quad \text{as} \quad x \to 0. \tag{7.2}$$

As before, we extend $f(t)$ by zero for $t > a$ and define $v(x)$ for $x > a$ by the left-hand side of (7.1).

The integral equation (7.1) is supplemented by the conditions (1.7); we write $f(0) = f_0$, an unknown constant. It follows that

$$\tilde{f}(z) \quad \text{is analytic for} \quad \sigma > 0 \tag{7.3}$$

(cf. (5.7)). Moreover, since $f(0)$ is required to be bounded, $\tilde{f}(z)$ can only have a simple pole at $z = 0$,

$$\tilde{f}(z) = \frac{f_0}{z} + \hat{f}_0 + \sum_{m=1} f_0^{(m)} z^m \tag{7.4}$$

in a neighbourhood of $z = 0$; if $\tilde{f}$ had a higher-order pole, $f(x)$ would be logarithmically infinite at $x = 0$ (Theorem 2).

7.1. *Rigid plate meeting a rigid wall*

For this problem, L is given by (3.1):

$$L(x,t) = t^{-2} l(x/t) \tag{7.5}$$

where
$$l(x) = \frac{(x^2+1)\cos 2\alpha + 2x}{(x^2+1+2x\cos 2\alpha)^2} = \frac{1}{2}\left\{\frac{\mathrm{e}^{2\mathrm{i}\alpha}}{(x+\mathrm{e}^{2\mathrm{i}\alpha})^2} + \frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{(x+\mathrm{e}^{-2\mathrm{i}\alpha})^2}\right\}$$

and $0 \leqslant \alpha < \frac{1}{2}\pi$. We know that

$$\int_0^\infty \frac{x^z}{x+t}\mathrm{d}x = \frac{-\pi t^z}{\sin(\pi z)} \tag{7.6}$$

for $-1 < \sigma < 0$, whence
$$\int_0^\infty \frac{x^z}{(x+t)^2}\mathrm{d}x = \frac{\pi z t^{z-1}}{\sin(\pi z)} \tag{7.7}$$

for $-1 < \sigma < 1$ and
$$\tilde{l}(z+1) = \frac{\pi z}{\sin(\pi z)}\cos(2\alpha z).$$

Hence, taking the Mellin transform of (7.1), using (6.2), we obtain

$$zS(z)\tilde{f}(z)/\sin(\pi z) = \tilde{v}(z+1), \tag{7.8}$$

where $$S(z) = \sin((\tfrac{1}{2}\pi+\alpha)z)\sin((\tfrac{1}{2}\pi-\alpha)z).$$

Equation (7.8) is valid for $-1 < \sigma < 1$. Within this strip, $S(z)$ only vanishes at $z = 0$, where it has a double zero. Hence

$$\tilde{f}(z) = \tilde{v}(z+1)\sin(\pi z)/zS(z) \tag{7.9}$$

for $-1 < \sigma < 1$, $z \neq 0$. In fact, by (7.3), we can deduce some properties of $\tilde{v}(z)$ in $\sigma > 1$, as in §5.

We can take the inversion contour along $\sigma = c$, with $c > 0$. Moving the contour leftwards, we cross the pole at $z = 0$. This must be a simple pole, whence $\tilde{v}(1) = 0$ and

$$f_0 = \pi\tilde{v}'(1)/(\tfrac{1}{4}\pi^2-\alpha^2).$$

The result $\tilde{v}(1) = 0$ can be verified independently. Green's theorem gives

$$\int_C \frac{\partial\phi}{\partial n}\,\mathrm{d}s = 0$$

where C is any closed contour in the fluid. Choose C to bound a large sector, with one straight side along half of the rigid wall and the other along the plate and its extension, and a small indentation where the plate meets the wall. There is no contribution from the indentation (because ϕ is bounded there) or from the wall (where $\partial\phi/\partial y = 0$), and the contribution from the large circular arc closing the sector vanishes as the arc recedes. The remaining contribution is proportional to $\tilde{v}(1)$.

For $\alpha > 0$, the next pole encountered is at $z = -a_1$, where

$$1 < a_1 = 2\pi/(\pi+2\alpha) \leqslant 2,$$

whence $$f(x) \sim f_0+f_1 x^{a_1} \quad \text{as} \quad x \to 0, \tag{7.10}$$

where $$f_1 = \tilde{v}(1-a_1)\sin(\pi a_1)/\pi\sin(\tfrac{1}{2}\pi-\alpha)\,a_1.$$

The result (7.10) could also have been obtained formally by the method of separation of variables in plane polar coordinates.

The third term in the expansion depends on the angle α; we merely examine the zeros of $\sin(\frac{1}{2}\pi\pm\alpha)z$ in $\sigma < -1$. In general, all the poles of $\tilde{f}(z)$ in $\sigma < 0$ will be simple if α is an irrational multiple of $\frac{1}{2}\pi$; if

$$\alpha = \tfrac{1}{2}\pi p/q$$

for some integers $0 \leqslant p < q$, $\tilde{f}(z)$ will have a double pole at $z = -2q$, leading to logarithmic terms in the expansion of $f(x)$.

Finally, consider the special case $\alpha = 0$, when (7.9) reduces to

$$\tilde{f}(z) = (2/z)\,\tilde{v}(z+1)\cot(\tfrac{1}{2}\pi z).$$

We know that $\tilde{v}(z)$ has simple poles at $z = -N$ with known residues v_N. It follows that

$$f(x) = f_0+(2/\pi)\,v_1 x^2\ln x+O(x^2) \quad \text{as} \quad x \to 0, \tag{7.11}$$

i.e. the coefficient of $x^2 \ln x$ is known explicitly.

We can understand this last result as follows. Recall that, since $\alpha = 0$, the plate is perpendicular to the rigid wall. By reflection in the wall, we can consider the equivalent problem of a flat plate of length $2a$ (along $-a \leqslant x \leqslant a$, $y = 0$) in an unbounded fluid. The data $v(x)$ is given for $-a \leqslant x \leqslant a$, but is symmetric about $x = 0$. The presence of the term v_1 in (7.11) is due to data $v_1|x|$, which is not differentiable at $x = 0$ and so gives rise to a weak singularity in the solution.

7.2. *Pressurized edge-crack in a half-plane*

For this problem, L is also a Mellin convolution, given by (3.4) and (7.5), with

$$l(x) = -1/(x+1)^2 + 12x/(x+1)^4,$$

whence
$$\tilde{l}(z+1) = \pi z(1-2z^2)/\sin(\pi z)$$

for $-1 < \sigma < 1$, using (7.7) and its second derivative with respect to t. Hence, the Mellin transform of (7.1) is given by (7.8), where

$$S(z) = \tfrac{1}{2} - z^2 - \tfrac{1}{2}\cos(\pi z) = \sin^2(\tfrac{1}{2}\pi z) - z^2 = S_+(z)\, S_-(z),$$

say, where
$$S_\pm(z) = \sin(\tfrac{1}{2}\pi z) \pm z.$$

As in §7.1, $S(z)$ has a double zero at $z = 0$, and no other zeros in $-1 < \sigma < 1$. Hence, we obtain (7.9), $\tilde{v}(1) = 0$ (which can also be verified independently) and

$$f_0 = \pi\tilde{v}'(1)/(\tfrac{1}{4}\pi^2 - 1).$$

The next zero of $S(z)$ is given by $S_-(-1) = 0$; noting that $\sin\pi z$ also has a zero at $z = -1$ whilst $\tilde{v}(z+1)$ has a pole there, we deduce that

$$f(x) \sim f_0 + \tfrac{1}{2}\pi v_0 x.$$

This gives an explicit formula for the slope of the crack-opening displacement at the mouth of the crack.

The remaining zeros of $S(z)$ occur in complex-conjugate pairs. The next satisfy $S_+(z) = 0$ and are given approximately by

$$z_0 = -2.740 - 1.119\mathrm{i} = -\sigma_0 - \mathrm{i}\tau_0,$$

say, and its complex conjugate. Then, we obtain

$$f(x) \sim f_0 + \tfrac{1}{2}\pi v_0 x + F x^{\sigma_0} \cos\{\tau_0 \ln x + \delta\} \tag{7.12}$$

where the real quantities F and δ are defined by

$$F\,\mathrm{e}^{\mathrm{i}\delta} = \tilde{v}(z_0+1)\sin\pi z_0 / z_0 S_-(z_0)\, S'_+(z_0).$$

The occurrence of the functions $S_\pm(z)$ is not surprising. They arise when the method of separation of variables is used inside an infinite elastic right-angled wedge, with traction-free boundaries; locally, the behaviour near the mouth of the crack is expected to be given by appropriate solutions of the wedge problem. The solutions of the transcendental equation $S_\pm(z) = 0$ have been tabulated in the literature (see, for example, Karp & Karal 1962; Gregory 1979; and references therein).

7.3. *A vertical surface-piercing barrier*

For this problem, L is given by (3.3), whence (7.1) can be written as

$$\frac{1}{2\pi} \fint_0^a f(t)\left\{\frac{1}{(y-t)^2} + \frac{1}{(y+t)^2}\right\}\mathrm{d}t + \frac{K}{\pi}\int_0^a f(t)\left\{\frac{1}{y+t} + K\varPhi(y+t)\right\}\mathrm{d}t = v(y), \tag{7.13}$$

for $0 < y < a$, where
$$\varPhi(Y) = \varPhi_0(0, Y) = \fint_0^\infty \mathrm{e}^{-kY}\frac{\mathrm{d}k}{k-K} \tag{7.14}$$

and we have used y for the independent variable, since the plate occupies a segment of the y-axis.

The equation (7.13) is more difficult to treat, as it is not a Mellin convolution. Nevertheless, we can find its Mellin transform. First, we note that the first term on the left-hand side of (7.13) corresponds to a plate perpendicular to a rigid wall, as studied in §7.1. Next, we can show that $v(y) \sim Cy^{-3}$ as $y \to \infty$, where

$$C = \frac{2}{\pi K}\int_0^a \{f'(t)+Kf(t)\}\,t\,\mathrm{d}t$$

and, as usual, $v(y)$ is defined for $y > a$ by (7.13). For, by rotating the contour onto the positive imaginary axis, we have

$$\Phi(Y) = 2\pi\mathrm{i}\,\mathrm{e}^{-KY}+\int_0^\infty \mathrm{e}^{-\mathrm{i}kY}\frac{\mathrm{d}k}{k+\mathrm{i}K} \sim -\frac{1}{KY}-\frac{1}{(KY)^2}-\frac{2}{(KY)^3} \quad \text{as} \quad y \to \infty.$$

It follows that $\tilde{v}(z)$ is analytic for $0 < \sigma < 3$.

In Appendix B, the Mellin transform of the second term in (7.13) is calculated. Thus,

$$\tilde{f}(z) = N(z)/zS(z) \tag{7.15}$$

for $-1 < \sigma < 1$, $z \neq 0$, where

$$S(z) = \sin^2(\tfrac{1}{2}\pi z), \quad N(z) = -KQ(z)+\tilde{v}(z+1)\sin(\pi z)$$

and
$$Q(z) = \Gamma(z+1)\left\{\frac{\mathrm{e}^{\mathrm{i}\pi z}}{K^z}\int_0^a f(t)\,\mathrm{e}^{-Kt}\,\mathrm{d}t - \sum_{n=0}^{\infty}\frac{(-K)^n}{\Gamma(z+n+1)}\tilde{f}(z+n+1)\right\}. \tag{7.16}$$

We proceed as before, choosing the inversion contour along $\sigma = c > 0$. Moving leftwards, we meet the pole of $\tilde{f}(z)$ at $z = 0$. At first sight, this appears to be a triple pole, whereas it must be a simple pole (with Laurent expansion (7.14)); this implies that $N(0) = 0$ and $N'(0) = 0$; these imply that $Q(0) = 0$, which is easily seen to be true from (7.16), and

$$KQ'(0) = \pi\tilde{v}(1),$$

which can also be verified independently (see Appendix B).

The next pole encountered is at $z = -1$. To determine the behaviour of $\tilde{f}(z)$ there, we use the recurrence relation

$$Q(z) = -(K/(z+1))\,Q(z+1)-\tilde{f}(z+1),$$

which is easily derived from (7.16). Thus, near $z = -1$, $zS(z) \sim -1+(z+1)$ and

$$\begin{aligned} N(z) &= (K^2/(z+1))\,Q(z+1)+K\tilde{f}(z+1)+\tilde{v}(z+1)\sin(\pi z) \\ &\sim K^2Q'(0)+K(f_0/(z+1)+\hat{f}_0)-\pi v_0, \end{aligned}$$

whence
$$\tilde{f}(z) \sim f_1/(z+1)+\hat{f}_1,$$

where
$$f_1 = -Kf_0 \tag{7.17}$$

and
$$\hat{f}_1 = -K^2Q'(0)-Kf_0-K\hat{f}_0+\pi v_0. \tag{7.18}$$

Note that there is a simple pole at $z = -1$, and that we have also calculated the constant term $\hat{f}_1$ in the Laurent expansion of $\tilde{f}(z)$ around the pole.

The next pole is at $z = -2$; it appears to be a triple pole. Using the recurrence relation again, and known properties of $\tilde{f}(z)$ around $z = 0$ and $z = -1$, we find that, near $z = -2$,

$$zS(z) \sim -\tfrac{1}{2}\pi^2(z+2)^2 \tag{7.19}$$

and

and $$N(z) = -\frac{K^3Q(z+2)}{(z+1)(z+2)} - \frac{K^2}{z+1}\tilde{f}(z+2) + K\tilde{f}(z+1) + \tilde{v}(z+1)\sin(\pi z)$$

$$\sim K^3Q'(0) + K^2(1+(z+2))(f_0/(z+2)+\hat{f}_0) + K(f_1/(z+2)+\hat{f}_1) + \pi v_1$$
$$= (K/(z+2))(Kf_0+f_1) + K^3Q'(0) + K^2(f_0+\hat{f}_0) + K\hat{f}_1 + \pi v_1.$$

But, from (7.17), we see that the first term vanishes, whence there is only a double pole; we have

$$N(-2) = K^3Q'(0) + K^2(f_0+\hat{f}_0) + K\hat{f}_1 + \pi v_1 = \pi(Kv_0+v_1)$$

and $$\tilde{f}(z) \sim -\frac{2}{\pi}\frac{Kv_0+v_1}{(z+2)^2}.$$

It follows that

$$f(y) \sim f_0(1-Ky) + (2/\pi)(Kv_0+v_1)(y^2\ln y + \hat{f}_2 y^2) \tag{7.20}$$

as $y \to 0$. It is noteworthy that the coefficient of $y^2 \ln y$ is given explicitly as

$$(2/\pi)(Kv(0)+v'(0)), \tag{7.21}$$

where $v(y)$ is given for $0 \leqslant y < a$. In fact, this coefficient was given many years ago by Kravtchenko (1954, p. 58) in his analysis of the motion generated by a wavemaker.

If the barrier is scattering an incident regular surface wave, $v(y)$ is proportional to e^{-Ky}, whence $Kv+v'$ vanishes identically, and so the logarithmic term in (7.20) is absent. Indeed, it can be shown that, for this particular problem, all logarithmic terms are absent; this is in accord with Ursell's exact solution (1947).

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Appendix A. Equality of two expressions for f_0

Write Barenblatt's formula (5.4) as

$$f_0 = (4/\pi)(Q_1 - \tilde{v}(\tfrac{1}{2})),$$

where $$Q_1 = \int_a^\infty v(t)\frac{dt}{\sqrt{t}} - \int_0^a \frac{\sqrt{(a-t)}-\sqrt{a}}{\sqrt{(at)}}v(t)\,dt \tag{A 1}$$

We show that $Q_1 = 0$ (cf. (5.10)). From (1.4), we have

$$v(t) = \frac{-1}{2\pi}\frac{d}{dt}\fint_0^a \frac{f(\xi)}{t-\xi}d\xi.$$

Substituting into (A 1) and integrating both integrals by parts gives

$$Q_1 = \frac{-1}{4\pi}\int_0^a f(\xi)\,Q_2(\xi)\,d\xi,$$

where
$$Q_2(\xi) = \int_a^\infty \frac{\mathrm{d}t}{t^{\frac{3}{2}}(t-\xi)} + \int_0^a \frac{\sqrt{(a-t)}-\sqrt{a}}{t^{\frac{3}{2}}\sqrt{(a-t)}}\frac{\mathrm{d}t}{t-\xi}.$$

$Q_2(\xi)$ vanishes identically for $0<\xi<1$. This can be shown using contour integration. In a complex t-plane, with a cut along the positive real axis, consider
$$\int_C \frac{\sqrt{(t-a)}-\mathrm{i}\sqrt{a}}{t^{\frac{3}{2}}\sqrt{(t-a)}}\frac{\mathrm{d}t}{t-\xi},$$
where the closed contour C consists of the large circular arc $t = R\,\mathrm{e}^{\mathrm{i}\theta}, 0<\theta<2\pi$, together with the two sides of the cut and suitable indentations around the two branch points ($t=0$ and $t=a$) and the pole at $t=\xi$. Routine calculations now show that $Q_2 = 0$, whence $Q_1 = 0$ and (5.4) reduces to (5.10).

Appendix B. The vertical barrier

(a) *A Mellin transform*

Using (7.6) and noting that
$$\Phi(y) = \pi\mathrm{i}\mathrm{e}^{-Ky} + \int_0^\infty \mathrm{e}^{-ky}\frac{\mathrm{d}k}{k-K},$$
and $\mathscr{M}\{\mathrm{e}^{-kx}\} = k^{-z}\Gamma(z)$, we deduce that
$$\begin{aligned}\frac{Q(z)}{\sin(\pi z)} &= \frac{1}{\pi}\int_0^\infty y^z \int_0^a f(t)\left\{\frac{1}{y+t}+K\Phi(y+t)\right\}\mathrm{d}t\,\mathrm{d}y \\ &= -\frac{\tilde{f}(z+1)}{\sin(\pi z)} + \frac{\mathrm{i}}{K^z}\Gamma(z+1)\int_0^a f(t)\,\mathrm{e}^{-Kt}\,\mathrm{d}t + \frac{K}{\pi}\Gamma(z+1)\int_0^a f(t)\,\Psi(t;z+1)\,\mathrm{d}t\end{aligned}$$
where
$$\Psi(t;z) = \int_0^\infty \frac{\mathrm{e}^{-kt}}{k^z}\frac{\mathrm{d}k}{k-K}.$$

We evaluate Ψ by taking its Laplace transform:
$$\begin{aligned}\mathscr{L}\{\Psi\} &= \int_0^\infty \Psi(t;z)\,\mathrm{e}^{-pt}\,\mathrm{d}t = \int_0^\infty \frac{k^{-z}}{(k-K)(p+K)}\mathrm{d}k \\ &= \frac{1}{p+K}\int_0^\infty \left\{\frac{k^{-z}}{k-K}-\frac{k^{-z}}{p+k}\right\}\mathrm{d}k = \frac{\pi}{p+K}\left\{K^{-z}\cot(\pi z) - \frac{p^{-z}}{\sin(\pi z)}\right\}.\end{aligned}$$

Now, $(p+K)^{-1} = \mathscr{L}\{\mathrm{e}^{-Kt}\}$ and
$$\frac{p^{-z}}{p+K} = \frac{1}{p^{z+1}}\sum_{n=0}^\infty\left(\frac{-K}{p}\right)^n = \sum_{n=0}^\infty \frac{(-K)^n}{\Gamma(n+z+1)}\mathscr{L}\{t^{n+z}\},$$
whence
$$\Psi(t;z) = \frac{\pi}{\sin(\pi z)}\left\{\frac{\cos(\pi z)}{K^z}\mathrm{e}^{-Kt} - \sum_{n=0}^\infty \frac{(-K)^n t^{n+z}}{\Gamma(n+z+1)}\right\}$$
and the stated result for $Q(z)$, namely (7.16), follows after some simple algebra.

(b) *Proof that $KQ'(0) = \pi\tilde{v}(1)$*

We have
$$\partial^2 G_2/\partial x\,\partial\xi = -\partial^2 G_2/\partial x^2 = \partial^2 G_2/\partial y^2$$

whence $$\int_0^\infty \frac{\partial^2 G_2}{\partial x\,\partial\xi}\,\mathrm{d}y = -\left.\frac{\partial G_2}{\partial y}\right|_{y=0} = KG_2|_{y=0} = -2K\Phi(\eta)$$

on the barrier. Integrating the governing integral equation with respect to y then gives

$$K\int_0^a f(t)\,\Phi(t)\,\mathrm{d}t = \pi\tilde{v}(1).$$

But the left-hand side of this equation is precisely $KQ'(0)$, as follows directly from (7.16) and the known expansion

$$\Phi(y) = -(\ln Ky - \mathrm{i}\pi + \gamma)\,\mathrm{e}^{-Ky} + \sum_{m=1}^{\infty}\frac{(-Ky)^m}{m!}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}\right)$$

(Yu & Ursell 1961), where $\gamma = 0.5772\ldots$ is Euler's constant.

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