Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle*

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Abstract

Time-harmonic electromagnetic waves are scattered by a homogeneous dielectric obstacle. The corresponding electromagnetic transmission problem is reduced to a single integral equation over S for a single unknown tangential vector field, where S is the interface between the obstacle and the surrounding medium. In fact, several different integral equations are derived and analysed, including two previously-known equations due to E. Marx and J.R. Mautz, and two new singular integral equations. Mautz's equation is shown to be uniquely solvable at all frequencies. A new uniquely-solvable singular integral equation is also found. The paper also includes a review of methods using pairs of coupled integral equations over S. It is these methods that are usually used in practice, although single integral equations seem to offer some computational advantages.

1 Introduction

It is well known that the problem of time-harmonic electromagnetic scattering by a perfectly-conducting obstacle can be reduced to a single integral equation over the boundary of the obstacle. It is also well known that the simplest of these equations suffer from *irregular frequencies*, at which they are not uniquely solvable. Various methods for eliminating irregular frequencies have been devised; see, for example, [18], [7, §6.17], [2, §4.6], [12], [20], [8].

The situation for dielectric obstacles is more complicated. If the obstacle is inhomogeneous, so that its material properties vary with position, integral equations can still be

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derived but the domain of integration is usually the whole volume occupied by the obstacle; see, for example, [19, §22], [7, §6.26], [3]. However, for homogeneous dielectrics, the problem can be reduced to integral equations over the interface between the two materials. It is equations of this type that we consider below.

The above electromagnetic transmission problem is usually reduced to a pair of coupled integral equations for a pair of unknown tangential vector fields. Such formulations have been reviewed recently by Harrington [6]. We give a complementary review in §6, where we use an operator notation familiar from the book [2]; for convenience, we include an appendix relating our notation with that used in [6]. Some pairs of integral equations have irregular frequencies, others do not. A new proof is given of a theorem in which all the irregular frequencies of the so-called E-field formulation are identified. This proof uses some properties of a certain operator ($\mathcal{A}_\lambda$, defined by (24) below), involving products of the standard electromagnetic boundary integral operators. These properties are proved in §5, using the theory of pseudodifferential operators.

In §§7 and 8, we consider methods for solving the electromagnetic transmission problem using a single integral equation for a single unknown tangential vector field. A systematic derivation is given (in §7) of two different two-parameter integral equations. As a special case, we recover a known hypersingular integral equation due to Marx [14], [15]. In fact, for almost all values of the two parameters, our single integral equations are hypersingular integral equations. Exceptionally, we find two new singular integral equations, although these are shown to suffer from irregular frequencies.

In §8, we derive single integral equations that are shown to be uniquely solvable at all frequencies. Thus, in §8.1 we derive an equation previously obtained by Mautz [16]. We establish the existence of a unique solution to this hypersingular integral equation by adapting a regularization method due to Kress [11]. Mautz's equation is fairly simple and so is worth investigating for computational work. In §8.2, we derive a new uniquely-solvable singular integral equation by adapting another method due to Kress [12]; this integral equation is attractive theoretically, although the kernel is rather complicated.

The paper can be viewed as the electromagnetic counterpart of [10], in which the acoustic transmission problem was studied. In [10], we found an abundance of Fredholm integral equations of the second kind; here, hypersingular integral equations are the norm, and it is this difference that makes existence results more difficult to establish.

2 Statement of the problem

Let B_i denote a bounded three-dimensional domain with a smooth closed boundary, S, and simply-connected exterior, B_e. We consider the following problem.

TRANSMISSION PROBLEM. Find electric fields $\mathbf{E}_e$ and $\mathbf{E}_i$, and magnetic fields $\mathbf{H}_e$ and $\mathbf{H}_i$, which satisfy Maxwell's equations

$$\operatorname{curl}\mathbf{E}_e - \mathrm{i}\mu_e\omega\mathbf{H}_e = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\mathbf{H}_e + \mathrm{i}\varepsilon_e\omega\mathbf{E}_e = \mathbf{0}, \quad P \in B_e,$$

$$\operatorname{curl}\mathbf{E}_i - \mathrm{i}\mu_i\omega\mathbf{H}_i = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\mathbf{H}_i + \mathrm{i}\varepsilon_i\omega\mathbf{E}_i = \mathbf{0}, \quad P \in B_i,$$

and two transmission conditions on the interface,

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{E}_i \quad \text{and} \quad \mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_i, \quad p \in S, \tag{1}$$

where the total fields in B_e are given by

$$\mathbf{E}(P) = \mathbf{E}_e + \mathbf{E}_{\text{inc}}, \quad \mathbf{H}(P) = \mathbf{H}_e + \mathbf{H}_{\text{inc}}, \quad P \in B_e, \tag{2}$$

and $\{\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}}\}$ is the given incident field. In addition, the scattered fields $\{\mathbf{E}_e, \mathbf{H}_e\}$ must satisfy a Silver-Müller radiation condition [2, §4.2],

$$\sqrt{\mu_e}\hat{\mathbf{r}}_P \times \mathbf{H}_e + \sqrt{\varepsilon_e}\mathbf{E}_e = o(r_P^{-1}) \quad \text{as} r_P \to \infty, \tag{3}$$

uniformly for all directions $\hat{\mathbf{r}}_P$.

We have suppressed a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ throughout. We assume that the electric permittivities ε_e and ε_i, and the magnetic permeabilities μ_e and μ_i are given positive real constants.

We shall use the following notation: capital letters P, Q denote points of $B_e \cup B_i$; lower-case letters p, q denote points of S; and $\mathbf{n}(q)$ denotes the unit normal at q pointing into B_e. We choose the origin O at some point in B_i; $\mathbf{r}_P$ is the position vector of P with respect to O, $r_P = |\mathbf{r}_P|$ and $\hat{\mathbf{r}}_P = \mathbf{r}_P / r_P$.

It is known that the transmission problem has precisely one solution. These existence and uniqueness results are proved in [19, §§21 and 23].

We shall also need to consider two interior problems.

Interior Maxwell Problem. Find a field $\{\mathbf{E}, \mathbf{H}\}$ which satisfies Maxwell's equations

$$\operatorname{curl} \mathbf{E} - \mathrm{i}\mu\omega\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + \mathrm{i}\varepsilon\omega\mathbf{E} = \mathbf{0}, \quad P \in B_i, \tag{4}$$

and the boundary condition

$$\mathbf{n} \times \mathbf{E} = \mathbf{0}, \quad p \in S.$$

If this problem has a non-trivial solution, we say that $k^2 = \omega^2\mu\varepsilon$ is an eigenvalue of the interior Maxwell problem. All such eigenvalues are known to be real [2, p. 125]. Physically, the interior Maxwell problem corresponds to a perfectly-conducting cavity resonator. It is a special case of the next problem.

Associated Interior Problem. Find a field $\{\mathbf{E}, \mathbf{H}\}$ which satisfies Maxwell's equations (4) in B_i and the boundary condition

$$a(\mathbf{n} \times \mathbf{E}) + b\{\mathbf{n} \times (\mathbf{n} \times \mathbf{H})\} = \mathbf{0}, \quad p \in S. \tag{5}$$

Here, a and b are constants. This problem is equivalent to the interior Maxwell problem if $a = 0$ or $b = 0$. Suppose $a \neq 0$ and set $\lambda = b/a$, whence the associated interior problem reduces to an impedance problem. From Maxwell's equations and the divergence theorem, we have

$$\operatorname{Re} \int_S (\mathbf{E} \times \overline{\mathbf{H}}).\mathbf{n}\, ds = 0,$$

where the overbar denotes complex conjugation. Then, the boundary condition (5) implies that

$$\operatorname{Re}(\lambda) \int_S |\mathbf{n} \times \mathbf{H}|^2\, ds = 0.$$

It follows that the associated interior problem only has the trivial solution if $\operatorname{Re}\lambda \neq 0$.

3 Potential theory

Introduce two free-space fundamental solutions, G_α, defined by

$$G_\alpha(P,Q) = \exp(\mathrm{i}k_\alpha R)/(2\pi R),$$

where $R = |\mathbf{r}_P - \mathbf{r}_Q|$ is the distance between P and Q,

$$k_\alpha = \omega\sqrt{\varepsilon_\alpha \mu_\alpha}$$

and $\alpha = e$ or i. Next, define a single-layer potential by

$$\left(S_\alpha \nu\right)(P) = \int_S \nu(q) G_\alpha(P,q)\, ds_q, \quad P \in B_e \cup B_i. \tag{6}$$

In electromagnetic theory, we usually apply S_α to a vector-valued function of position, $\mathbf{a}(q)$, say; we define

$$\left(C_\alpha \mathbf{a}\right)(P) = \text{curl}\ \{S_\alpha \mathbf{a}\} \qquad \text{and} \qquad \left(F_\alpha \mathbf{a}\right)(P) = \text{curl}\ \{C_\alpha \mathbf{a}\}\,.$$

We are interested in the tangential components of these vector fields evaluated on S when $\mathbf{a}(q)$ itself is a tangential density (so that $\mathbf{a}(q).\mathbf{n}(q) = 0$ for all $q \in S$). For continuous tangential densities, we have

$$\mathbf{n} \times C_\alpha \mathbf{a} = \pm \mathbf{a} + M_\alpha \mathbf{a},$$

where the upper (lower) sign corresponds to $P \to p \in S$ from B_e (B_i) and M_α is a boundary integral operator defined by

$$\left(M_\alpha \mathbf{a}\right)(p) = \mathbf{n}(p) \times \text{curl}\ \{S_\alpha \mathbf{a}\}\,, \quad p \in S.$$

For sufficiently smooth tangential densities $\mathbf{a}$ (we shall be more precise later), we also have

$$\mathbf{n} \times F_\alpha \mathbf{a} = P_\alpha \mathbf{a}$$

on S, where

$$\left(P_\alpha \mathbf{a}\right)(p) = \mathbf{n}(p) \times \text{curl}\,\text{curl}\ \{S_\alpha \mathbf{a}\}\,, \quad p \in S.$$

Note that M_α and P_α are related to the operators $\mathbf{M}_\alpha$ and $\mathbf{N}_\alpha$ in [2, §2.7] by

$$\mathbf{M}_\alpha \mathbf{a} = 2M_\alpha \mathbf{a} \qquad \text{and} \qquad \mathbf{N}_\alpha \mathbf{a} = 2P_\alpha\{\mathbf{n} \times \mathbf{a}\}. \tag{7}$$

We shall make extensive use of the Stratton-Chu representation. Applied in B_e to $\{\mathbf{E}_e, \mathbf{H}_e\}$, it gives

$$C_e\{\mathbf{n} \times \mathbf{E}_e\} + \tfrac{\mathrm{i}}{\omega\varepsilon_e} F_e\{\mathbf{n} \times \mathbf{H}_e\} = \begin{cases} 2\mathbf{E}_e(P), & P \in B_e, \\ \mathbf{0}, & P \in B_i, \end{cases} \tag{8}$$

and

$$C_e\{\mathbf{n} \times \mathbf{H}_e\} - \tfrac{\mathrm{i}}{\omega\mu_e} F_e\{\mathbf{n} \times \mathbf{E}_e\} = \begin{cases} 2\mathbf{H}_e(P), & P \in B_e, \\ \mathbf{0}, & P \in B_i. \end{cases} \tag{9}$$

An application in B_i to $\{\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc}\}$ (with the exterior material) yields similar formulae which, when added to (8) and (9), give

$$C_e\{\mathbf{n}\times\mathbf{E}\} + \tfrac{\mathrm{i}}{\omega\varepsilon_e}F_e\{\mathbf{n}\times\mathbf{H}\} = \begin{cases} 2\mathbf{E}_e(P), & P\in B_e, \\ -2\mathbf{E}_{\rm inc}(P), & P\in B_i, \end{cases} \tag{10}$$

and

$$C_e\{\mathbf{n}\times\mathbf{H}\} - \tfrac{\mathrm{i}}{\omega\mu_e}F_e\{\mathbf{n}\times\mathbf{E}\} = \begin{cases} 2\mathbf{H}_e(P), & P\in B_e, \\ -2\mathbf{H}_{\rm inc}(P), & P\in B_i. \end{cases} \tag{11}$$

Finally, an application in B_i to $\{\mathbf{E}_i, \mathbf{H}_i\}$ gives

$$C_i\{\mathbf{n}\times\mathbf{E}_i\} + \tfrac{\mathrm{i}}{\omega\varepsilon_i}F_i\{\mathbf{n}\times\mathbf{H}_i\} = \begin{cases} \mathbf{0}, & P\in B_e, \\ -2\mathbf{E}_i(P), & P\in B_i, \end{cases} \tag{12}$$

and

$$C_i\{\mathbf{n}\times\mathbf{H}_i\} - \tfrac{\mathrm{i}}{\omega\mu_i}F_i\{\mathbf{n}\times\mathbf{E}_i\} = \begin{cases} \mathbf{0}, & P\in B_e, \\ -2\mathbf{H}_i(P), & P\in B_i. \end{cases} \tag{13}$$

Computing the tangential components of (10), (11), (12) and (13) on S, we obtain

$$(I-M_e)\{\mathbf{n}\times\mathbf{E}\} - \tfrac{\mathrm{i}}{\omega\varepsilon_e}P_e\{\mathbf{n}\times\mathbf{H}\} = 2\mathbf{n}\times\mathbf{E}_{\rm inc}, \tag{14}$$

$$(I-M_e)\{\mathbf{n}\times\mathbf{H}\} + \tfrac{\mathrm{i}}{\omega\mu_e}P_e\{\mathbf{n}\times\mathbf{E}\} = 2\mathbf{n}\times\mathbf{H}_{\rm inc}, \tag{15}$$

$$(I+M_i)\{\mathbf{n}\times\mathbf{E}_i\} + \tfrac{\mathrm{i}}{\omega\varepsilon_i}P_i\{\mathbf{n}\times\mathbf{H}_i\} = \mathbf{0}, \tag{16}$$

$$(I+M_i)\{\mathbf{n}\times\mathbf{H}_i\} - \tfrac{\mathrm{i}}{\omega\mu_i}P_i\{\mathbf{n}\times\mathbf{E}_i\} = \mathbf{0}. \tag{17}$$

4 Properties of boundary integral operators

In this section, we begin by defining appropriate function spaces. Properties of M_α and P_α, considered as operators acting on these spaces, are given; usually, we omit the subscript α in this section. Next, we introduce the adjoints of M and P. Finally, we obtain some identities satisfied by products of M and P.

4.1 Function spaces

We seek classical solutions of the transmission problem, that is

$$\{\mathbf{E}_e, \mathbf{H}_e\} \in C^1(B_e)\cap C^{0,\beta}(\overline{B_e}) \quad \text{and} \quad \{\mathbf{E}_i, \mathbf{H}_i\} \in C^1(B_i)\cap C^{0,\beta}(\overline{B_i}),$$

where $C^{0,\beta}(\Omega)$ is the usual Banach space of Hölder-continuous functions on Ω (with $0<\beta<1$) and $\overline{\Omega}$ is the closure of Ω. For functions defined on the interface S, we use the space $T^{0,\beta}(S)$, where

$$T^{0,\beta}(S) = \{\mathbf{a}(q) \ : \ \mathbf{a}\in C^{0,\beta}(S) \quad \text{and} \quad \mathbf{a}.\mathbf{n}=0\}$$

contains all Hölder-continuous tangential densities $\mathbf{a}(q)$; the spaces $T^{m,\beta}(S)$ are defined similarly. Kress [12] has pointed out that the natural space to use is $T_d^{0,\beta}(S)$, where

$$T_d^{0,\beta}(S) = \{\mathbf{a}(q) \ : \ \mathbf{a}\in T^{0,\beta}(S) \quad \text{and} \quad \operatorname{Div}\mathbf{a}\in C^{0,\beta}(S)\}$$

contains all $\mathbf{a}$ in $T^{0,\beta}(S)$ with a Hölder-continuous surface divergence, $\operatorname{Div}\mathbf{a}$. $T_d^{0,\beta}(S)$ is a normed space, with norm

$$\|\mathbf{a}\|_{T_d^{0,\beta}(S)} = \max\left\{\|\mathbf{a}\|_{T^{0,\beta}(S)}, \|\operatorname{Div}\mathbf{a}\|_{C^{0,\beta}(S)}\right\}.$$

From [2], [12] and [9], we have the following properties:

$$M : T^{0,\beta}(S) \longrightarrow T^{0,\beta}(S) \quad \text{and} \quad M : T_d^{0,\beta}(S) \longrightarrow T_d^{0,\beta}(S)$$

are compact;

$$P : T^{1,\beta}(S) \longrightarrow T^{0,\beta}(S) \quad \text{and} \quad P : T_d^{0,\beta}(S) \longrightarrow T_d^{0,\beta}(S)$$

are bounded; and

$$P_e - P_i : T^{0,\beta}(S) \longrightarrow T^{1,\beta}(S) \tag{18}$$

is bounded, whence

$$P_e - P_i : T^{0,\beta}(S) \longrightarrow T^{0,\beta}(S)$$

is compact. Note that, according to (18), $(P_e - P_i)\mathbf{a}$ is smoother than $\mathbf{a}$. However, $\operatorname{Div}\{(P_e - P_i)\mathbf{a}\}$ is not smoother than $\operatorname{Div}\mathbf{a}$, and so $(P_e - P_i)$ is *not* compact from $T_d^{0,\beta}(S)$ into itself.

4.2 Adjoints

We define the adjoint of M, namely M', so that

$$\langle M\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, M'\mathbf{b}\rangle$$

for every $\mathbf{a}$, $\mathbf{b} \in T^{0,\beta}(S)$, where $\langle \mathbf{a}, \mathbf{b}\rangle = \int_S \mathbf{a}.\mathbf{b}\, ds$. From [2, §2.7] and $(7)_1$, we have

$$M'\mathbf{a} = \mathbf{n} \times M\{\mathbf{n} \times \mathbf{a}\}.$$

From $(7)_2$, we have

$$\mathbf{N}\{\mathbf{n} \times \mathbf{a}\} = 2P\{\mathbf{n} \times (\mathbf{n} \times \mathbf{a})\} = -2P\mathbf{a}, \qquad \mathbf{a} \in T_d^{0,\beta}(S).$$

Since $\mathbf{N}$ is known to be self adjoint, we have

$$\langle P\mathbf{a}, \mathbf{b}\rangle = -\langle \mathbf{n} \times \mathbf{a}, P\{\mathbf{n} \times \mathbf{b}\}\rangle$$

whence

$$P'\mathbf{a} = \mathbf{n} \times P\{\mathbf{n} \times \mathbf{a}\}.$$

It follows that, for $\mathbf{a} \in T^{0,\beta}(S)$, we have

$$\mathbf{n} \times M'\mathbf{a} = -M\{\mathbf{n} \times \mathbf{a}\} \quad \text{and} \quad \mathbf{n} \times M_e'\{M_i'\mathbf{a}\} = M_e M_i\{\mathbf{n} \times \mathbf{a}\}; \tag{19}$$

the same relations also hold for P.

4.3 Operator products

For Maxwell's equations in B_i, the Stratton-Chu representation gives (16) and (17), namely

$$(I+M)\{\mathbf{n}\times\mathbf{E}\}+\tfrac{\mathrm{i}}{\omega\varepsilon}P\{\mathbf{n}\times\mathbf{H}\}=\mathbf{0}, \tag{20}$$

and

$$(I+M)\{\mathbf{n}\times\mathbf{H}\}-\tfrac{\mathrm{i}}{\omega\mu}P\{\mathbf{n}\times\mathbf{E}\}=\mathbf{0}. \tag{21}$$

Let us represent the field $\{\mathbf{E},\mathbf{H}\}$ as

$$\mathbf{E}(P)=(C\mathbf{m})(P), \quad \mathbf{H}(P)=-\tfrac{\mathrm{i}}{\omega\mu}(F\mathbf{m})(P), \quad P\in B_i,$$

where $\mathbf{m}\in T_d^{0,\beta}(S)$; $\{\mathbf{E},\mathbf{H}\}$ satisfies Maxwell's equations for any $\mathbf{m}$. On S, we have

$$\mathbf{n}\times\mathbf{E}=(-I+M)\mathbf{m} \quad \text{and} \quad \mathbf{n}\times\mathbf{H}=-\tfrac{\mathrm{i}}{\omega\mu}P\mathbf{m}.$$

Substituting these boundary values into (20) and (21), we obtain the formulae

$$P^2=k^2(I-M^2) \tag{22}$$

and

$$MP+PM=0, \tag{23}$$

where $k^2=\omega^2\mu\varepsilon$. The formula (22) is equivalent to [2, Eq. (4.56)]; it will be used later. The same formulae hold for the corresponding adjoint operators. Analogous formulae for the boundary integral operators of acoustics are well known [21], [9].

5 Pseudodifferential operators

In the sequel, we shall make use of the operator

$$\mathcal{A}_\lambda=(I+M_i)(I+M_e)+(\lambda/k_i^2)P_iP_e, \tag{24}$$

where $\lambda\geq 0$ is a real parameter. From (22), we have

$$P_iP_e=P_i(P_e-P_i)+k_i^2(I-M_i^2), \tag{25}$$

whence, using some results from §4.1, we see that

$$\mathcal{A}_\lambda: T^{0,\beta}(S)\longrightarrow T^{0,\beta}(S)$$

is a bounded operator.

The continuity of $\mathcal{A}_\lambda$ can also be shown by interpreting it as a pseudodifferential operator [22], [24]. Thus, at any $p\in S$, introduce a local system of orthonormal coordinates (x,y,z) so that $\mathbf{n}(p)=(0,0,1)$. Then, the principal symbol of $\mathcal{A}_\lambda$ for the smooth bounded surface S, $\sigma(\mathcal{A}_\lambda)$, is the same as when S is replaced by the tangent plane at p; the latter is readily calculated by taking Fourier transforms in the xy-plane. Let

$$\mathcal{F}w=\hat{w}(\xi_1,\xi_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\,\exp(\mathrm{i}\boldsymbol{\xi}.\mathbf{x})\,dx\,dy,$$

where $\mathbf{x} = (x, y)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Also, for any $\mathbf{u} = (u, v, 0)$, we have

$$\mathbf{n} \times \operatorname{curl}\operatorname{curl}\mathbf{u} = \left(-\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}, \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial z^2}, 0\right).$$

Since

$$\mathcal{F}\{\mathrm{e}^{\mathrm{i}k|\mathbf{x}|}/(2\pi|\mathbf{x}|)\} = (\xi^2 - k^2)^{-1/2}, \tag{26}$$

where $\xi^2 = |\boldsymbol{\xi}|^2$, we find that

$$\sigma(P) = \frac{1}{|\boldsymbol{\xi}|}\begin{pmatrix} \xi_1\xi_2 & \xi_2^2 \\ -\xi_1^2 & -\xi_1\xi_2 \end{pmatrix}, \tag{27}$$

in agreement with [23, Lemma 2.7]. Similarly,

$$\sigma(P_e - P_i) = \frac{k_e^2 - k_i^2}{2|\boldsymbol{\xi}|^3}\begin{pmatrix} \xi_1\xi_2 & \xi_2^2 - 2\xi^2 \\ -\xi_1^2 + 2\xi^2 & -\xi_1\xi_2 \end{pmatrix},$$

whence

$$\sigma(P_\alpha(P_e - P_i)) = \frac{k_e^2 - k_i^2}{\xi^2}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix}.$$

Then, from (25), we obtain

$$\sigma(P_iP_e) = \frac{1}{\xi^2}\begin{pmatrix} k_i^2\xi_1^2 + k_e^2\xi_2^2 & (k_i^2 - k_e^2)\xi_1\xi_2 \\ (k_i^2 - k_e^2)\xi_1\xi_2 & k_i^2\xi_1^2 + k_e^2\xi_2^2 \end{pmatrix}.$$

This shows that P_iP_e is a pseudodifferential operator of order zero. In general, if A is a classical pseudodifferential operator, of integer order m, defined on a compact C^∞-manifold S, we have [24, Chpt. 9]

$$A : C^{k,\beta}(S) \longrightarrow C^{k-m,\beta}(S), \qquad 0 < \beta < 1,$$

for every integer $k \geq m$.

Note that (27) shows that P_α is a pseudodifferential operator of order 1, whence P_iP_e is expected to be of order 2; since $\sigma(P_i)\sigma(P_e) \equiv 0$, we deduce merely that the order of P_iP_e is less than or equal to 1.

From (24), we have

$$\sigma(\mathcal{A}_\lambda) = (1 + \lambda)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda(k_e^2 - k_i^2)}{k_i^2\,\xi^2}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix},$$

which is independent of ω. Since

$$\det\{\sigma(\mathcal{A}_\lambda)\} = (1 + \lambda)(1 + \lambda k_e^2/k_i^2)$$

does not vanish (for $\lambda \geq 0$), we deduce that $\mathcal{A}_\lambda$ is an elliptic pseudodifferential operator of order zero. In fact, $\mathcal{A}_\lambda$ is a Fredholm operator with index zero (this means that the Fredholm alternative holds). We show this using a homotopy argument. Thus, consider the family of Fredholm operators

$$\mathcal{A}_{\lambda t} : T^{0,\beta}(S) \longrightarrow T^{0,\beta}(S),$$

parametrized by t, with $0 \le t \le 1$ and a fixed $\lambda > 0$. From [22, Proposition 8.1], we know that $\operatorname{index}(\mathcal{A}_{\lambda t})$ is locally constant as t is varied. Hence

$$\operatorname{index}(\mathcal{A}_\lambda) = \operatorname{index}(\mathcal{A}_0) = 0,$$

since $\mathcal{A}_0 - I$ is compact.

Since $\mathcal{A}_\lambda$ is an elliptic operator of order zero, we have the following regularity results: any solution in $T^{0,\beta}(S)$ of the inhomogeneous equation $\mathcal{A}_\lambda \mathbf{a} = \mathbf{f}$ inherits additional smoothness from $\mathbf{f}$, so that $\mathbf{f} \in T^{m,\beta}(S)$ implies that $\mathbf{a} \in T^{m,\beta}(S)$, where $m \ge 0$ and $0 < \beta < 1$; in particular, if $\mathbf{a}$ solves the homogeneous equation $\mathcal{A}_\lambda \mathbf{a} = \mathbf{0}$, then

$$\mathbf{a} \in \bigcap_{m \ge 0} T^{m,\beta}(S) \subset C^\infty(S).$$

6 Pairs of coupled integral equations

The usual method of solving the transmission problem is to reduce it to a pair of coupled boundary integral equations. There are two standard approaches, namely the *direct method* and the *indirect method*.

6.1 The direct method

If we use (1) in the Stratton-Chu representations (10)–(13), we obtain the representations

$$\begin{aligned} 2\mathbf{E}_e(P) &= \tfrac{\mathrm{i}}{\omega\varepsilon_e}(F_e\mathbf{J})(P) - (C_e\mathbf{M})(P), \quad \mathbf{H}_e(P) = -\tfrac{\mathrm{i}}{\omega\mu_e}\operatorname{curl}\mathbf{E}_e(P), \quad P \in B_e, && (28)\\ -2\mathbf{E}_i(P) &= \tfrac{\mathrm{i}}{\omega\varepsilon_i}(F_i\mathbf{J})(P) - (C_i\mathbf{M})(P), \quad \mathbf{H}_i(P) = -\tfrac{\mathrm{i}}{\omega\mu_i}\operatorname{curl}\mathbf{E}_i(P), \quad P \in B_i, && (29) \end{aligned}$$

where, as is customary, we have defined $\mathbf{J}$ and $\mathbf{M}$ by

$$\mathbf{J}(p) = \mathbf{n} \times \mathbf{H} \quad \text{and} \quad \mathbf{M}(p) = -\mathbf{n} \times \mathbf{E}, \quad p \in S.$$

Similarly, using (1) in (14)–(17), we obtain

$$(I - M_e)\mathbf{J} - \tfrac{\mathrm{i}}{\omega\mu_e}P_e\mathbf{M} = 2\mathbf{n} \times \mathbf{H}_{\mathrm{inc}}, \tag{30}$$

$$(I - M_e)\mathbf{M} + \tfrac{\mathrm{i}}{\omega\varepsilon_e}P_e\mathbf{J} = -2\mathbf{n} \times \mathbf{E}_{\mathrm{inc}}, \tag{31}$$

$$(I + M_i)\mathbf{J} + \tfrac{\mathrm{i}}{\omega\mu_i}P_i\mathbf{M} = \mathbf{0}, \tag{32}$$

$$(I + M_i)\mathbf{M} - \tfrac{\mathrm{i}}{\omega\varepsilon_i}P_i\mathbf{J} = \mathbf{0}. \tag{33}$$

These are four boundary integral equations for the two unknowns $\mathbf{J}(q)$ and $\mathbf{M}(q)$. To proceed, we must choose two equations or two linear combinations of equations. Let us consider the two combinations

$$\alpha_1(30) + \alpha_2(31) + \alpha_3(32) \quad \text{and} \quad \beta_1(30) + \beta_2(31) + \beta_3(33), \tag{34}$$

Formulation	α_1	α_2	α_3	β_1	β_2	β_3
E-field	0	1	0	0	0	1
H-field	0	0	1	1	0	0
Combined field	1	0	-1	0	1	-1
Mautz-Harrington	1	0	$-\beta$	0	1	$-\alpha$
Müller	μ_e	0	μ_i	0	ε_e	ε_i

Table 1: Direct method: choices of constants in (34).

where the α's and β's are constants to be chosen. Harrington [6] describes several possible choices; see Table 1. For all these choices, we always have *existence*: $\mathbf{J}$ and $\mathbf{M}$ are just the tangential components of $\mathbf{H}$ and $\mathbf{E}$, respectively, and we already know that the transmission problem always has precisely one solution. However, the question of *uniqueness* is less obvious.

Müller's system of equations is uniquely solvable [19]; see also [7, §6.27]. The combined-field formulation (also known as the PMCHW formulation) also gives a uniquely-solvable system of equations [17]. It turns out that we obtain the same system if we view the representations (28) and (29) as given, and then impose the transmission conditions, that is we use an indirect method with the Stratton-Chu representations; this approach has been used in [4].

The Mautz-Harrington system is uniquely solvable, provided the constants α and β are such that $\alpha\overline{\beta}$ is real and positive [17], [6]. This system includes the Müller and combined-field systems as special cases.

Let us now consider the E-field formulation, namely the pair (31) and (33). All the irregular frequencies of this system are identified in the next theorem.

Theorem 6.1 *The E-field system of integral equations,* (31) *and* (33)*, is uniquely solvable if, and only if, k_e^2 is not an eigenvalue of the interior Maxwell problem.*

Proof. Suppose that $\mathbf{J}_0$ and $\mathbf{M}_0$ solve the homogeneous forms of (31) and (33), namely

$$(I - M_e)\mathbf{M}_0 + \tfrac{\mathrm{i}}{\omega\varepsilon_e} P_e \mathbf{J}_0 = \mathbf{0}, \tag{35}$$

$$(I + M_i)\mathbf{M}_0 - \tfrac{\mathrm{i}}{\omega\varepsilon_i} P_i \mathbf{J}_0 = \mathbf{0}. \tag{36}$$

Assume that $\mathbf{J}_0$ and $\mathbf{M}_0$ are not both identically zero. We show that this can only occur if k_e^2 is an eigenvalue of the interior Maxwell problem. Define fields $\{\mathbf{E}_\alpha^0, \mathbf{H}_\alpha^0\}$ in B_α ($\alpha = e,\, i$) by (28) and (29) with $\mathbf{J}$ and $\mathbf{M}$ replaced by $\mathbf{J}_0$ and $\mathbf{M}_0$, respectively. On S, we find that

$$\begin{aligned} \mathbf{n} \times \mathbf{E}_e^0 &= \mathbf{n} \times \mathbf{E}_i^0 = -\mathbf{M}_0, \\ 2\mathbf{n} \times \mathbf{H}_e^0 &= (I + M_e)\mathbf{J}_0 + \tfrac{\mathrm{i}}{\omega\mu_e} P_e \mathbf{M}_0, \\ 2\mathbf{n} \times \mathbf{H}_i^0 &= (I - M_i)\mathbf{J}_0 - \tfrac{\mathrm{i}}{\omega\mu_i} P_i \mathbf{M}_0. \end{aligned} \tag{37}$$

Now, construct the following fields:

$$2\widetilde{\mathbf{E}}_e(P) = -\tfrac{\mathrm{i}}{\omega\varepsilon_i} F_i \mathbf{J}_0 + C_i \mathbf{M}_0, \quad 2\widetilde{\mathbf{H}}_e(P) = -C_i \mathbf{J}_0 - \tfrac{\mathrm{i}}{\omega\mu_i} F_i \mathbf{M}_0, \quad P \in B_e,$$

$$2\widetilde{\mathbf{E}}_i(P) = \tfrac{\mathrm{i}}{\omega\varepsilon_e} F_e \mathbf{J}_0 - C_e \mathbf{M}_0, \quad 2\widetilde{\mathbf{H}}_i(P) = C_e \mathbf{J}_0 + \tfrac{\mathrm{i}}{\omega\mu_e} F_e \mathbf{M}_0 + C_e \mathbf{m}_0, \quad P \in B_i.$$

By construction, $\{\widetilde{\mathbf{E}}_e, \widetilde{\mathbf{H}}_e\}$ satisfies Maxwell's equations (for the interior material) in B_e and the Silver-Müller radiation conditions. Also,

$$\mathbf{n} \times \widetilde{\mathbf{E}}_e = \mathbf{0}, \quad p \in S,$$

by (36). The uniqueness theorem for the exterior Maxwell problem [2, Thm. 4.18] then implies that $\{\widetilde{\mathbf{E}}_e, \widetilde{\mathbf{H}}_e\}$ vanishes identically in B_e. In particular,

$$\mathbf{0} = \mathbf{n} \times \widetilde{\mathbf{H}}_e = \mathbf{n} \times \mathbf{H}_i^0 - \mathbf{J}_0.$$

Similarly, $\{\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{H}}_i\}$ satisfies Maxwell's equations (for the exterior material) in B_i and

$$\mathbf{n} \times \widetilde{\mathbf{E}}_i = \mathbf{0}, \quad p \in S,$$

by (35). So, if k_e^2 is not an eigenvalue of the interior Maxwell problem, we deduce that $\{\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{H}}_i\}$ vanishes identically in B_i, whence

$$\mathbf{0} = \mathbf{n} \times \widetilde{\mathbf{H}}_i = \mathbf{n} \times \mathbf{H}_e^0 - \mathbf{J}_0.$$

Thus

$$\mathbf{n} \times \mathbf{H}_e^0 = \mathbf{n} \times \mathbf{H}_i^0 = \mathbf{J}_0, \tag{38}$$

the fields $\{\mathbf{E}_\alpha^0, \mathbf{H}_\alpha^0\}$ satisfy the homogeneous transmission problem and hence must vanish, whence (38) and (37) imply that $\mathbf{J}_0 = \mathbf{M}_0 = \mathbf{0}$, which is contrary to our assumptions.

We have just shown that non-uniqueness for the E-field formulation implies that k_e^2 is an eigenvalue of the interior Maxwell problem. We now prove the converse. At such a value of k_e^2, we know that there is a non-trivial tangential density $\mathbf{a}(q)$ satisfying

$$(I + M_e)\mathbf{a} = \mathbf{0} \quad \text{and} \quad P_e \mathbf{a} = \mathbf{0};$$

this follows by using the Stratton-Chu representation in B_i, with $\mathbf{a} = \mathbf{n} \times \mathbf{H}$. Clearly, $\mathbf{a}$ also satisfies the homogeneous equation

$$\mathcal{A}_\tau \mathbf{a} = \left\{ (I + M_i)(I + M_e) + \tfrac{1}{\omega^2 \mu_i \varepsilon_e} P_i P_e \right\} \mathbf{a} = \mathbf{0}, \tag{39}$$

where $\tau = \varepsilon_i/\varepsilon_e$. From §5, we know that $\mathcal{A}_\tau$ is a Fredholm operator with index zero. Hence, there exists a non-trivial solution, $\mathbf{b} \in T_d^{0,\beta}(S)$, of the corresponding adjoint homogeneous equation, namely

$$\mathcal{A}'_\tau \mathbf{b} = \left\{ (I + M'_e)(I + M'_i) + \tfrac{1}{\omega^2 \mu_i \varepsilon_e} P'_e P'_i \right\} \mathbf{b} = \mathbf{0}.$$

Using (19), this equation can be written as

$$\left\{ (I - M_e)(I - M_i) + \tfrac{1}{\omega^2 \mu_i \varepsilon_e} P_e P_i \right\} (\mathbf{n} \times \mathbf{b}) = \mathbf{0}. \tag{40}$$

Now, set

$$\mathbf{J}_0 = -\tfrac{\mathrm{i}}{\omega\mu_i} P_i \mathbf{c} \quad \text{and} \quad \mathbf{M}_0 = (I - M_i)\mathbf{c},$$

for some $\mathbf{c}(q) \in T_d^{0,\beta}(S)$. It follows from (22) that (36) is satisfied *identically*, for any such $\mathbf{c}$. Moreover, (35) is also seen to be satisfied if we choose

$$\mathbf{c} = \mathbf{n} \times \mathbf{b},$$

by comparison with (40). Thus, we have found a non-trivial solution to the homogeneous E-field system, (35) and (36). This concludes the proof of Theorem 6.1.

A different proof of this theorem is given in [6]; the non-trivial solutions at irregular values of k_e^2 are also given in [1]. We can also give a similar argument to show that the fields $\{\mathbf{E}_\alpha, \mathbf{H}_\alpha\}$, given by (28) and (29), will also solve the transmission problem if $\mathbf{J}$ and $\mathbf{M}$ solve the E-field system and k_e^2 is not an eigenvalue; see [13] for analogous arguments for the acoustic transmission problem. The usual electromagnetic duality argument gives exactly the same result for the H-field formulation.

6.2 The indirect method

Suppose that we can write

$$\mathbf{E}_\alpha(P) = \frac{\mathrm{i}}{\omega\varepsilon_\alpha}(F_\alpha \mathbf{j}_\alpha)(P) - (C_\alpha \mathbf{m}_\alpha)(P), \quad \mathbf{H}_\alpha(P) = -\frac{\mathrm{i}}{\omega\mu_\alpha}\operatorname{curl}\mathbf{E}_\alpha(P), \quad P \in B_\alpha, \tag{41}$$

where $\mathbf{j}_\alpha(q)$ and $\mathbf{m}_\alpha(q)$ are tangential densities and $\alpha = e, i$. The fields $\{\mathbf{E}_e, \mathbf{H}_e\}$ and $\{\mathbf{E}_i, \mathbf{H}_i\}$ satisfy the appropriate form of Maxwell's equations in B_e and B_i, respectively. The Silver-Müller radiation conditions are also satisfied. Imposing the transmission conditions gives

$$\left.\begin{aligned} (I + M_e)\mathbf{m}_e + (I - M_i)\mathbf{m}_i - \frac{\mathrm{i}}{\omega}\left(\frac{1}{\varepsilon_e}P_e\mathbf{j}_e - \frac{1}{\varepsilon_i}P_i\mathbf{j}_i\right) &= \mathbf{n} \times \mathbf{E}_{\text{inc}}, \\ (I + M_e)\mathbf{j}_e + (I - M_i)\mathbf{j}_i + \frac{\mathrm{i}}{\omega}\left(\frac{1}{\mu_e}P_e\mathbf{m}_e - \frac{1}{\mu_i}P_i\mathbf{m}_i\right) &= -\mathbf{n} \times \mathbf{H}_{\text{inc}}. \end{aligned}\right\}$$

These are two boundary integral equations for the determination of four unknowns, namely $\mathbf{j}_e$, $\mathbf{j}_i$, $\mathbf{m}_e$ and $\mathbf{m}_i$; we need two constraints.

We know that $(P_e - P_i)$ is compact in $T^{0,\beta}(S)$, so a good theoretical choice is

$$\frac{1}{\varepsilon_e}\mathbf{j}_e = \frac{1}{\varepsilon_i}\mathbf{j}_i = \mathbf{j}(q) \quad \text{and} \quad \frac{1}{\mu_e}\mathbf{m}_e = \frac{1}{\mu_i}\mathbf{m}_i = \mathbf{m}(q),$$

say. This leads to the Fredholm system of the second kind

$$\left.\begin{aligned} (\varepsilon_e + \varepsilon_i)\mathbf{j} + (\varepsilon_e M_e - \varepsilon_i M_i)\mathbf{j} + \frac{\mathrm{i}}{\omega}(P_e - P_i)\mathbf{m} &= -\mathbf{n} \times \mathbf{H}_{\text{inc}}, \\ (\mu_e + \mu_i)\mathbf{m} + (\mu_e M_e - \mu_i M_i)\mathbf{m} - \frac{\mathrm{i}}{\omega}(P_e - P_i)\mathbf{j} &= \mathbf{n} \times \mathbf{E}_{\text{inc}}, \end{aligned}\right\} \tag{42}$$

which is to be solved for $\mathbf{j}$, $\mathbf{m} \in T_d^{0,\beta}(S)$; note that $\mathbf{n} \times \mathbf{E}_{\text{inc}}$, $\mathbf{n} \times \mathbf{H}_{\text{inc}} \in T_d^{0,\beta}(S)$. If solvable, we can then construct the solution of the transmission problem, using

$$\mathbf{E}_\alpha = \frac{\mathrm{i}}{\omega}F_\alpha\mathbf{j} - \mu_\alpha C_\alpha\mathbf{m} \quad \text{and} \quad \mathbf{H}_\alpha = \varepsilon_\alpha C_\alpha\mathbf{j} + \frac{\mathrm{i}}{\omega}F_\alpha\mathbf{m}, \quad P \in B_\alpha, \quad \alpha = e, i. \tag{43}$$

It turns out that if we set $\mathbf{j} = -\mathbf{M}$ and $\mathbf{m} = \mathbf{J}$, and interchange the materials ($e \rightleftharpoons i$), we obtain Müller's system of equations; hence the system of integral equations (42) is uniquely solvable.

Formulation	$\mathbf{j}_e$	$\mathbf{j}_i$	$\mathbf{m}_e$	$\mathbf{m}_i$
Electric current	$\mathbf{j}_e$	$\mathbf{j}_i$	$\mathbf{0}$	$\mathbf{0}$
Magnetic current	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{m}_e$	$\mathbf{m}_i$
Combined current	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{m}$	$-\mathbf{m}$
Combined source	$\mathbf{j}_e$	$\mathbf{j}_i$	$\alpha_e \mathbf{n} \times \mathbf{j}_e$	$\alpha_i \mathbf{n} \times \mathbf{j}_i$
2nd-kind Fredholm	$\varepsilon_e \mathbf{j}$	$\varepsilon_i \mathbf{j}$	$\mu_e \mathbf{m}$	$\mu_i \mathbf{m}$

Table 2: Indirect method: constraints on the surface currents.

Other choices for the surface currents $\mathbf{j}_\alpha$ and $\mathbf{m}_\alpha$ in the representations (41) are possible. Harrington [6] describes four; see Table 2, wherein α_e and α_i are constants. The first and second formulations both exhibit irregular frequencies, whereas the third and fourth do not [6]. We have existence and uniqueness for the fifth formulation. Similar results obtain for the combined-current formulation; if we set $\mathbf{j} = 2\mathbf{J}$ and $\mathbf{m} = 2\mathbf{M}$, we obtain exactly the same system of equations as with the combined-field formulation [6]. We are not aware of any existence results for the other three, although it may be possible to adapt the analysis of Kress [11] to the combined-source formulation.

7 Single integral equations

In this section, we use a hybrid of the indirect and direct methods, leading to single integral equations. Specifically, we use a representation in B_e involving a single unknown tangential density $\mathbf{j}$, and the Stratton-Chu representation in B_i. Thus, assume that we can write

$$\mathbf{E}_e(P) = a\tfrac{\mathrm{i}}{\omega\varepsilon_e} F_e \mathbf{j} - bC_e\{\mathbf{n} \times \mathbf{j}\}, \quad \mathbf{H}_e(P) = aC_e\mathbf{j} + b\tfrac{\mathrm{i}}{\omega\mu_e} F_e\{\mathbf{n} \times \mathbf{j}\}, \quad P \in B_e, \tag{44}$$

where the constants a and b are at our disposal; the use of $\mathbf{n} \times \mathbf{j}$ rather than $\mathbf{j}$ for two of the densities is convenient, and facilitates comparison with [16]. The Stratton-Chu representations in B_i, (12) and (13), together with (1), give

$$-2\mathbf{E}_i(P) = \tfrac{\mathrm{i}}{\omega\varepsilon_i} F_i\{\mathbf{n} \times \mathbf{H}\} + C_i\{\mathbf{n} \times \mathbf{E}\}, \quad \mathbf{H}_i(P) = -\tfrac{\mathrm{i}}{\omega\mu_i} \operatorname{curl} \mathbf{E}_i, \quad P \in B_i. \tag{45}$$

Computing the tangential components on S, (44) gives

$$\mathbf{n} \times \mathbf{E}_e = a\tfrac{\mathrm{i}}{\omega\varepsilon_e} P_e \mathbf{j} - b(I + M_e)(\mathbf{n} \times \mathbf{j}) \equiv K_e \mathbf{j}, \tag{46}$$

say, and

$$\mathbf{n} \times \mathbf{H}_e = a(I + M_e)\mathbf{j} + b\tfrac{\mathrm{i}}{\omega\mu_e} P_e(\mathbf{n} \times \mathbf{j}) \equiv L_e \mathbf{j}, \tag{47}$$

say. Similarly, (45) gives (32) and (33), namely

$$(I + M_i)(\mathbf{n} \times \mathbf{E}) + \tfrac{\mathrm{i}}{\omega\varepsilon_i} P_i(\mathbf{n} \times \mathbf{H}) = \mathbf{0}, \tag{48}$$

$$(I + M_i)(\mathbf{n} \times \mathbf{H}) - \tfrac{\mathrm{i}}{\omega\mu_i} P_i(\mathbf{n} \times \mathbf{E}) = \mathbf{0}. \tag{49}$$

If we substitute from (46) and (47) into (48), using (2), we obtain

$$\left\{(I + M_i)K_e + \tfrac{\mathrm{i}}{\omega\varepsilon_i} P_i L_e\right\} \mathbf{j} = \mathbf{f} \tag{50}$$

where

$$\mathbf{f}(p) = -(I+M_i)(\mathbf{n}\times\mathbf{E}_{\rm inc}) - \tfrac{\rm i}{\omega\varepsilon_i}P_i(\mathbf{n}\times\mathbf{H}_{\rm inc}). \tag{51}$$

Similarly, (49) gives

$$\left\{(I+M_i)L_e - \tfrac{\rm i}{\omega\mu_i}P_iK_e\right\}\mathbf{j} = \mathbf{g} \tag{52}$$

where

$$\mathbf{g}(p) = -(I+M_i)(\mathbf{n}\times\mathbf{H}_{\rm inc}) + \tfrac{\rm i}{\omega\mu_i}P_i(\mathbf{n}\times\mathbf{E}_{\rm inc}). \tag{53}$$

Equation (50) is a boundary integral equation for $\mathbf{j}(q)$. Equation (52) is another boundary integral equation for $\mathbf{j}(q)$. Having solved either, $\{\mathbf{E}_e,\mathbf{H}_e\}$ and $\{\mathbf{E}_i,\mathbf{H}_i\}$ are to be constructed from (44) and

$$-2\mathbf{E}_i(P) = \tfrac{\rm i}{\omega\varepsilon_i}F_i\{\mathbf{n}\times\mathbf{H}_{\rm inc} + L_e\mathbf{j}\} + C_i\{\mathbf{n}\times\mathbf{E}_{\rm inc} + K_e\mathbf{j}\}, \quad \mathbf{H}_i(P) = -\tfrac{\rm i}{\omega\mu_i}\operatorname{curl}\mathbf{E}_i, \tag{54}$$

respectively. Note that $\mathbf{f}$ and $\mathbf{g}$ are both in $T_d^{0,\beta}(S)$. We have the following two theorems concerning solvability of the transmission problem and uniqueness.

Theorem 7.1 *If $\mathbf{j}(q)\in T_d^{0,\beta}(S)$ solves* (50) *or* (52)*, $\{\mathbf{E}_e,\mathbf{H}_e\}$ and $\{\mathbf{E}_i,\mathbf{H}_i\}$, given by* (44) *and* (54)*, respectively, solve the transmission problem.*

Proof. This is similar to the proof of Theorem 5.1 in [10]. We have to check that (1) are satisfied. We have

$$2(\mathbf{n}\times\mathbf{E}_e + \mathbf{n}\times\mathbf{E}_{\rm inc} - \mathbf{n}\times\mathbf{E}_i) = \left\{(I+M_i)K_e + \tfrac{\rm i}{\omega\varepsilon_i}P_iL_e\right\}\mathbf{j} - \mathbf{f}$$

and

$$2(\mathbf{n}\times\mathbf{H}_e + \mathbf{n}\times\mathbf{H}_{\rm inc} - \mathbf{n}\times\mathbf{H}_i) = \left\{(I+M_i)L_e - \tfrac{\rm i}{\omega\mu_i}P_iK_e\right\}\mathbf{j} - \mathbf{g}.$$

Thus, if $\mathbf{j}$ solves (50), then $(1)_1$ is satisfied, whereas if $\mathbf{j}$ solves (52), then $(1)_2$ is satisfied. Next, construct the radiating fields

$$\widetilde{\mathbf{E}}_e(P) = \tfrac{\rm i}{\omega\varepsilon_i}F_i\{\mathbf{n}\times\mathbf{H}_{\rm inc} + L_e\mathbf{j}\} + C_i\{\mathbf{n}\times\mathbf{E}_{\rm inc} + K_e\mathbf{j}\}, \quad \widetilde{\mathbf{H}}_e(P) = -\tfrac{\rm i}{\omega\mu_i}\operatorname{curl}\widetilde{\mathbf{E}}_e,$$

for $P\in B_e$. On S, we find that $\mathbf{n}\times\widetilde{\mathbf{E}}_e = \mathbf{0}$ if $\mathbf{j}$ solves (50), or $\mathbf{n}\times\widetilde{\mathbf{H}}_e = \mathbf{0}$ if $\mathbf{j}$ solves (52). In either case, the exterior uniqueness theorem implies that $\{\widetilde{\mathbf{E}}_e,\widetilde{\mathbf{H}}_e\}$ vanishes identically. Then, in the first case, $\mathbf{n}\times\widetilde{\mathbf{H}}_e = \mathbf{0}$ implies that $(1)_2$ is satisfied, whereas in the second case, $\mathbf{n}\times\widetilde{\mathbf{E}}_e = \mathbf{0}$ implies that $(1)_1$ is satisfied.

The next theorem is concerned with non-trivial solutions of the homogeneous forms of (50) and (52), namely

$$\left\{(I+M_i)K_e + \tfrac{\rm i}{\omega\varepsilon_i}P_iL_e\right\}\mathbf{j}_0 = \mathbf{0} \tag{55}$$

and

$$\left\{(I+M_i)L_e - \tfrac{\rm i}{\omega\mu_i}P_iK_e\right\}\mathbf{j}_0 = \mathbf{0}. \tag{56}$$

We show that uniqueness depends on the eigenvalues of the associated interior problem (§2).

Theorem 7.2 *The homogeneous equations* (55) *and* (56) *have a non-trivial solution if, and only if, k_e^2 is an eigenvalue of the associated interior problem.*

Proof. Suppose that $\mathbf{j}_0 \not\equiv \mathbf{0}$ solves (55) or (56). Define fields $\{\mathbf{E}_\alpha^0, \mathbf{H}_\alpha^0\}$ in B_α ($\alpha = e,\ i$) using (44) and (54), with $\mathbf{j}$ replaced by $\mathbf{j}_0$, $\mathbf{n} \times \mathbf{E}_{\text{inc}} = \mathbf{0}$ and $\mathbf{n} \times \mathbf{H}_{\text{inc}} = \mathbf{0}$. These fields solve the homogeneous transmission problem, and so vanish identically. Now, construct the following fields:

$$\widetilde{\mathbf{E}}_i(P) = a\tfrac{\mathrm{i}}{\omega\varepsilon_e} F_e \mathbf{j}_0 - bC_e\{\mathbf{n} \times \mathbf{j}_0\}, \quad \widetilde{\mathbf{H}}_i(P) = aC_e\mathbf{j}_0 + b\tfrac{\mathrm{i}}{\omega\mu_e} F_e\{\mathbf{n} \times \mathbf{j}_0\}, \quad P \in B_i.$$

On S, we find that

$$\mathbf{n} \times \widetilde{\mathbf{E}}_i = \mathbf{n} \times \mathbf{E}_e^0 + 2b(\mathbf{n} \times \mathbf{j}_0) = 2b(\mathbf{n} \times \mathbf{j}_0), \quad \mathbf{n} \times \widetilde{\mathbf{H}}_i = \mathbf{n} \times \mathbf{H}_e^0 - 2a\mathbf{j}_0 = -2a\mathbf{j}_0;$$

hence,

$$a(\mathbf{n} \times \widetilde{\mathbf{E}}_i) + b\{\mathbf{n} \times (\mathbf{n} \times \widetilde{\mathbf{H}}_i)\} = \mathbf{0}, \quad p \in S. \tag{57}$$

Thus, either $\{\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{H}}_i\}$ is an eigenfunction of the associated interior problem, or it vanishes identically; we can eliminate the second possibility since it implies that $\mathbf{j}_0 \equiv \mathbf{0}$, contrary to hypothesis.

Conversely, suppose that k_e^2 is an eigenvalue of the associated interior problem. Then the Stratton-Chu representations give

$$(I + M_e)(\mathbf{n} \times \widetilde{\mathbf{E}}_i) + \tfrac{\mathrm{i}}{\omega\varepsilon_e} P_e(\mathbf{n} \times \widetilde{\mathbf{H}}_i) = \mathbf{0},$$

$$(I + M_e)(\mathbf{n} \times \widetilde{\mathbf{H}}_i) - \tfrac{\mathrm{i}}{\omega\mu_e} P_e(\mathbf{n} \times \widetilde{\mathbf{E}}_i) = \mathbf{0}.$$

Using the boundary condition (57), these give $K_e\{\mathbf{n} \times \widetilde{\mathbf{H}}_i\} = \mathbf{0}$ and $L_e\{\mathbf{n} \times \widetilde{\mathbf{H}}_i\} = \mathbf{0}$, respectively. Hence, $\mathbf{n} \times \widetilde{\mathbf{H}}_i$ is a non-trivial solution of both (55) and (56), if $a \neq 0$; if $a = 0$, $\mathbf{n} \times \widetilde{\mathbf{E}}_i$ is a non-trivial solution.

Let us now examine the structure of (50) and (52) in more detail, with the intention of establishing existence. Set

$$\rho = \mu_i/\mu_e \quad \text{and} \quad \tau = \varepsilon_i/\varepsilon_e.$$

Then (50) and (52) can be written as

$$a\tfrac{\mathrm{i}}{\omega\varepsilon_i}\mathcal{B}_\tau\mathbf{j} - b\mathcal{A}_\rho(\mathbf{n} \times \mathbf{j}) = \mathbf{f} \tag{58}$$

and

$$a\mathcal{A}_\tau\mathbf{j} + b\tfrac{\mathrm{i}}{\omega\mu_i}\mathcal{B}_\rho(\mathbf{n} \times \mathbf{j}) = \mathbf{g}, \tag{59}$$

where $\mathcal{A}_\lambda$ is defined by (24) and

$$\mathcal{B}_\lambda = P_i(I + M_e) + \lambda(I + M_i)P_e. \tag{60}$$

$\mathcal{B}_\lambda$ is a pseudodifferential operator of order $+1$; the determinant of its principal symbol is identically zero.

Consider (58). If we want only Fredholm operators, we must take $a = 0$; without loss of generality, we can set $b = -1$. This gives

$$\mathcal{A}_\rho(\mathbf{n} \times \mathbf{j}) = \mathbf{f}, \tag{61}$$

whilst (59) becomes

$$\mathcal{B}_\rho(\mathbf{n}\times\mathbf{j}) = \mathrm{i}\omega\mu_i\mathbf{g}. \tag{62}$$

Having solved either,

$$\mathbf{E}_e(P) = C_e\{\mathbf{n}\times\mathbf{j}\}, \quad \mathbf{H}_e(P) = -\tfrac{\mathrm{i}}{\omega\mu_e}F_e\{\mathbf{n}\times\mathbf{j}\}, \quad P\in B_e, \tag{63}$$

and $\{\mathbf{E}_i,\mathbf{H}_i\}$ is given by (54), wherein $K_e\mathbf{j} = (I+M_e)(\mathbf{n}\times\mathbf{j})$ and $L_e\mathbf{j} = -\frac{\mathrm{i}}{\omega\mu_e}P_e(\mathbf{n}\times\mathbf{j})$.

Similar considerations for (59) lead to $a=1$ and $b=0$. This gives

$$\mathcal{A}_\tau\mathbf{j} = \mathbf{g} \tag{64}$$

and

$$\mathcal{B}_\tau\mathbf{j} = -\mathrm{i}\omega\varepsilon_i\mathbf{f}; \tag{65}$$

having solved either,

$$\mathbf{E}_e(P) = \tfrac{\mathrm{i}}{\omega\varepsilon_e}F_e\mathbf{j}, \quad \mathbf{H}_e(P) = C_e\mathbf{j}, \quad P\in B_e, \tag{66}$$

and $\{\mathbf{E}_i,\mathbf{H}_i\}$ is given by (54), where now $K_e = \frac{\mathrm{i}}{\omega\varepsilon_e}P_e$ and $L_e = (I+M_e)$.

Equation (65) is a hypersingular integral equation. It was derived previously by Marx [14], [15], using a different method. Glisson [5] rederived Marx's equation and noted that irregular frequencies would occur when k_e^2 was an eigenvalue of the interior Maxwell problem; this is consistent with Theorem 7.2. He also suggested using the representation (63), but did not derive any associated integral equations. The two singularintegral equations (61) and (64) are preferable theoretically and are new. We have the following results.

Theorem 7.3 *Assume that k_e^2 is not an eigenvalue of the interior Maxwell problem. Then (61) is uniquely solvable in $T^{0,\beta}(S)$ for any $\mathbf{f}\in T^{0,\beta}(S)$ and (64) is uniquely solvable in $T^{0,\beta}(S)$ for any $\mathbf{g}\in T^{0,\beta}(S)$. Moreover, if $\mathbf{f}$ is given by (51) (or $\mathbf{g}$ by (53)), the solution will be in $T_d^{0,\beta}(S)$.*

Proof. The first part follows from the Fredholm alternative, which gives existence from uniqueness; we have the latter from Theorem 7.2. For the second part, consider (61). Let $\{\widetilde{\mathbf{E}}_e,\widetilde{\mathbf{H}}_e\}$, $\{\widetilde{\mathbf{E}}_i,\widetilde{\mathbf{H}}_i\}$ solve the transmission problem. We know that such a solution exists (§6), and that

$$\mathbf{n}\times\widetilde{\mathbf{E}}_e \in T_d^{0,\beta}(S),$$

using Maxwell's equations and the formula

$$\mathrm{Div}\,(\mathbf{n}\times\widetilde{\mathbf{E}}_e) = -\mathbf{n}.\mathrm{curl}\,\widetilde{\mathbf{E}}_e.$$

Next, consider the Fredholm integral equation of the second kind

$$(I+M_e)(\mathbf{n}\times\mathbf{j}) = \mathbf{n}\times\widetilde{\mathbf{E}}_e.$$

Since k_e^2 is not an eigenvalue of the interior Maxwell problem, this equation has a unique solution $\mathbf{j}\in T_d^{0,\beta}(S)$ [12]. It remains to show that $\mathbf{j}$ solves (61). To this end, construct the fields $\{\mathbf{E}_\alpha, \mathbf{H}_\alpha\}$, using (63) and (54). We find that $\{\mathbf{E}_e, \mathbf{H}_e\}$ solves the exterior Maxwell problem, with

$$\mathbf{n} \times \mathbf{E}_e = \mathbf{n} \times \widetilde{\mathbf{E}}_e \qquad \text{on } S,$$

whence $\{\mathbf{E}_e, \mathbf{H}_e\} \equiv \{\widetilde{\mathbf{E}}_e, \widetilde{\mathbf{H}}_e\}$, by uniqueness for the exterior problem. The Stratton-Chu formulae, (12) and (13), then show that $\{\mathbf{E}_i, \mathbf{H}_i\} \equiv \{\widetilde{\mathbf{E}}_i, \widetilde{\mathbf{H}}_i\}$. Finally, the interface condition

$$\mathbf{0} = 2(\widetilde{\mathbf{E}}_e + \mathbf{E}_{\text{inc}} - \widetilde{\mathbf{E}}_i) = 2(\mathbf{E}_e + \mathbf{E}_{\text{inc}} - \mathbf{E}_i) = \mathcal{A}_\rho(\mathbf{n} \times \mathbf{j}) - \mathbf{f}$$

gives the desired result. A similar argument succeeds for (64).

Theorem 7.4 *Assume that k_e^2 is not an eigenvalue of the interior Maxwell problem. Then Marx's equation,* (65)*, and* (62) *are both uniquely solvable.*

Proof. We cannot prove existence for every $\mathbf{f}$ and $\mathbf{g}$ in $T_d^{0,\beta}(S)$, but only for those $\mathbf{f}$ and $\mathbf{g}$ defined by (51) and (53), respectively. Thus, given an incident field $\{\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}}\}$, Theorem 7.3 says that we can find $\mathbf{j}$ uniquely by solving (64). We then construct the fields $\{\mathbf{E}_\alpha, \mathbf{H}_\alpha\}$, using (66) and (54). By Theorem 7.1, these fields will solve the transmission problem. In particular, the transmission condition $(1)_1$ implies that $\mathbf{j}$ satisfies (65). Uniqueness follows from Theorem 7.2. A similar argument succeeds for (62).

We would like to obtain a single integral equation that does not suffer from irregular frequencies. By Theorem 7.2 and the remarks at the end of §2, we see that we can secure *uniqueness* by making different choices for a and b. However, if a and b are both non-zero, we always obtain integral equations involving non-Fredholm operators, and so the question of *existence* remains. We discuss this in the next section.

8 Single integral equations without irregular frequencies

In this section, we describe two methods of obtaining single integral equations that are uniquely solvable at all frequencies.

8.1 Mautz's equation

Mautz [16] has suggested choosing $a = 1$ and $b = \alpha$, where α is a non-zero constant. This leads to the representations

$$\mathbf{E}_e(P) = \tfrac{\mathrm{i}}{\omega\varepsilon_e} F_e \mathbf{j} - \alpha C_e\{\mathbf{n} \times \mathbf{j}\}, \quad \mathbf{H}_e(P) = C_e \mathbf{j} + \alpha \tfrac{\mathrm{i}}{\omega\mu_e} F_e\{\mathbf{n} \times \mathbf{j}\}, \quad P \in B_e, \tag{67}$$

and to two integral equations,

$$\mathcal{L}_1 \mathbf{j} \equiv \tfrac{\mathrm{i}}{\omega\varepsilon_i} \mathcal{B}_\tau \mathbf{j} - \alpha \mathcal{A}_\rho(\mathbf{n} \times \mathbf{j}) = \mathbf{f} \tag{68}$$

and

$$\mathcal{L}_2 \mathbf{j} \equiv \mathcal{A}_\tau \mathbf{j} + \alpha \tfrac{\mathrm{i}}{\omega\mu_i} \mathcal{B}_\rho(\mathbf{n} \times \mathbf{j}) = \mathbf{g}. \tag{69}$$

Mautz derived (68) and noted that his equation has at most one solution if $\operatorname{Re}\alpha > 0$. In fact, we have uniqueness for both (68) and (69) if $\operatorname{Re}\alpha \neq 0$.

In order to prove existence at all frequencies, we modify an ingenious argument used by Kress [11], [2, §4.6] to regularise a related singular integral equation for the exterior Maxwell problem.

We start by choosing a frequency ω^* so that $k_e^* = \omega^* \sqrt{\mu_e \varepsilon_e}$ is such that

$$(k_e^*)^2 \text{ is not an eigenvalue of the interior Maxwell problem} \tag{70}$$

and

$$(k_e^*)^2 \text{ is not an eigenvalue of the interior Dirichlet problem} \tag{71}$$

(so that the only solution of $(\nabla^2 + (k_e^*)^2)u = 0$ in B_i with $u = 0$ on S is $u \equiv 0$). We use an asterisk to denote any quantity evaluated at ω^*. The aim is to show that (68) and (69) have a solution at ω^* by actually constructing it.

By Theorem 7.1, Theorem 7.3 and (70), we know that we can solve the transmission problem at ω^* by solving (61). Thus, we have

$$\mathbf{m}^* = \left\{\mathcal{A}_\rho^*\right\}^{-1} \mathbf{f}^*,$$

where the exterior field is given by

$$\mathbf{E}_e^*(P) = C_e^* \mathbf{m}^*, \quad \mathbf{H}_e^*(P) = -\tfrac{\mathrm{i}}{\omega^* \mu_e} F_e^* \mathbf{m}^*, \quad P \in B_e,$$

and the interior field is given by (54). Now, construct the potential

$$\mathbf{F}(P) = (S_e^* \mathbf{m}^*)(P), \quad P \in B_e.$$

$\mathbf{F}$ satisfies the vector Helmholtz equation in B_e and, by [2, Corollary 4.14], the radiation condition

$$\hat{\mathbf{r}}_P \times \operatorname{curl} \mathbf{F} - \hat{\mathbf{r}}_P \operatorname{div} \mathbf{F} + \mathrm{i} k_e^* \mathbf{F} = o(r_P^{-1}) \quad \text{as } r_P \to \infty.$$

On S, we can compute

$$\mathbf{n} \times \mathbf{F} = \mathbf{c}(q), \text{ say,} \quad \text{and} \quad \operatorname{div} \mathbf{F} + \tfrac{\mathrm{i}\omega^* \mu_e}{\alpha^*} \mathbf{n}.\mathbf{F} = \gamma(q), \text{ say,}$$

where we have written $\alpha^* = \alpha(\omega^*)$, since α can depend on ω. From the proof of Theorem 4.42 in [2], we know that $\mathbf{F}$ has an alternative representation, namely

$$\mathbf{F}(P) = C_e^* \mathbf{b} + \tfrac{\mathrm{i}\omega^* \mu_e}{\alpha^*} \left\{ S_e^*(\mathbf{n} \times \mathbf{b}) + \operatorname{grad}(S_e^* \lambda) \right\}, \quad P \in B_e,$$

(it is here that we use (71)), where the densities $\mathbf{b} \in T_d^{0,\beta}(S)$ and $\lambda \in C^{0,\beta}(S)$ are given by

$$\begin{pmatrix} \mathbf{b} \\ \lambda \end{pmatrix} = \mathcal{M}^* \begin{pmatrix} \mathbf{c} \\ \gamma \end{pmatrix},$$

for a certain matrix of bounded operators, $\mathcal{M}^*$ (see [2, Eq. (4.71)]). If we set

$$\mathbf{j}^* = \tfrac{\mathrm{i}\omega^* \mu_e}{\alpha^*} \mathbf{n} \times \mathbf{b},$$

we see that

$$\operatorname{curl}\mathbf{F} = C_e^*\mathbf{j}^* + \alpha^*\tfrac{\mathrm{i}}{\omega^*\mu_e}F_e^*(\mathbf{n}\times\mathbf{j}^*).$$

So, comparing with (67), we take

$$\mathbf{H}_e^*(P) = \operatorname{curl}\mathbf{F}, \quad \mathbf{E}_e^*(P) = \tfrac{\mathrm{i}}{\omega^*\varepsilon_e}\operatorname{curl}\mathbf{H}_e^*, \quad P\in B_e,$$

giving a representation for the exterior field in the required form. Retracing our steps shows that we can construct $\mathbf{j}^*(q)\in T_d^{0,\beta}(S)$ from $\mathbf{f}^*(q)\in T_d^{0,\beta}(S)$ (subject to (70) and (71)) in the form

$$\mathbf{j}^* = \mathcal{C}^*\mathbf{f}^*,$$

where $\mathcal{C}^*$ is a known bounded operator. But, from the derivation of (68), we know that $\mathbf{j}^*$ solves

$$\mathcal{L}_1^*\mathbf{j}^* = \mathbf{f}^*.$$

We also know from Theorem 7.2 that there can be at most one $\mathbf{j}^*$ that solves this equation, and so it follows that $\mathcal{C}^* = (\mathcal{L}_1^*)^{-1}$.

Next, consider an arbitrary frequency ω. We have

$$\mathcal{L}_1\mathbf{j} = \beta\mathcal{L}_1^*\mathbf{j} + (\mathcal{L}_1 - \beta\mathcal{L}_1^*)\mathbf{j} = \mathbf{f},$$

where β is a constant, whence

$$\beta\mathbf{j} + \mathcal{C}^*(\mathcal{L}_1 - \beta\mathcal{L}_1^*)\mathbf{j} = \mathcal{C}^*\mathbf{f}. \tag{72}$$

This is a Fredholm integral equation of the second kind for $\mathbf{j}(q)$ if $\beta\neq 0$ and $(\mathcal{L}_1 - \beta\mathcal{L}_1^*)$ is compact. Now,

$$(\mathcal{L}_1 - \beta\mathcal{L}_1^*)\mathbf{j} = \left(\beta\alpha^*\mathcal{A}_\rho^* - \alpha\mathcal{A}_\rho\right)(\mathbf{n}\times\mathbf{j}) + \tfrac{\mathrm{i}}{\varepsilon_i}\left(\tfrac{1}{\omega}\mathcal{B}_\tau - \tfrac{\beta}{\omega^*}\mathcal{B}_\tau^*\right)\mathbf{j}.$$

This will be compact from $T^{0,\beta}(S)$ into itself if

$$\beta\alpha^* = \alpha \quad \text{and} \quad \omega^* = \beta\omega; \tag{73}$$

thus, we require that Mautz's 'constant' α be given by

$$\alpha(\omega) = \eta/\omega,$$

where η is a frequency-independent constant. Since we already have uniqueness for (68), the Riesz theory gives the existence of $\mathbf{j}\in T^{0,\beta}(S)$. It remains to show that $\mathbf{j}$ is actually in $T_d^{0,\beta}(S)$. Consider (72). We know that its right-hand side $\mathcal{C}^*\mathbf{f}\in T_d^{0,\beta}(S)$, since $\mathbf{f}\in T_d^{0,\beta}(S)$. Also, the choices (73) ensure that $\mathcal{L}_1 - \beta\mathcal{L}_1^*$ is a pseudodifferential operator of order -1; thus, it maps $T^{0,\beta}(S)\longrightarrow T^{1,\beta}(S)\subset T_d^{0,\beta}(S)$. Hence, $\mathbf{j}\in T_d^{0,\beta}(S)$, as required, and so we obtain the following result.

Theorem 8.1 *Mautz's single integral equation* (68)*, in which $\alpha = \eta/\omega$ and η is a frequency-independent constant with* $\operatorname{Re}\eta\neq 0$*, is uniquely solvable in $T_d^{0,\beta}(S)$ at all frequencies.*

A similar argument works for the other single integral equation, (69); the starting point is the Fredholm equation (64).

Theorem 8.2 *The single integral equation* (69)*, in which $\alpha = \eta\omega$ and η is a frequency-independent constant with* $\operatorname{Re}\eta\neq 0$*, is uniquely solvable in $T_d^{0,\beta}(S)$ at all frequencies.*

8.2 A singular integral equation without irregular frequencies

Consider the hypersingular integral equation (59). If $\mathcal{B}_\rho$ was replaced by $\mathcal{B}_\rho V$, where V is chosen so that $\mathcal{B}_\rho V$ is compact, (59) would be a Fredholm equation. This idea is the basis of the paper by Kress [12] on the exterior Maxwell problem. Thus, assume that we can write (cf. (44))

$$\mathbf{E}_e(P) = \frac{\mathrm{i}}{\omega\varepsilon_e}F_e\mathbf{j} - bC_eV\mathbf{j}, \quad \mathbf{H}_e(P) = C_e\mathbf{j} + b\frac{\mathrm{i}}{\omega\mu_e}F_eV\mathbf{j}, \quad P \in B_e, \tag{74}$$

where the constant b and the operator V will be specified later. Proceeding as in §7, we use the Stratton-Chu representations in B_i and then obtain (cf. (59))

$$\left\{\mathcal{A}_\tau + b\frac{\mathrm{i}}{\omega\mu_i}\mathcal{B}_\rho V\right\}\mathbf{j} = \mathbf{g}, \tag{75}$$

where $\mathbf{g}$ is defined by (53). Having solved (75), $\{\mathbf{E}_e, \mathbf{H}_e\}$ and $\{\mathbf{E}_i, \mathbf{H}_i\}$ are to be constructed from (74) and (54), respectively, with

$$K_e = \frac{\mathrm{i}}{\omega\varepsilon_e}P_e - b(I + M_e)V \quad \text{and} \quad L_e = I + M_e + b\frac{\mathrm{i}}{\omega\mu_e}P_eV \tag{76}$$

in (54). We have the following result.

Theorem 8.3 *If $\mathbf{j}(q) \in T_d^{0,\beta}(S)$ solves* (75)*, $\{\mathbf{E}_e, \mathbf{H}_e\}$ and $\{\mathbf{E}_i, \mathbf{H}_i\}$, given by* (74) *and* (54) *(with* (76)*), respectively, solve the transmission problem.*

Proof. Straightforward adaptation of proof of Theorem 7.1.

The next step is to examine uniqueness. It depends on the eigenvalues of the following problem.

INTERIOR V-PROBLEM. Find a field $\{\mathbf{E}, \mathbf{H}\}$ which satisfies Maxwell's equations (4) in B_i and the boundary condition

$$\mathbf{n} \times \mathbf{E} + bV\{\mathbf{n} \times \mathbf{H}\} = \mathbf{0}, \quad p \in S. \tag{77}$$

Theorem 8.4 *The homogeneous form of* (75) *has a non-trivial solution if, and only if, k_e^2 is an eigenvalue of the interior V-problem.*

Proof. Straightforward adaptation of the proof of Theorem 7.2.

We eliminate eigenvalues of the interior V-problem by making appropriate choices for b and V. Thus, suppose that $\{\mathbf{E}, \mathbf{H}\}$ solves the interior V-problem. As at the end of §2, we deduce that

$$\mathrm{Re}\left\{b\int_S \overline{H}.\{V(\mathbf{n} \times \mathbf{H})\}\,ds\right\} = 0,$$

where we have used (77). If, following Kress [12], we set

$$V\mathbf{a} = \mathbf{n} \times \left\{\overline{S}_0 S_0 \mathbf{a}\right\}, \tag{78}$$

where k_0 will be specified and

$$(\overline{S}_0\nu)(p) = \int_S \nu(q)\overline{G}_0(p,q)\,ds_q,$$

we find that

$$\operatorname{Re}(b) \int_S |S_0(\mathbf{n} \times \mathbf{H})|^2 \, ds = 0.$$

So, if $\operatorname{Re} b \neq 0$, we deduce that

$$S_0(\mathbf{n} \times \mathbf{H}) = \mathbf{0}, \quad p \in S.$$

But, if we choose k_0 so that k_0^2 is not an eigenvalue of the interior Dirichlet problem, S_0 is invertible. Hence, $\mathbf{n} \times \mathbf{H} = \mathbf{0}$ on S, (77) gives $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ and so $\{\mathbf{E}, \mathbf{H}\}$ vanishes identically in B_i.

Now, from (26) and (78), we have

$$\sigma(V) = \frac{1}{\xi^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

whence (27) and (60) give

$$\sigma(\mathcal{B}_\rho V) = \frac{1+\rho}{|\boldsymbol{\xi}|^3} \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_1\xi_2 & \xi_1^2 \end{pmatrix}.$$

Hence, $\mathcal{B}_\rho V$ is compact from $T^{0,\beta}(S)$ into itself. The Fredholm alternative then guarantees the unique solvability of (75) in $T^{0,\beta}(S)$. It remains to show that $\mathbf{j} \in T_d^{0,\beta}(S)$. We proceed as in the proof of Theorem 7.3. Thus, let $\{\widetilde{\mathbf{E}}_\alpha, \widetilde{\mathbf{H}}_\alpha\}$ solve the transmission problem, whence $\mathbf{n} \times \widetilde{\mathbf{H}}_e \in T_d^{0,\beta}(S)$. Then

$$\left\{ I + M_e + b \tfrac{\mathrm{i}}{\omega\mu_e} P_e V \right\} \mathbf{j} = \mathbf{n} \times \widetilde{\mathbf{H}}_e$$

is uniquely solvable for $\mathbf{j} \in T_d^{0,\beta}(S)$ [12, Eqn. (10′)]. The result follows as before, using uniqueness for the exterior problem, and so we obtain the following.

Theorem 8.5 *The single integral equation* (75)*, wherein V is given by* (78)*, is a Fredholm equation. It is uniquely solvable in $T^{0,\beta}(S)$ for any $\mathbf{g} \in T^{0,\beta}(S)$ if the two constants b and k_0 are chosen so that* $\operatorname{Re} b \neq 0$ *and k_0 is not an eigenvalue of the interior Dirichlet problem. Moreover, if $\mathbf{g}$ is given by* (53)*, the solution will be in $T_d^{0,\beta}(S)$.*

9 Discussion

We have seen that there are various methods for reducing the electromagnetic transmission problem for a homogeneous dielectric obstacle to boundary integral equations: one can use a pair of coupled integral equations for a pair of unknowns (§§5 and 6) or a single integral equation for a single unknown (§§7 and 8). Clearly, we have not exhausted all the possibilities. Thus, we derived single integral equations by using an assumed representation (or *ansatz*) in B_e and the Stratton-Chu representation in B_i. We could reverse this procedure, leading to different integral equations; a systematic investigation of this approach for the acoustic transmission problem is given in [10, §6]. We could also derive single integral equations without irregular frequencies by modifying the Green's function, as done in [20] and [8] for the exterior Maxwell problem.

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Appendix. Notational comparisons

For a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, Harrington's notation [6] for the basic potentials is as follows:

$$\begin{aligned}
\overline{E}(\mathbf{J},\mathbf{0}) &= \mathrm{i}\omega\overline{A}(\mathbf{J}) - \operatorname{grad}\phi(\mathbf{J}),\\
\overline{H}(\mathbf{J},\mathbf{0}) &= \tfrac{1}{\mu}\operatorname{curl}\overline{A}(\mathbf{J}),\\
\overline{E}(\mathbf{0},\mathbf{M}) &= -\tfrac{1}{\varepsilon}\overline{F}(\mathbf{M}),\\
\overline{H}(\mathbf{0},\mathbf{M}) &= \mathrm{i}\omega\overline{F}(\mathbf{M}) - \operatorname{grad}\psi(\mathbf{M}),
\end{aligned}$$

where

$$\overline{A}(\mathbf{J}) = \mu S\mathbf{J}, \quad \phi(\mathbf{J}) = \tfrac{1}{\varepsilon}Sq, \quad q = \tfrac{1}{\mathrm{i}\omega}\operatorname{Div}\mathbf{J},$$
$$\overline{F}(\mathbf{M}) = \varepsilon S\mathbf{M}, \quad \psi(\mathbf{M}) = \tfrac{1}{\mu}Sm, \quad m = \tfrac{1}{\mathrm{i}\omega}\operatorname{Div}\mathbf{M},$$

and S is defined by (6). By Theorem 2.29 of [2], we have

$$\phi(\mathbf{J}) = -\tfrac{\mathrm{i}}{\omega\varepsilon}\operatorname{div}\{S\mathbf{J}\} \quad \text{and} \quad \psi(\mathbf{M}) = -\tfrac{\mathrm{i}}{\omega\mu}\operatorname{div}\{S\mathbf{M}\}.$$

Then, we can simplify $\overline{E}(\mathbf{J},\mathbf{0})$ and $\overline{H}(\mathbf{0},\mathbf{M})$ using $(\nabla^2 + k^2)G = 0$ and the vector identity grad div $=$ curl curl $+ \nabla^2$. The results are

$$\overline{E}(\mathbf{J},\mathbf{0}) = \tfrac{\mathrm{i}}{\omega\varepsilon}\operatorname{curl}\{\overline{H}(\mathbf{J},\mathbf{0})\}, \quad \overline{H}(\mathbf{J},\mathbf{0}) = \operatorname{curl}\{S\mathbf{J}\},$$

$$\overline{E}(\mathbf{0},\mathbf{M}) = -\operatorname{curl}\{S\mathbf{M}\}, \quad \overline{H}(\mathbf{0},\mathbf{M}) = -\tfrac{\mathrm{i}}{\omega\mu}\operatorname{curl}\{\overline{E}(\mathbf{0},\mathbf{M})\}.$$

Note that $\overline{E}(\mathbf{0},\mathbf{M}) = -\overline{H}(\mathbf{M},\mathbf{0})$ and $\overline{H}(\mathbf{0},\mathbf{M}) = \frac{\varepsilon}{\mu}\overline{E}(\mathbf{M},\mathbf{0})$.