On amphidromic points†

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Amphidromic points are isolated points at which the wave amplitude vanishes. We investigate the consequences of their existence in a wave field. For example, one method for solving the mild-slope equation (this models the propagation of water waves over a variable bathymetry) begins by writing the complex potential in terms of a real amplitude A and a real phase S, both of which are functions of position. We show that S is not continuous at amphidromic points, whereas its gradient is singular there. We also find local approximations for A and S. We discuss various differential equations governing A and S, with emphasis on their properties in the presence of amphidromic points, and find a new pair that is well behaved there. We discuss two simple examples for which the amphidromic points can be found explicitly. Finally, we show that our analysis can also be extended to Laplace's tidal equations.

1. Introduction

Consider the propagation of time-harmonic waves, so that a typical dependent variable (such as a velocity potential) can be written as

$$\mathrm{Re}\,\{\psi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}\},$$

where ψ is a complex-valued function of position $\boldsymbol{x}$ and ω is the circular frequency. It is conventional to write

$$\psi(\boldsymbol{x}) = A\mathrm{e}^{\mathrm{i}S}, \tag{1.1}$$

where $A(\boldsymbol{x})$ and $S(\boldsymbol{x})$ are real-valued functions of position. Examples occur in the refraction of short water waves (Keller 1958; Meyer 1979) and in numerical treatments of the mild-slope equation (Berkhoff *et al.* 1982; Ebersole 1985; Li & Anastasiou 1992).

$S(\boldsymbol{x})$ is called the *phase function*. Any feature, such as a crest, can be identified with a constant value of S and will move in the direction of grad S: 'the sense of propagation is also that of S increasing' (Meyer 1979, p. 104). In simple ray-theoretic models of refraction, S increases smoothly as one moves along a ray, apart from known discontinuities at boundaries and caustics (Meyer 1979, p. 81). This usually implies that $S(\boldsymbol{x})$ is not single valued. We can easily recover single-

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valuedness by restricting S to lie within an interval of length 2π,

$$0 \leqslant S < 2\pi, \tag{1.2}$$

say. The price of this restriction is that S is now discontinuous as one moves along a ray; however, $\operatorname{grad} S$ is unchanged (it can be defined by continuity at these discontinuities in S).

If one substitutes (1.1) into the governing partial differential equation for ψ, one obtains a pair of coupled partial differential equations for A and S (see §6), which may be solved numerically. Typically, these equations involve A and $\operatorname{grad} S$; the latter is single valued, in general.

The use of the representation (1.1) is supposed to aid the determination of ψ. If we suppose that ψ is *given*, then A and S can be determined uniquely from ψ, subject to the restriction (1.2), *except at places where* $A = |\psi| = 0$. Isolated points where $A = 0$ are called *amphidromic points* or *amphidromes* or *dislocations* (Nye & Berry 1974); it is shown in §3 that other possibilities can be excluded. (We remark that simple ray theory fails at caustics and shore-lines, where it predicts that A is infinite; we are not concerned with those defects here).

In this paper, we describe what happens near an amphidromic point, $\boldsymbol{x}_0$, in the context of water-wave propagation, as modelled by the mild-slope equation. First, we show that $S(\boldsymbol{x})$ is not continuous at $\boldsymbol{x}_0$. Second, we show that $\operatorname{grad} S$ is singular at $\boldsymbol{x}_0$:

$$|\operatorname{grad} S| \sim |\boldsymbol{x} - \boldsymbol{x}_0|^{-1}$$

near $\boldsymbol{x}_0$; one should be aware of this fact in numerical work. Third, we deduce that S must increase by $2m\pi$ when one circuit of $\boldsymbol{x}_0$ is made, where the integer m is such that $A(\boldsymbol{x})$ has a zero of order $|m|$ at $\boldsymbol{x}_0$; this result may be useful in the location of amphidromic points. It follows that $S(\boldsymbol{x})$ is single valued, for $\boldsymbol{x} \neq \boldsymbol{x}_0$, *provided* one imposes the restriction (1.2). Local approximations for $A(\boldsymbol{x})$ near $\boldsymbol{x}_0$ are also found. Further consequences of the presence of amphidromic points are discussed in §5.

In § 6 we derive a pair of coupled partial differential equations for A and S. One of these is well known, the other is often written in an alternative form; the step leading to this alternative is not valid if there are amphidromic points, because it involves multiplication by A. We also develop other differential equations, motivated by the recent paper of Radder (1992).

In § 7*a* we study the simplest physical problem involving amphidromic points: the superposition of three regular wavetrains, propagating in three different directions on water of constant finite depth. (We remark that amphidromic points cannot be created with two such wavetrains on water of constant depth; if the depth varies, amphidromic points can, of course, be caused by the refraction of a single wave.) This problem can be solved exactly. We give the solution for a particular symmetric choice of propagation directions: there is an infinite, hexagonal array of amphidromic points; the behaviour of the field near the amphidromic points is in accord with our simple theory. This solution may also be a useful benchmark for numerical calculations.

A comparison with the paper by Berkhoff *et al.* (1982) is made in § 7*b*. They gave graphical results, for a particular configuration, suggesting that S is not a monotonic function of the polar angle at $\boldsymbol{x}_0$; this is not consistent with our leading-order approximation. We conjecture that this is due to the presence of a stationary point (where grad S vanishes) near the amphidromic point. We support this conjecture in two ways: (i) we give an explicit solution (due to Nye *et al.* 1988) that has the same qualitative features; and (ii) we consider higher-order approximations. Comparisons with some other published computations are made in § 7*c*.

Finally, in § 7*d* we consider the related (and older) topic of tidal amphidromic points. The governing equations for horizontal motions, taking gravity and the Earth's rotation into account are well known (Laplace's tidal equations) and lead to an equation that is similar to (but more complicated than) the mild-slope equation (see (7.6) below). A similar analysis of this equation can be made; the results are in accord with the well-known exact solution for a rectangular basin found by G. I. Taylor (1922).

After submission of this paper, Radder's short paper (1992) appeared. He discusses some of the effects of amphidromic points, and proposes a system of differential equations for A and a function G related to S. In § 6 we examine this system, using our local analysis near amphidromic points. This leads us to propose an alternative system, which seems worthy of further study.

2. Formulation: the mild-slope equation

The mild-slope equation (Berkhoff 1973) is

$$\operatorname{div}(p \operatorname{grad} \psi) + k^2 p \psi = 0, \tag{2.1}$$

where $\psi(\boldsymbol{x})$ is a complex function of two horizontal coordinates, x and y,

$$\boldsymbol{x} = (x, y), \quad k = \omega / c, \quad p = c c_{\mathrm{g}},$$

$c(\boldsymbol{x})$ is the phase velocity and $c_{\mathrm{g}}(\boldsymbol{x})$ is the group velocity; a time-dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ has been suppressed. The wavenumber $k(\boldsymbol{x})$ is defined as the positive real root of the dispersion relation

$$\omega^2 = gk \tanh kh,$$

where $h(\boldsymbol{x})$ is the water depth at $\boldsymbol{x}$ and g is the acceleration due to gravity. The group velocity is given by

$$c_{\mathrm{g}} = \tfrac{1}{2} c \left(1 + \frac{2kh}{\sinh 2kh}\right).$$

The coefficient $p(\boldsymbol{x})$ is assumed to be a smooth function of $\boldsymbol{x}$.

In the special case of constant finite depth h, the mild-slope equation (2.1) reduces to the two-dimensional Helmholtz equation,

$$(\nabla^2 + k^2)\psi = 0. \tag{2.2}$$

(*a*) *A path-independent integral*

Because $\bar{\psi}$, the complex conjugate of ψ, also satisfies (2.1) (p and k^2 are real), we have

$$\bar{\psi} \operatorname{div}(p \operatorname{grad} \psi) - \psi \operatorname{div}(p \operatorname{grad} \bar{\psi}) = 0. \tag{2.3}$$

Let Γ denote a simple closed curve, bounding a region Ω of the mean free surface. Integrating (2.3) over Ω, using the (two-dimensional) divergence theorem, then gives

$$\int_{\Gamma} p\left(\psi\frac{\partial\bar{\psi}}{\partial n}-\bar{\psi}\frac{\partial\psi}{\partial n}\right)\,\mathrm{d}s=0, \tag{2.4}$$

where $\partial/\partial n$ denotes normal differentiation on Γ, out of Ω. We shall make use of (2.4) below.

3. Amplitude and phase

Suppose we write the complex function ψ as

$$\psi(\boldsymbol{x})=\psi_{\mathrm{R}}+\mathrm{i}\psi_{\mathrm{I}}=A\mathrm{e}^{\mathrm{i}S}, \tag{3.1}$$

where ψ_{R}, ψ_{I}, A and S are all real-valued functions of x and y. A is the *amplitude* and S is the *phase*. Clearly, A is uniquely determined in terms of ψ:

$$A=|\psi|=\sqrt{\psi_{\mathrm{R}}^2+\psi_{\mathrm{I}}^2}\geqslant 0. \tag{3.2}$$

On the other hand, S cannot be determined uniquely from ψ because it can only be determined within a multiple of 2π. We eliminate this indeterminacy by restricting S to satisfy (1.2). Then, just as with plane polar coordinates, we see that S is uniquely determined by

$$\psi_{\mathrm{R}}=A\cos S \quad \text{and} \quad \psi_{\mathrm{I}}=A\sin S, \tag{3.3}$$

unless $A=0$.

Now, there are three possibilities. First, suppose that ψ_{R} or ψ_{I} vanishes identically, so that ψ corresponds to a *standing wave*. If $\psi_{\mathrm{R}}\equiv 0$, (1.2) and (3.3) imply that $S=\frac{1}{2}\pi$ when $\psi_{\mathrm{I}}>0$ and $S=\frac{3}{2}\pi$ when $\psi_{\mathrm{I}}<0$; if $\psi_{\mathrm{I}}\equiv 0$, $S=0$ when $\psi>0$ and $S=\pi$ when $\psi<0$. Thus, in either event, $S(\boldsymbol{x})$ is piecewise constant and the continuous extension of $\operatorname{grad} S=\mathbf{0}$. Henceforth, we assume that neither ψ_{R} nor ψ_{I} vanishes identically.

Second, there are the spurious discontinuities in S induced by the restriction (1.2). Thus, $S(\boldsymbol{x})$ will suffer a discontinuity of magnitude 2π whenever $\psi_{\mathrm{I}}(\boldsymbol{x})=0$ but $\psi_{\mathrm{R}}(\boldsymbol{x})>0$. However, because any solution of (2.1) has continuous second partial derivatives, it follows that the derivatives of S can be continuously extended across such discontinuities.

Third, suppose that $A(\boldsymbol{x})=0$ for $\boldsymbol{x}\in D$; the domain D consists of patches, curves and isolated points; the latter are called *amphidromic points.* When $\boldsymbol{x}\in D$, $S(\boldsymbol{x})$ is arbitrary. However, $S(\boldsymbol{x})$ can be defined in D by continuity, *except* at amphidromic points. To see this, we consider three cases.

Case 1. Suppose that $A(\boldsymbol{x})=0$ for all $\boldsymbol{x}$ in a patch D_0 of the xy-plane. Thus, $\psi(\boldsymbol{x})=0$ in D_0. But ψ solves an elliptic partial differential equation, whence analytic continuation shows that ψ vanishes in a larger region; if the coefficient $p(\boldsymbol{x})$ is sufficiently smooth, ψ will vanish everywhere. In general, this solution is either not interesting or contradicts some known property of ψ, for example its behaviour on a boundary or at infinity. Hence, we can forego the possibility that A vanishes on a patch.

Case 2. Suppose that $A(\boldsymbol{x})=0$ for all $\boldsymbol{x}$ on a curve C. Parametrize C as

$$\mathrm{C}:\boldsymbol{x}=\boldsymbol{x}(\tau)=(x(\tau),y(\tau)),\quad 0\leqslant\tau\leqslant 1.$$

From (3.2), we have $\psi_\alpha(\boldsymbol{x}(\tau)) = 0$ for $0 \leqslant \tau \leqslant 1$ and $\alpha = \mathrm{R}, \mathrm{I}$. We use this information to calculate S near C. A unit normal to C at $\boldsymbol{x}(\tau)$ is $\boldsymbol{n}(\tau) = (y'(\tau), -x'(\tau))/|\boldsymbol{x}'(\tau)|$. Thus, the point P at $\boldsymbol{y} \equiv \boldsymbol{x}(\tau) + h\boldsymbol{n}(\tau)$, where h is small, is near C. If we assume that ψ_R and ψ_I both have simple zeros on C, we find that $\psi_\alpha(\boldsymbol{y}) \sim h\partial\psi_\alpha/\partial n$ evaluated at $\boldsymbol{x}(\tau)$, for small h. Hence

$$\tan\{S(\boldsymbol{y})\} = \frac{\psi_\mathrm{I}(\boldsymbol{y})}{\psi_\mathrm{R}(\boldsymbol{y})} \sim \frac{\partial\psi_\mathrm{I}/\partial n}{\partial\psi_\mathrm{R}/\partial n} \quad \text{evaluated at } \boldsymbol{x}(\tau). \tag{3.4}$$

It follows that $S(\boldsymbol{x}(\tau))$ can be defined by letting $h \to 0$. Moreover, because the numerator and the denominator on the right-hand side of (3.4) are continuous functions of τ, it follows that the resulting expression for $S(\boldsymbol{x}(\tau))$ is also a continuous function of τ (apart from possible jumps of 2π). Finally, similar arguments show that $S(\boldsymbol{x})$ has continuous first and second partial derivatives on C. The same conclusions are obtained also if ψ_R or ψ_I have zeros on C of any finite order.

Case 3. Suppose that $A(\boldsymbol{x}) = 0$ at an isolated point, $\boldsymbol{x}_0$ say. Introduce plane polar coordinates (ρ, φ) at $\boldsymbol{x}_0$, so that if ρ is small,

$$\boldsymbol{y} = \boldsymbol{x}_0 + \rho(\cos\varphi, \sin\varphi)$$

is a point close to $\boldsymbol{x}_0$. Then, we find that

$$\frac{\psi_\mathrm{I}(\boldsymbol{y})}{\psi_\mathrm{R}(\boldsymbol{y})} \sim \frac{(\partial\psi_\mathrm{I}/\partial x)\cos\varphi + (\partial\psi_\mathrm{I}/\partial y)\sin\varphi}{(\partial\psi_\mathrm{R}/\partial x)\cos\varphi + (\partial\psi_\mathrm{R}/\partial y)\sin\varphi}$$

for small ρ, where the four partial derivatives are evaluated at the amphidromic point $\boldsymbol{x}_0$. If we now let $\rho \to 0$, we obtain different results according to the angle of approach, φ. Thus, $S(\boldsymbol{x})$ is not continuous at $\boldsymbol{x}_0$, in general. We give a more detailed local analysis around $\boldsymbol{x}_0$ in the next section.

4. Local analysis near an amphidromic point

Suppose that $h(\boldsymbol{x})$ is constant in a neighbourhood of the amphidromic point $\boldsymbol{x}_0$,

$$h(\boldsymbol{y}) \approx h_0 \quad \text{for small } \rho,$$

where $h_0 = h(\boldsymbol{x}_0)$. Then, locally, $\psi(\boldsymbol{y})$ solves the Helmholtz equation (2.2). Hence, if $\boldsymbol{x}_0$ is an mth-order zero, we have solutions of the form

$$\psi(\boldsymbol{y}) \approx c_m J_m(k_0\rho)\,\mathrm{e}^{\pm\mathrm{i}m\varphi} \tag{4.1}$$

$$\approx a_m \rho^m\,\mathrm{e}^{\pm\mathrm{i}m\varphi}, \tag{4.2}$$

for small ρ, where a_m and c_m are unknown complex coefficients, $k_0 = k(\boldsymbol{x}_0)$ and J_m is a Bessel function.

We can obtain (4.2) by a systematic method. Let L be the diameter of our neighbourhood of $\boldsymbol{x}_0$. We assume that $\varepsilon \equiv k_0 L \ll 1$ and that $h(\boldsymbol{x})$ can vary slowly, so that $\delta \equiv |\boldsymbol{l}_0|L/h_0 \ll 1$, where $\boldsymbol{l}_0 = \operatorname{grad} h$ evaluated at $\boldsymbol{x}_0$. If we then look for a solution in the form

$$\psi = \psi_0 + \varepsilon\psi_{01} + \delta\psi_{10} + O(\varepsilon^2, \varepsilon\delta, \delta^2),$$

we find that ψ_0 satisfies Laplace's equation, and this leads to (4.2). Furthermore, we find that $\psi_{01} = 0$, and ψ_{10} satisfies

$$\nabla^2 \psi_{10} = -\hat{\boldsymbol{l}}_0 \cdot \operatorname{grad} \psi_0, \tag{4.3}$$

where $\hat{\boldsymbol{l}}_0 = \boldsymbol{l}_0/|\boldsymbol{l}_0|$; if ψ_0 is given by (4.2), a particular integral of (4.3) is

$$\psi_{10} = -\tfrac{1}{4} a_m \rho^{m+1} \exp\{\pm \mathrm{i}((m-1)\varphi + \alpha_0)\},$$

where $\hat{\boldsymbol{l}}_0 = (\cos\alpha_0, \sin\alpha_0)$.

In general, we expect the local solution for ψ to be given by a linear combination of terms such as (4.2): suppose that

$$\psi(\boldsymbol{y}) = A_1 \rho^m \mathrm{e}^{\mathrm{i}m\varphi} + A_2 \rho^m \mathrm{e}^{-\mathrm{i}m\varphi}, \tag{4.4}$$

where A_1 and A_2 are complex coefficients. It is convenient to write (4.4) as

$$\psi(\boldsymbol{y}) = A_2 \rho^m \mathrm{e}^{-\mathrm{i}m\varphi} \chi(2m\varphi + \gamma; a), \tag{4.5}$$

where $A_1/A_2 = a\mathrm{e}^{\mathrm{i}\gamma}$, a and γ are real, and χ is defined by

$$\chi(\theta; a) = 1 + a\,\mathrm{e}^{\mathrm{i}\theta} = |\chi| \mathrm{e}^{\mathrm{i}T}, \tag{4.6}$$

say, with $a \geq 0$. Clearly, comparison with (1.1) gives

$$\begin{aligned} S(\boldsymbol{y}) &= +m\varphi + \arg A_1 \quad \text{if } A_2 = 0, \\ S(\boldsymbol{y}) &= -m\varphi + \arg A_2 \quad \text{if } A_1 = 0. \end{aligned}$$

These are both monotonic functions of φ. In particular, S increases monotonically by $2m\pi$ after one anticlockwise circuit of $\boldsymbol{x}_0$ (φ increasing) if $A_2 = 0$ ($a = \infty$), whereas it decreases monotonically by $2m\pi$ if $A_1 = 0$ ($a = 0$). It turns out that this monotonic behaviour also occurs with (4.5), wherein $0 < a < \infty$; the changeover occurs at $a = 1$. To see this, use (4.6) in (4.5) to give

$$S(\boldsymbol{y}) = -m\varphi + T(2m\varphi + \gamma; a) + \arg A_2. \tag{4.7}$$

We cannot write down an explicit formula for T. However, we have

$$\tan\{T(\theta; a)\} = \frac{a \sin\theta}{1 + a\cos\theta},$$

and we observe that if θ increases by 2π, then T also increases by 2π if $a > 1$, but T is single valued if $0 \leqslant a < 1$. Moreover, we can calculate the rate of change of $T(\theta; a)$ with θ, for fixed a:

$$T'(\theta; a) = \frac{\mathrm{d}T}{\mathrm{d}\theta} = \frac{a(\cos\theta + a)}{1 + 2a\cos\theta + a^2},$$

whence

$$S'(\varphi) = \frac{\mathrm{d}S}{\mathrm{d}\varphi} = -m + 2mT'(2m\varphi + \gamma; a) = \frac{m(a^2 - 1)}{1 + 2a\cos(2m\varphi + \gamma) + a^2}. \tag{4.8}$$

Thus, if $0 < a < 1$, $S'(\varphi) < 0$ for all φ and S decreases by $2m\pi$ if φ increases by 2π. Similarly, if $a > 1$, $S'(\varphi) > 0$ for all φ and S increases by $2m\pi$ if φ increases by 2π. At the changeover, $a = 1$ and S is piecewise constant: it is given by $S(\boldsymbol{y}) = \frac{1}{2}\gamma + \arg A_2$ if $\cos(m\varphi + \frac{1}{2}\gamma) > 0$ and by $S(\boldsymbol{y}) = \frac{1}{2}\gamma + \arg A_2 + \pi$ if $\cos(m\varphi + \frac{1}{2}\gamma) < 0$.

In summary, equation (4.7) gives the phase S corresponding to the local expansion (4.4). S is independent of ρ, it is a monotonic function of φ (for $a \neq 1$) and it is not continuous at $\boldsymbol{x}_0$.

5. Some consequences of amphidromic points

If we substitute (3.1) directly into the integral (2.4), we obtain

$$\int_\Gamma pA^2 \frac{\partial S}{\partial n}\,\mathrm{d}s = \int_\Gamma pA^2 \boldsymbol{s}\cdot\boldsymbol{n}\,\mathrm{d}s = 0 \tag{5.1}$$

for any simple closed curve Γ, where we have written

$$\boldsymbol{s}(\boldsymbol{x}) = \operatorname{grad} S. \tag{5.2}$$

The vector field $pA^2\boldsymbol{s}$ is an energy-flux vector, whence (5.1) expresses conservation of energy. Note that the two-dimensional vector field $\boldsymbol{s}(\boldsymbol{x})$ is continuously differentiable everywhere, except at amphidromic points where it is singular, in general. Specifically, for an amphidromic point at $\boldsymbol{x}_0$, (4.7) and (4.8) give

$$\boldsymbol{s}(\boldsymbol{y}) \approx S'(\varphi)\hat{\varphi}/\rho \tag{5.3}$$

for small ρ, where $\hat{\varphi}$ is a unit vector in the φ-direction. This shows the nature of the singularity in $\boldsymbol{s}(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}_0$ ($\rho = 0$).

The integral relation (5.1) is valid for any simple closed curve Γ, even if there is an amphidromic point in Ω. To verify this, we take Γ as C_ρ, a small circle of radius ρ and centre $\boldsymbol{x}_0$. Then, from (5.3), we find that $\boldsymbol{s}\cdot\boldsymbol{n} \sim 0$ on C_ρ, as required.

Next consider

$$\int_\Gamma \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{r},$$

where Γ is any simple closed curve bounding a region Ω. Suppose that Ω does not contain any amphidromic points. Then, the definition (5.2) implies that

$$\operatorname{curl}\boldsymbol{s} = \boldsymbol{0}, \tag{5.4}$$

everywhere in Ω, whence Stokes's theorem gives

$$\int_\Gamma \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{r} = 0:$$

'conservation of waves requires the total number (with correct sign) crossing any closed curve be zero' (Whitham 1960, p. 349). Equivalently, we can assert that the line integral,

$$\int_{\mathrm{P}}^{\mathrm{Q}} \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{r},$$

between the two fixed points P and Q is independent of the path chosen between P and Q, implying that S is single valued.

Suppose, now, that Ω contains a single amphidromic point at $\boldsymbol{x}_0$. Then

$$\int_\Gamma \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{r} = \int_{\mathrm{C}_\rho} \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{r} = \begin{cases} -2m\pi & \text{if } 0 \leqslant a < 1, \\ +2m\pi & \text{if } a > 1, \end{cases}$$

for an mth-order zero at $\boldsymbol{x}_0$, where we have used (5.3). This shows that $S(\boldsymbol{x})$

(without the restriction (1.2)) increases by an integer multiple of 2π after making one circuit around one amphidromic point. Imposing (1.2) eliminates this increase and reasserts the single-valuedness of $\boldsymbol{s}(\boldsymbol{x})$ for $\boldsymbol{x} \neq \boldsymbol{x}_0$.

6. Governing equations for amplitude and phase

Let us substitute (3.1) into (2.1) and split the result into real and imaginary parts, giving

$$\operatorname{div}(p \operatorname{grad} A) + (k^2 - |\boldsymbol{s}|^2)pA = 0 \tag{6.1}$$

and

$$A \operatorname{div}(p\boldsymbol{s}) + 2p\boldsymbol{s} \cdot \operatorname{grad} A = 0, \tag{6.2}$$

respectively, where $\boldsymbol{s}(\boldsymbol{x}) = \operatorname{grad} S$. These are the coupled partial differential equations governing $A(\boldsymbol{x})$ and $S(\boldsymbol{x})$.

We note here that, in the theory of refraction of short waves there is a small parameter, which makes the first term in (6.1) negligible compared to the second. This leads to $|\boldsymbol{s}| = k$, which is known as the *eikonal equation.* In our case, this equation is not satisfied and so we have chosen to use $\boldsymbol{s}$ rather than the more conventional $\boldsymbol{k}$.

Suppose that $A(\boldsymbol{x})$ does not vanish. Then we can write (6.2) as

$$\operatorname{div}(pA^2\boldsymbol{s}) = 0. \tag{6.3}$$

Alternatively, as Γ does not contain any amphidromic points, $pA^2\boldsymbol{s}$ is continuously differentiable within Γ and so equation (6.3) follows from an application of the divergence theorem to the integral in (5.1). Equations (6.1) and (6.3) are well known; see, for example, Berkhoff *et al.* (1982, equations (7) and (8), apart from a sign error in (7)) and Ebersole (1985, equations (4) and (5)). Although one can view (6.1) and (6.3) as equations for A and S, we have already noted that the phase S can suffer discontinuities. Thus, it seems preferable to solve the three equations (6.1), (6.3) and (5.4) for $A(\boldsymbol{x})$ and the two components of $\boldsymbol{s}(\boldsymbol{x})$. This method has been used by Ebersole (1985).

Suppose now that $A(\boldsymbol{x})$ does vanish at some isolated amphidromic points. If we use the local approximations obtained in § 4, we find that the vector field

$$pA^2\boldsymbol{s}$$

is continuously differentiable, even at amphidromic points. It follows that (6.3), which asserts the conservation of energy, is always valid. Thus, we could attempt to solve (6.1), (6.3) and (5.4); but (5.4) is not valid at amphidromic points. Alternatively, (6.3) implies the existence of an 'energy stream-function' $G(\boldsymbol{x})$, so that

$$pA^2\frac{\partial S}{\partial x} = \frac{\partial G}{\partial y} \quad \text{and} \quad pA^2\frac{\partial S}{\partial y} = -\frac{\partial G}{\partial x} \tag{6.4}$$

(Radder 1992) and (6.3) is satisfied identically.

To proceed, we need an equation for G. We know that G has continuous second partial derivatives everywhere. Moreover, apart from amphidromic points, $\boldsymbol{s}$ must satisfy (5.4). Thus, following Radder (1992; he considered p to be constant), we substitute (6.4) into (5.4) and into (6.1) divided by pA, giving

$$\operatorname{div}\left(\frac{\operatorname{grad} G}{pE}\right) = 0 \tag{6.5}$$

and

$$\left(\frac{\operatorname{grad} G}{pE}\right)^2 = k^2 + \frac{1}{2p}\operatorname{div}\left(p\frac{\operatorname{grad} E}{E}\right) + \frac{1}{4}\left(\frac{\operatorname{grad} E}{E}\right)^2; \tag{6.6}$$

here, $E = A^2$ is called the 'energy density' by Radder (1992).

A disadvantage of Radder's scheme is that most terms are singular at amphidromic points (these singularities can be seen easily using the local expansions given in § 4). Instead, we may formally multiply (6.5) and (6.6) by p^2E^2 to give

$$pE\nabla^2 G = \operatorname{grad} G \cdot \operatorname{grad}(pE) \tag{6.7}$$

and

$$(\operatorname{grad} G)^2 = (kpE)^2 + \tfrac{1}{4}p\operatorname{div}(p\operatorname{grad} E^2) - \tfrac{3}{4}p^2(\operatorname{grad} E)^2. \tag{6.8}$$

This pair of equations is equivalent to Radder's pair and to the three equations ((6.1), (6.3) and (5.4)) used by Ebersole (1985), in the absence of amphidromic points. However, every term in (6.7) and (6.8) is well behaved at amphidromic points, and so this pair should be better computationally.

7. Discussion

In this section we give two simple explicit examples in which amphidromic points occur (§ 7*a*, *b*), we make some comparisons with previous work (§ 7*b*, *c*) and we extend the theory to tidal amphidromic points (§ 7*d*).

(*a*) *A simple example: three-wave superposition*

Probably the simplest example (of physical relevance) exhibiting amphidromic points is obtained by adding three regular wavetrains together, where the water has constant depth (see Nicholls & Nye 1987, appendix 1). Thus, consider three waves, given by

$$\psi_j(\boldsymbol{x}) = \exp\{\mathrm{i}k(x\cos\alpha_j + y\sin\alpha_j)\}, \quad j = 1, 2, 3,$$

so that the jth wave propagates at an angle α_j to the x-axis. Let us superpose these waves in a symmetric manner, with

$$\alpha_1 = \pi, \quad \alpha_2 = \tfrac{1}{3}\pi, \quad \alpha_3 = -\tfrac{1}{3}\pi.$$

Then, the wave field is given by

$$\psi = \psi_1 + \psi_2 + \psi_3 = \mathrm{e}^{-\mathrm{i}kx} + 2\mathrm{e}^{\mathrm{i}kx\cos\alpha_2}\cos(ky\sin\alpha_2). \tag{7.1}$$

The amphidromic points can be found explicitly. Set

$$X = kx\cos\alpha_2 = \tfrac{1}{2}kx \quad \text{and} \quad Y = ky\sin\alpha_2 = \tfrac{\sqrt{3}}{2}ky.$$

Then, the amphidromic points correspond to

$$(X, Y) = (X_0 + \tfrac{2}{3}m\pi, Y_0 + 2n\pi)$$

Figure 1

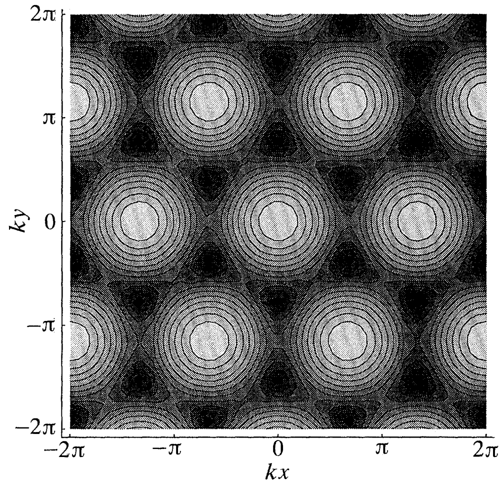

Figure 2

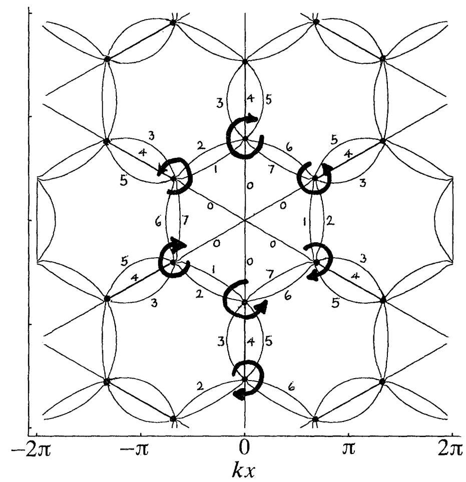

Figure 1. The amplitude A corresponding to the three-wave superposition (7.1). The scale ranges from dark ($A=0$) to light ($A=3$).

Figure 2. The phase S corresponding to the three-wave superposition (7.1). The arrows indicate the direction in which S increases around the amphidromic points. Some of the phase lines are labelled with an integer n, where $S=\frac{1}{4}n\pi$.

where m and n are arbitrary integers, and (X_0, Y_0) is any of

$$(0, \tfrac{2}{3}\pi), \quad (0, \tfrac{4}{3}\pi), \quad (\tfrac{1}{3}\pi, \tfrac{1}{3}\pi) \quad \text{and} \quad (\tfrac{1}{3}\pi, \tfrac{5}{3}\pi).$$

We observe that the amphidromic points form a regular hexagonal array in the xy-plane. We also find that each amphidromic point corresponds to a simple zero of $A(\boldsymbol{x})$; near (X_0, Y_0), we find that

$$\psi(\boldsymbol{y}) \approx k\rho\{3\mathrm{i}\cos Y_0 \cos\varphi - \sqrt{3}\sin Y_0 \sin\varphi\}\,\mathrm{e}^{\mathrm{i}X_0},$$

from which the phase can be obtained easily. The amplitude and phase are illustrated in figures 1 and 2 respectively (see also Nicholls & Nye 1987, figs 3 and 16*a*).

Similar calculations can be made for other values of α_j, $j = 1, 2, 3$.

(*b*) *Comparison with Berkhoff* et al. *(1982)*

Berkhoff *et al.* (1982) have sketched the lines of constant phase for wave propagation over a sloping bottom with an elliptical shoal. They present experimental results (their fig. 3) and numerical results based on the mild-slope equation (2.1) (their fig. 11a). In both cases, it appears that two amphidromic points are visible. However, the behaviour of the phase near each amphidromic point is not consistent with (4.7), although the curves themselves stop short of the amphidromic points. The striking feature of their results is that the phase is not a monotonic function of φ as $\boldsymbol{y}$ makes a circuit around either amphidromic point (if it was, such a circuit would intersect the solid and dashed lines in their figures alternately). We conjecture that this behaviour is due to the presence of a stationary point (where $\boldsymbol{s} = \boldsymbol{0}$ but $A \neq 0$) in the vicinity of each amphidromic point. We support this conjecture with (i) a local model solution and (ii) a refined local analysis.

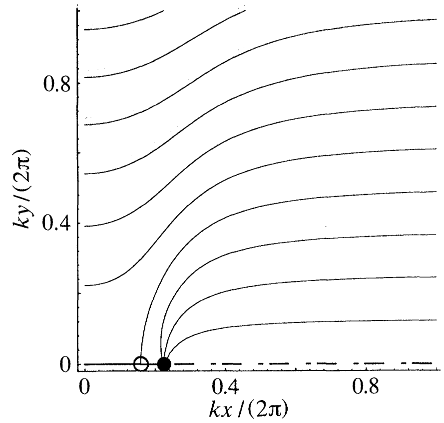

Figure 3. The phase S corresponding to the model solution (7.2), for $c = \surd 2$, at intervals of $\frac{1}{4}\pi$. Only the first quadrant is shown, as the curves are symmetric about both $x = 0$ and $y = 0$. On the positive x-axis, there is an amphidromic point at $kx = \surd 2$ (marked with a black dot) and a stationary point at $kx = 1$ (marked with a circle). $S = \frac{1}{2}\pi$ on the dashed line to the right of the amphidromic point, and then decreases as we move upwards; $S = \frac{3}{2}\pi$ on the curves passing through the stationary point (recall (1.2)).

(i) An exact model solution

Nye *et al.* (1988, appendix B) consider functions of the form

$$\psi(\boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}ky}\{ky + \mathrm{i}[(kx)^2 - c^2]\}, \tag{7.2}$$

where c is a real parameter. ψ solves the Helmholtz equation (2.2). If we choose $c > 1$, we find that ψ has amphidromic points at $(kx, ky) = (\pm c, 0)$ and stationary points (saddles) at $(kx, ky) = (\pm\surd(c^2 - 1), 0)$. The curves of constant phase when $c = \surd 2$ are shown in figure 3. The picture is symmetric about both $x = 0$ and $y = 0$. It can be interpreted as showing a wave crest along the dashed line, and wave troughs along the lines passing through the stationary point.

Although the solution (7.2) has no physical relevance for large x or y, its phase is qualitatively similar to that found by Berkhoff *et al.* (1982), near the amphidromic points. In addition, we can use it to verify some of the results in § 6. For example, since $p(\boldsymbol{x}) \equiv 1$, we find that the energy-flux vector is given by

$$pA^2\boldsymbol{s} = k(2XY, c^2 - X^2 - Y^2 - (X^2 - c^2)^2),$$

where $X = kx$ and $Y = ky$. This vector field satisfies (6.3). The corresponding energy stream-function (defined by (6.4)) is given by

$$G = X\{Y^2 + \tfrac{1}{5}X^4 + X^2(1 - \tfrac{2}{3}c^2) + c^4 - c^2\};$$

note that there is no corresponding energy potential.

(ii) Higher-order terms

We can include higher-order terms in (4.4). Thus, for example, suppose that

$$\psi(\boldsymbol{y}) \approx A_1\rho^m\mathrm{e}^{+\mathrm{i}m\varphi} + B\rho^{m+1}\mathrm{e}^{\mathrm{i}(m+1)\varphi} = A_1\rho^m\mathrm{e}^{\mathrm{i}m\varphi}\chi(\varphi + \gamma; a\rho), \tag{7.3}$$

where $B/A_1 = a\mathrm{e}^{\mathrm{i}\gamma}$; hence

$$S(\boldsymbol{y}) \approx m\varphi + T(\varphi + \gamma; a\rho) + \arg A_1,$$

which depends on φ and ρ. For simplicity, we take $\gamma = 0$. Then, it is easy to check that ψ, given by (7.3), has amphidromic points at $\rho = 0$ and at $(a\rho, \varphi) = (1, \pi)$, and a stationary point in between at $(a\rho, \varphi) = (r_m, \pi)$, where $r_m = m/(m+1)$.

Now, for fixed ρ, we have

$$S'(\varphi) = \mathrm{d}S/\mathrm{d}\varphi = m + T'(\varphi; a\rho) = \mathcal{S}'_m(\varphi; a\rho), \tag{7.4}$$

say, where

$$\mathcal{S}'_m(\varphi; r) = \frac{m + (2m+1)r\cos\varphi + (m+1)r^2}{1 + 2r\cos\varphi + r^2}.$$

We will see that, for a certain finite range of values of r, $\mathcal{S}'_m$ is not a monotonic function of φ: it changes sign twice. For $\mathcal{S}'_m$ to vanish, we must have

$$\cos\varphi = -\frac{m + (m+1)r^2}{(2m+1)r}; \tag{7.5}$$

this gives real solutions for φ provided $r_m \leqslant r \leqslant 1$. So, for such values of r, let $\varphi_m(r)$ solve (7.5), with $0 < \varphi_m(r) \leqslant \pi$. Then

$$\begin{aligned} \mathcal{S}'_m(\varphi; r) &> 0 \quad \text{for } 0 \leqslant \varphi < \varphi_m(r), \\ \mathcal{S}'_m(\varphi; r) &< 0 \quad \text{for } \varphi_m(r) < \varphi < 2\pi - \varphi_m(r), \\ \mathcal{S}'_m(\varphi; r) &> 0 \quad \text{for } 2\pi - \varphi_m(r) < \varphi \leqslant 2\pi. \end{aligned}$$

As r increases from r_m, the angle $\varphi_m(r)$ decreases from π to a minimum value, φ_{min} say, at $r = \sqrt{r_m}$, and then increases back to π at $r = 1$; moreover, $\varphi_{min} > \frac{1}{2}\pi$. If $r < r_m$ or $r > 1$, $\mathcal{S}'_m(\varphi; r)$ is a monotonic function of φ.

In our application, $r = a\rho$, where a can be any positive number. Thus, the above argument gives an explanation for the results of Berkhoff *et al.* (1982): it gives two sign changes in $S'(\varphi)$, for a certain interval of ρ, and, within this interval, the two points at which S' changes sign subtend an angle at $\boldsymbol{x}_0$ of $2\varphi_{min}(a\rho) < \pi$.

(*c*) *Comparisons with other published results*

The simple prediction (4.7), giving the behaviour of the phase around amphidromic points, can be compared with other published results. Skovgaard & Jonsson (1981) have calculated the scattering of waves by an ideal axisymmetric island, known as Homma's island, by solving the mild-slope equation (2.1). They give phase contours in the xy-plane for $y \geqslant 0$ (the motion is symmetric about $y = 0$) for two frequencies: in one (their fig. 7), five amphidromic points are clearly visible; in another (their fig. 6), there are 23 amphidromic points! The phase increases by 2π around each amphidromic point, implying simple zeros in the amplitude. Thus, these results are consistent with our simple theory. Similar results have been obtained by Sprinks & Smith (1983, fig. 6) for scattering by a conical island; they take viscous effects into account and solve a modified mild-slope equation.

Ebersole (1985) has made computations for the same bathymetry as used by Berkhoff *et al.* (1982), using a finite-difference approximation to (6.1), (6.3) and (5.4). He noted that his results were not in good agreement with their experimental data, especially downwave of the shoal. He conjectured that this error was 'related to the occurrence of an "amphidromic" point in the wavefield' (Ebersole 1985, p. 946). In fact, one of his dependent variables, $|\boldsymbol{s}|$ in our notation, is infinite at amphidromic points and so cannot be approximated using finite differences in their vicinity.

(*d*) *Tidal waves*

The simple analysis described above for gravity waves governed by the mild-slope equation can also be extended to treat tidal waves. Lamb (1932, §168) defines tidal waves as 'waves in which the motion of the fluid is mainly horizontal', although rotational effects are usually included. Thus, let $\boldsymbol{u}(\boldsymbol{x},t) = (u,v)$ be the horizontal fluid velocity and let $\zeta(\boldsymbol{x},t)$ be the surface elevation. Then, the governing equations are known as Laplace's tidal equations; in their homogeneous planar form, they are (see, for example, Lamb 1932, §207; Bowden 1983, §2.3.3; Hendershott 1977, p. 64)

$$\frac{\partial \zeta}{\partial t} + \operatorname{div}(h\boldsymbol{u}) = 0, \quad \frac{\partial u}{\partial t} - fv = -g\frac{\partial \zeta}{\partial x},$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \zeta}{\partial y}$$

where $h(\boldsymbol{x})$ is the water depth at $\boldsymbol{x}$ and f is the Coriolis parameter. For time-harmonic motions, with

$$\zeta(\boldsymbol{x},t) = \operatorname{Re}\{\psi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega t}\},$$

we can eliminate u and v to give

$$\operatorname{div}(h \operatorname{grad}\psi) + \frac{\mathrm{i}f}{\omega}\left(\frac{\partial h}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial h}{\partial y}\frac{\partial \psi}{\partial x}\right) + \frac{\omega^2 - f^2}{g}\psi = 0. \tag{7.6}$$

If h is constant, (7.6) reduces to

$$(\nabla^2 + \kappa)\psi = 0, \tag{7.7}$$

where the parameter

$$\kappa = (\omega^2 - f^2)/(gh)$$

can be positive or negative.

Taylor (1922) solved (7.7) in a rectangular bay, for $\kappa > 0$, and showed that amphidromic points could occur. See Bowden (1983, §2.5) and LeBlond & Mysak (1978, §28) for more information and further references on tidal amphidromic points.

For water of constant depth, the local behaviour near an amphidromic point $\boldsymbol{x}_0$ depends on the value of κ at $\boldsymbol{x}_0$: if $\kappa > 0$, then ψ is given by (4.1) wherein $k_0^2 = \kappa$; if $\kappa < 0$, then ψ solves the modified Helmholtz equation, locally, whence

$$\psi(\boldsymbol{y}) \approx c_m I_m(k_0\rho)\,\mathrm{e}^{\pm \mathrm{i}m\varphi}$$

for small ρ, where $k_0^2 = -\kappa$ and I_m is a modified Bessel function. In either case, the local solutions are given by (4.2), even when $h(\boldsymbol{x})$ varies slowly near $\boldsymbol{x}_0$. It follows that the phase is given by (4.7).

The governing equations for the amplitude and phase can be found, as before, by substituting (1.1) into (7.6). Again, the resulting equations involve only A and $\operatorname{grad} S$, so they may be solved by supplementing them with (5.4); again, $\operatorname{grad} S$ is singular at amphidromic points.

Finally, it is perhaps worth noting that there is an extensive literature on the finite-difference solution of (7.6); see Hendershott (1977) for a review. Within this context, it is notable that the representation in terms of an amplitude and a phase (1.1) does not seem to have been used.

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Figure 1

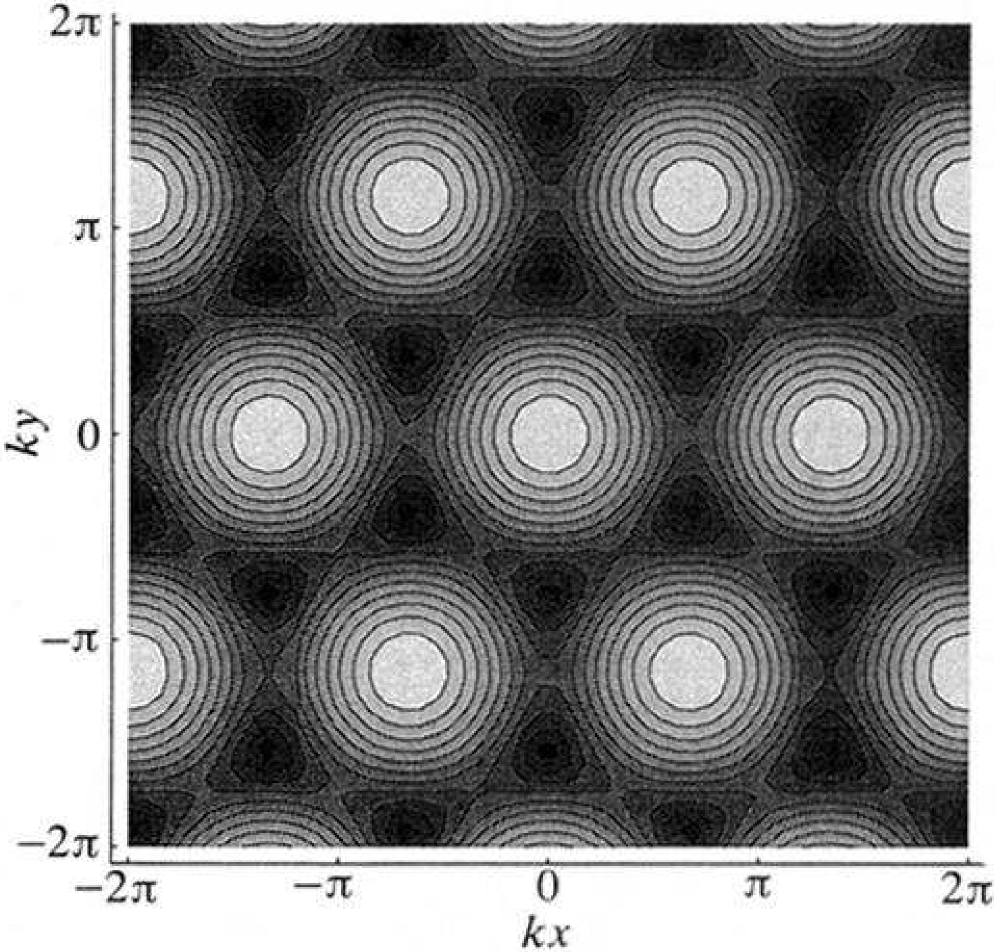

Figure 2

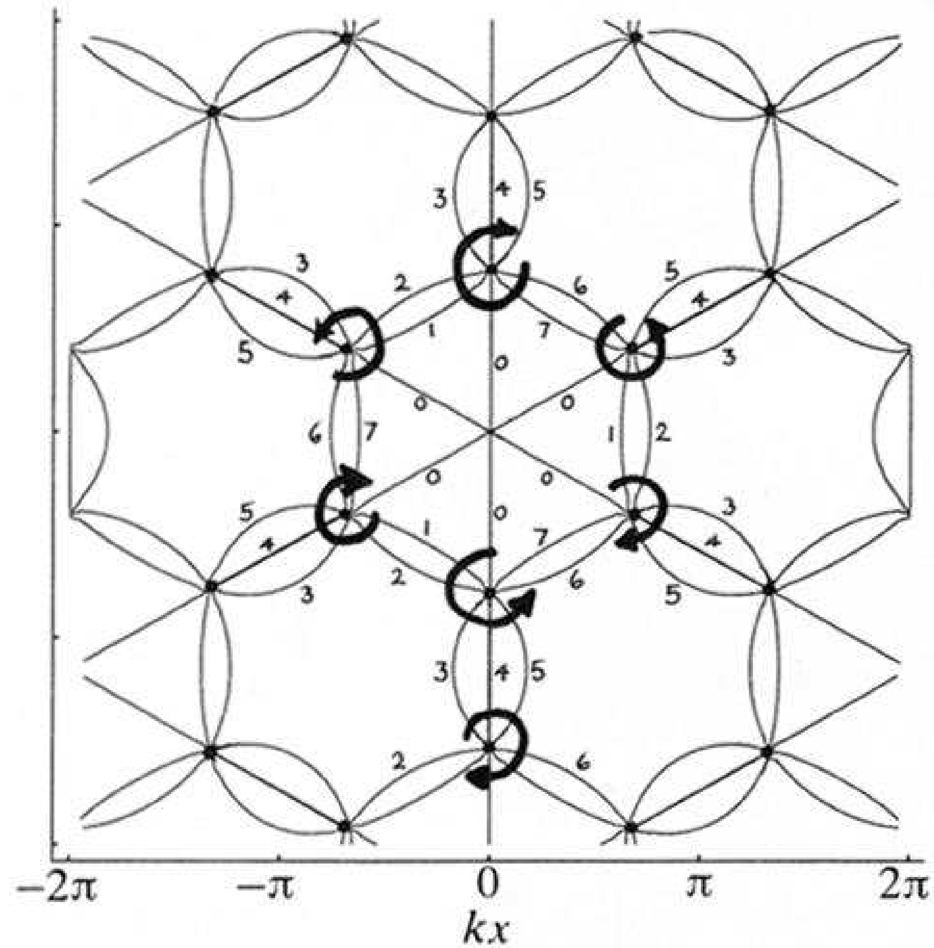

Figure 1. The amplitude A corresponding to the three-wave superposition (7.1). The scale ranges from dark ($A = 0$) to light ($A = 3$).

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