On the propagation of water waves along a porous-walled channel†

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The propagation of linear water waves in regions bounded by porous media is examined analytically. Two cases are considered: a single porous medium and a channel with porous sidewalls. For the first case, a porous medium boundary condition is developed for use in numerical parabolic models that allows for scattered wave energy to leave the computational domain without reflection.

1. Introduction

Parabolic approximations are now routinely used to model the propagation of waves over large domains. In the context of three-dimensional water waves, we cite Radder (1979), Mei & Tuck (1980), Booij (1981), Tsay & Liu (1982), Kirby & Dalrymple (1983), Kirby (1986) and Panchang *et al.* (1988). These approximations are appropriate when the waves propagate mainly in one direction, which we take to be the x-direction, and lead to parabolic differential equations; these can be solved numerically by marching in x.

A disadvantage of numerical parabolic models has been the efficacy of the lateral boundary conditions, which are imposed to allow the study of very wide domains by computing within only a narrow domain. Waves should propagate cleanly across these lateral boundaries, and so the corresponding boundary conditions are referred to as *transmitting boundary conditions*. Yet, in practice, for most imposed boundary conditions, partial reflection occurs for scattered waves generated within the model domain. Dalrymple & Martin (1992) developed a *perfect boundary condition* for parabolic wave models to permit these scattered waves to exit the modelling domain without reflection. For waves in domains bounded by a porous medium, no transmitting boundary condition has existed previously.

In this paper, we examine the propagation of water waves in regions bounded laterally by porous media modelled by the equations of Sollitt & Cross (1972). The simplest case we treat is for waves obliquely incident on a thick porous breakwater lying along the x-axis. For this case, an analytic expression is obtained for the wave field, which diffracts into the porous medium. A perfect boundary condition,

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similar to that developed by Dalrymple & Martin (1992) is provided for use in numerical models.

A more difficult case corresponds to waves propagating normally into a channel composed of two thick porous breakwaters. Melo & Guza (1991 a, b) developed a numerical parabolic model for waves within parallel jetties and made field observations of the wave-height decay down the channel (Mission Bay, California). Following a suggestion from D. H. Peregrine, they showed that the wave-height decay with distance did not differ drastically from that predicted for the centre-line decay of height behind a breakwater gap of the same width as the channel, showing that diffraction of waves into the jetties plays a major role in the wave-energy reduction. Dalrymple (1992) examined this energy dissipation by using a simple impedance boundary condition at the jetties and found a simple exponential decay of wave height down the channel, depending on the specific admittance of the jetties and the relative width of the channel (ratio of channel width to incident wave length). For this case, we provide an analytic expression for the wave field, using more exact matching conditions along the jetties, and find that the wave-height decay in the far field is proportional to $1/\sqrt{x}$. We further compare the solution to the result for waves passing through a breakwater gap of similar width. We find that, in the far field, the two solutions differ only by a multiplicative factor that depends on properties of the porous medium.

(a) Parabolic models

All of the sequel is based on the use of simple parabolic approximations to the governing equations for small-amplitude time-harmonic waves. Thus, we start by sketching the derivation of such approximations. Consider the irrotational motion of an incompressible, inviscid fluid, which we suppose has constant depth h. A harmonic velocity potential Φ exists; write it as

$$\Phi(x, y, z, t) = \mathrm{Re}\left\{\phi(x, y)\frac{\cosh k(h + z)}{\cosh kh}\,\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \qquad (1.1)$$

where the bottom is at $z = -h$ ($h > 0$); the wavenumber k and the circular frequency ω are related by the dispersion relation,

$$\omega^2 = gk\tanh kh,$$

where g is the acceleration due to gravity; and (x, y) are cartesian coordinates.

By substituting (1.1) into Laplace's equation, we obtain

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k^2\phi = 0, \qquad (1.2)$$

which is the Helmholtz equation; it is an elliptic equation for $\phi(x, y)$.

To obtain a parabolic equation, we write

$$\phi(x, y) = A(x, y)\,\mathrm{e}^{\mathrm{i}kx}$$

and then discard a term proportional to $\partial^2 A/\partial x^2$, leaving

$$2\mathrm{i}k\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial y^2} = 0, \qquad (1.3)$$

which is a parabolic equation for $A(x, y)$. This approach may be justified by

supposing that $A(x, y)$ is a slowly-varying function of x, so that

$$|\partial A/\partial x| \ll |kA|. \tag{1.4}$$

Various solutions of (1.3), the *simple parabolic equation*, are obtained below.

2. A composite semi-infinite breakwater

We consider a thin impermeable rigid semi-infinite breakwater along $x = 0$, $y < 0$, which protects a porous medium that lies in the fourth quadrant of the xy-plane. We suppose that the first, second and third quadrants are filled with water. We suppose further that waves are incident from the region $x < 0$; thus, the exact incident field is given by

$$\phi_{\mathrm{inc}}(x, y) = \mathrm{e}^{\mathrm{i}k(x \cos \theta + y \sin \theta)}. \tag{2.1}$$

ϕ_{inc} is a solution of (1.2), corresponding to a plane wave propagating at an angle θ to the x-axis. The incident waves are reflected by the rigid face of the breakwater $(x = 0, y < 0)$, diffracted by the corner at the origin, and refracted into the porous quadrant. The exact solution, then, is governed by (1.2) in the water, a different Helmholtz equation in the porous medium (with k replaced by K_1, defined below), continuity conditions across the interface $(y = 0, x > 0)$ and zero-velocity conditions on the breakwater $(x = 0, y < 0)$. This problem has been formulated by Meister (1987), but it has not been solved.

To make progress, we start by invoking a Kirchhoff approximation (Born & Wolf 1980, ch. 8). Thus, we suppose that $\phi(0, y)$ is known for $y > 0$, and then try to calculate the transmitted field in $x > 0$. Usually, one assumes that $\phi(0, y) = \phi_{\mathrm{inc}}(0, y)$ for $y > 0$, but one could suppose that $\phi(0, y)$ is known from experimental measurements. This leads to a problem posed in the half-plane $x > 0$, which is itself composed of two quadrants, one filled with water and one filled with the porous medium. This problem is still complicated (it has been studied theoretically by Meister *et al.* 1992) and the generalization to a porous-walled channel seems to be intractable.

To make further progress, we retain the Kirchhoff approximation, but also invoke the parabolic approximation. Thus, the incident field is now given by

$$A_{\mathrm{inc}}(x, y) = \mathrm{e}^{-\gamma x + \mathrm{i}\lambda y}, \tag{2.2}$$

where

$$\gamma = \tfrac{1}{2}\mathrm{i}k \sin^2 \theta \quad \text{and} \quad \lambda = k \sin \theta.$$

A_{inc} is a solution of (1.3), corresponding to a plane wave propagating at an angle θ to the x-axis; we assume that $|\theta| < \tfrac{1}{2}\pi$. We calculate the wave field in $x > 0$, using simple parabolic models.

Denote the solution in the first quadrant $Q_+ = \{(x, y) : x > 0 \text{ and } y > 0\}$ by $A_+(x, y)$, and that in the fourth quadrant $Q_- = \{(x, y) : x > 0 \text{ and } y < 0\}$ by $A_-(x, y)$; when convenient, we shall also use A, defined by

$$A(x, y) = \begin{cases} A_+(x, y) & \text{in } Q_+, \\ A_-(x, y) & \text{in } Q_-. \end{cases}$$

We suppose that Q_+ is occupied by water, so that

$$\frac{\partial^2 A_+}{\partial y^2} + \pi \Omega^2 \frac{\partial A_+}{\partial x} = 0 \quad \text{in } Q_+, \tag{2.3}$$

where

$$\Omega^2 = 2\mathrm{i}k/\pi. \tag{2.4}$$

We suppose further that Q_- is occupied by a rigid porous medium. The fluid motion within Q_- may also be described using a potential and a modified free-surface boundary condition. These equations have been derived by Sollitt & Cross (1972); see Dalrymple *et al.* (1991, Appendix A) for a summary. The porous medium is characterized by three parameters: the porosity, ϵ, the linear friction factor, f, and the inertial term, s; all these parameters are taken to be constant here. All wave motion in Q_- is damped if $f > 0$. For our parabolic model in Q_-, we choose the 'least-damped' wavenumber, K_1; this is the root of the complex dispersion relation,

$$\omega^2(s + \mathrm{i}f) = gK_1 \tanh K_1 h, \tag{2.5}$$

in the first quadrant of the complex K_1-plane with smallest imaginary part. (Note that Dalrymple *et al.* (1991) have $(s - \mathrm{i}f)$ throughout, as their time-dependence is $\mathrm{e}^{+\mathrm{i}\omega t}$. Furthermore, they examined all the roots of (2.5) and showed that the modes can 'swap' for various combinations of wave and porous media parameters.) Then, the governing equation in the porous medium is taken as

$$\frac{\partial^2 A_-}{\partial y^2} + \pi \Omega_1^2 \frac{\partial A_-}{\partial x} = 0 \quad \text{in } Q_-, \tag{2.6}$$

where

$$\Omega_1^2 = 2\mathrm{i}K_1/\pi. \tag{2.7}$$

The boundary conditions on $x = 0$ are

$$A_+(0, y) = A_{\mathrm{inc}}(0, y) = \mathrm{e}^{\mathrm{i}\lambda y} \quad \text{for } y > 0 \tag{2.8}$$

and

$$A_-(0, y) = 0 \quad \text{for } y < 0, \tag{2.9}$$

where we have used (2.2). Note that, because of the assumption (1.4), (2.9) is a comparable approximation to $\partial \phi / \partial x = 0$ on $x = 0$, which is itself the appropriate boundary condition on a rigid wall or impermeable breakwater. There are also continuity conditions across the interface between the water and the porous medium. These are (see Dalrymple *et al.* 1991)

$$\partial A_+/\partial y = \epsilon \, \partial A_-/\partial y, \ A_+ = (s + \mathrm{i}f)A_- \quad \text{on } y = 0, \ x > 0. \tag{2.10}$$

Finally, we also assume that

$$A(x, y) \text{ is bounded as } |y| \to \infty. \tag{2.11}$$

(a) Analytical solution for A

We solve for A using a Laplace transform in x (a similar method was used by Dalrymple & Martin (1992)):

$$\mathcal{L}\{A\} \equiv \overline{A}(p, y) = \int_0^\infty A(x, y) \, \mathrm{e}^{-px} \, \mathrm{d}x,$$

where we suppose that $\mathrm{Re}\, p > 0$.

(*i*) *Solution in* Q_+

Because $\mathcal{L}\{\partial A/\partial x\} = p\overline{A}(p,y) - A(0,y)$, (2.3) is transformed into

$$\frac{\partial^2 \overline{A}_+}{\partial y^2} + \pi p \Omega^2 \overline{A}_+ = \pi \Omega^2 e^{i\lambda y}, \tag{2.12}$$

where we have used (2.8). A particular integral of (2.12) is

$$\pi \Omega^2 e^{i\lambda y}(\pi p \Omega^2 - \lambda^2)^{-1} = e^{i\lambda y}(p+\gamma)^{-1} = \overline{A}_{\mathrm{inc}}(p,y),$$

and so the general solution of (2.12) is

$$\overline{A}_+(p,y) = C_+(p)\exp\{iy\Omega\sqrt{\pi p}\} + D_+(p)\exp\{-iy\Omega\sqrt{\pi p}\} + (p+\gamma)^{-1}e^{i\lambda y}. \tag{2.13}$$

Given (2.4), we define Ω by

$$\Omega = (1+i)\sqrt{(k/\pi)}.$$

Then, the condition (2.11) for $y \to \infty$ implies that $D_+(p) \equiv 0$. $C_+(p)$ will be determined below by imposing the interface conditions (2.10). Note that, on $y = 0$, we have

$$\overline{A}_+(p,0) = C_+(p) + (p+\gamma)^{-1}, \tag{2.14}$$

$$\partial \overline{A}_+/\partial y = i\Omega\sqrt{\pi p}\, C_+(p) + i\lambda(p+\gamma)^{-1}. \tag{2.15}$$

(*ii*) *Solution in* Q_-

Transforming the differential equation (2.6), making use of (2.9), gives

$$\frac{\partial^2 \overline{A}_-}{\partial y^2} + \pi p \Omega_1^2 \overline{A}_- = 0$$

with general solution

$$\overline{A}_-(p,y) = C_-(p)\exp\{iy\Omega_1\sqrt{\pi p}\} + D_-(p)\exp\{-iy\Omega_1\sqrt{\pi p}\}. \tag{2.16}$$

Now, from the definition of the complex wavenumber K_1, we have

$$K_1 = |K_1|e^{i\delta} \quad \text{with} \quad 0 \leqslant \delta < \tfrac{1}{2}\pi,$$

and so, given (2.7), we define Ω_1 by

$$\Omega_1 = (1+i)\sqrt{(|K_1|/\pi)}\, e^{i\delta/2}.$$

Also, (2.11) for $y \to -\infty$ implies that $C_-(p) \equiv 0$. Hence, on $y = 0$, we have

$$\overline{A}_-(p,0) = D_-(p), \tag{2.17}$$

$$\partial \overline{A}_-/\partial y = -i\Omega_1\sqrt{\pi p}\, D_-(p). \tag{2.18}$$

(*iii*) *Determination of* $C_+(p)$ *and* $D_-(p)$

Transforming the interface conditions (2.10), and using (2.14), (2.15), (2.17) and (2.18) gives

$$C_+(p) + (p+\gamma)^{-1} = (s+if)D_-(p),$$

$$\Omega\sqrt{\pi p}\, C_+(p) + \lambda(p+\gamma)^{-1} = -\epsilon\Omega_1\sqrt{\pi p}\, D_-(p).$$

These can be solved to give

$$C_+(p) = \frac{-\Delta(s+\mathrm{i}f)}{p+\gamma}\left\{m + \frac{\lambda}{\Omega\sqrt{\pi p}}\right\} \quad \text{and} \quad D_-(p) = \frac{\Delta}{p+\gamma}\left\{1 - \frac{\lambda}{\Omega\sqrt{\pi p}}\right\},$$

where

$$\Delta = (s+\mathrm{i}f+\epsilon\Lambda)^{-1}, \quad \Lambda = \Omega_1/\Omega = \sqrt{(K_1/k)} \quad \text{and} \quad m = \epsilon\Lambda/(s+\mathrm{i}f); \quad (2.19)$$

these complex quantities involve all the parameters of the porous medium. Dalrymple *et al.* (1991) called m the dimensionless admittance of the porous medium. Typically m satisfies $|m| < 1$. Note that if $\Phi(x, y)$ is any solution of (2.3), then $\Phi(x, \Lambda y)$ is a solution of (2.6). This completes the determination of the transforms $\overline{A}_+(p, y)$ and $\overline{A}_-(p, y)$.

(iv) Inversion of $\overline{A}_+$ *and* $\overline{A}_-$

From Gradshteyn & Ryzhik (1980, §17.13, eq. (32)), we have

$$\exp\{\mathrm{i}\alpha\sqrt{\pi p}\} = \mathcal{L}\left\{-\tfrac{1}{2}\mathrm{i}\alpha x^{-3/2}\exp\left(\tfrac{1}{4}\pi\alpha^2/x\right)\right\}.$$

Integrating with respect to α from $\mathrm{i}\infty$ to α gives

$$\exp\{\mathrm{i}\alpha\sqrt{\pi p}\}/\sqrt{\pi p} = \mathcal{L}\left\{\pi^{-1}x^{-1/2}\exp\left(\tfrac{1}{4}\pi\alpha^2/x\right)\right\}. \tag{2.20}$$

Hence, from (2.13), we have

$$\overline{A}_+(p, y) = \overline{A}_{\mathrm{inc}}(p, y) + C_+(p)\exp\{\mathrm{i}y\Omega\sqrt{\pi p}\}$$
$$= \overline{A}_{\mathrm{inc}}(p, y) + \tfrac{1}{2}\mathrm{i}y\Omega(1+m)^{-1}\mathcal{L}\{\mathrm{e}^{-\gamma x}\}\,\mathcal{L}\{\mathcal{A}_+(x, y)\} \tag{2.21}$$

for $y > 0$, where

$$\mathcal{A}_+(x, y) = \{mx^{-3/2} + (\lambda/(ky))x^{-1/2}\}\exp\{\tfrac{1}{2}\mathrm{i}ky^2/x\}. \tag{2.22}$$

Similarly, from (2.16), we have

$$\overline{A}_-(p, y) = D_-(p)\exp\{\mathrm{i}|y|\Omega_1\sqrt{\pi p}\}$$
$$= -\tfrac{1}{2}\mathrm{i}|y|\Omega\Delta\mathcal{L}\{\mathrm{e}^{-\gamma x}\}\,\mathcal{L}\{\mathcal{A}_-(x, y)\} \tag{2.23}$$

for $y < 0$, where

$$\mathcal{A}_-(x, y) = \{\Lambda x^{-3/2} + (\lambda/(ky))x^{-1/2}\}\exp\{\tfrac{1}{2}\mathrm{i}K_1 y^2/x\}. \tag{2.24}$$

These expressions for $\overline{A}_+$ and $\overline{A}_-$ can be inverted using the convolution theorem, namely

$$\mathcal{L}\{u\}\mathcal{L}\{v\} = \mathcal{L}\left\{\int_0^x u(x-\xi)v(\xi)\,\mathrm{d}\xi\right\},$$

to give the solutions

$$A_+(x, y) = A_{\mathrm{inc}}(x, y) + \tfrac{1}{2}\mathrm{i}y\Omega(1+m)^{-1}\,\mathrm{e}^{-\gamma x}I_+(x, y), \tag{2.25}$$
$$A_-(x, y) = -\tfrac{1}{2}\mathrm{i}|y|\Omega\Delta\,\mathrm{e}^{-\gamma x}I_-(x, y), \tag{2.26}$$

where

$$I_\pm(x, y) = \int_0^x \mathrm{e}^{\gamma\xi}\mathcal{A}_\pm(\xi, y)\,\mathrm{d}\xi. \tag{2.27}$$

We can express $I_\pm$ in terms of the Fresnel integral

$$F(v) = \int_v^\infty e^{iu^2}\, du. \tag{2.28}$$

We find that (see Appendix A)

$$I_+(x, y) = \sqrt{2/(ky^2)}\left\{(m+1)\, e^{i\lambda y} F(\nu_-) + (m-1)\, e^{-i\lambda y} F(\nu_+)\right\} \tag{2.29}$$

for $y > 0$, where $\nu_\pm(x, y) = \sqrt{\tfrac{1}{2}k/x}\,(y \pm x \sin\theta)$, and

$$I_-(x, y) = 2\sqrt{2/(ky^2)}\,\exp[i\lambda\Lambda y]\, F(-\nu_1) \tag{2.30}$$

for $y < 0$, where $\nu_1 = \sqrt{\tfrac{1}{2}k/x}\,(\Lambda y - x \sin\theta) = \nu_-(x, \Lambda y)$.

Substituting for I_+ in (2.25) and simplifying gives

$$A_+(x, y) = A_{\text{diff}}(x, y) + A_{\text{por}}(x, y), \tag{2.31}$$

where

$$A_{\text{diff}}(x, y) = A_{\text{inc}}(x, y)\left\{1 + \frac{i(1 + i)}{\sqrt{2\pi}}\, F(\nu_-)\right\} \tag{2.32}$$

is solely due to diffraction (it does not depend on the properties of the porous medium) and

$$A_{\text{por}}(x, y) = A_{\text{inc}}(x, -y)\,\frac{i(1+i)}{\sqrt{2\pi}}\,\frac{m-1}{m+1}\, F(\nu_+)$$

incorporates effects in Q_+ due to the porous medium in Q_-. Similarly, substituting for I_- in (2.26), noting that $F(-v) = \sqrt{\pi}\, e^{i\pi/4} - F(v)$, gives the solution in Q_- as

$$A_-(x, y) = 2\Delta\, A_{\text{diff}}(x, \Lambda y). \tag{2.33}$$

Finally, it is sometimes convenient to express the solution in terms of the real Fresnel integrals

$$C(v) = \int_0^v \cos\left(\tfrac{1}{2}\pi u^2\right) du \quad \text{and} \quad S(v) = \int_0^v \sin\left(\tfrac{1}{2}\pi u^2\right) du.$$

Because

$$F(v) = \sqrt{\tfrac{1}{2}\pi}\left\{\tfrac{1}{2}(1 + i) - C\left(v\sqrt{2/\pi}\right) - iS\left(v\sqrt{2/\pi}\right)\right\},$$

we obtain

$$A_{\text{diff}}(x, y) = \frac{e^{-i\pi/4}}{\sqrt{2}}\, A_{\text{inc}}(x, y)\,\left\{[\tfrac{1}{2} + C(\sigma_-)] + i[\tfrac{1}{2} + S(\sigma_-)]\right\} \tag{2.34}$$

and

$$A_{\text{por}}(x, y) = -\frac{e^{-i\pi/4}}{\sqrt{2}}\, A_{\text{inc}}(x, -y)\,\frac{m-1}{m+1}\,\left\{[\tfrac{1}{2} - C(\sigma_+)] + i[\tfrac{1}{2} - S(\sigma_+)]\right\}, \tag{2.35}$$

where $\sigma_\pm(x, y) = \sqrt{k/(\pi x)}\,(y \pm x \sin\theta)$.

(b) *Comparisons and special cases*

Here, we specialize the solution obtained in §2a, and compare with published results.

For normal incidence ($\theta = 0$), we have $A_{\mathrm{inc}}(x,y) \equiv 1$,

$$\sigma_+ = \sigma_- = y\sqrt{k/(\pi x)} = \sigma,$$

say, and $\sigma_1 = \Lambda\sigma$. This leads to slight simplifications in (2.34) and (2.35). In particular, if we suppose that Q_- is filled with water, we have

$$s = \epsilon = 1 \quad \text{and} \quad f = 0,$$

whence

$$K_1 = k, \quad m = \Lambda = 1, \quad \Delta = \tfrac{1}{2}, \quad A_{\mathrm{por}} \equiv 0.$$

Hence, for all y, we have

$$A(x,y) = A_{\mathrm{diff}}(x,y) = 2^{-1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\left\{[\tfrac{1}{2} + C(\sigma)] + \mathrm{i}[\tfrac{1}{2} + S(\sigma)]\right\}.$$

This agrees with equation (7.16) in §10.7.1 of Mei's book (1983), obtained there by looking for a similarity solution.

More generally, for oblique incidence but still filling Q_- with water, we obtain

$$A(x,y) = A_{\mathrm{diff}}(x,y)$$

for all y, where A_{diff} is given by (2.32) or (2.34). In Appendix B, we show that this is equivalent to a result obtained previously by Kirby & Dalrymple (1986), using a Fourier transform in y.

(i) *Comparison with Sommerfeld's solution*

When the fourth quadrant Q_- is filled with water, the exact problem formulated at the beginning of this section can be solved exactly: this is Sommerfeld's solution of (1.2) for scattering by a thin rigid semi-infinite barrier. It is of interest to compare the solution (2.32), obtained by solving the parabolic equation (2.3) in $x > 0$, with Sommerfeld's exact solution, $A_{\mathrm{ex}}(x,y) = \mathrm{e}^{-\mathrm{i}kx}\phi(x,y)$. Thus, from Noble (1988, p. 73), we have

$$A_{\mathrm{ex}}(x,y) = A_{\mathrm{ex}}^{(1)}(x,y) + A_{\mathrm{ex}}^{(2)}(x,y), \tag{2.36}$$

where

$$A_{\mathrm{ex}}^{(1)}(x,y) = \pi^{-1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\lambda y}\mathrm{e}^{\mathrm{i}kx(\cos\theta-1)}\,F(a_1),$$
$$A_{\mathrm{ex}}^{(2)}(x,y) = \pi^{-1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\lambda y}\mathrm{e}^{-\mathrm{i}kx(\cos\theta+1)}\,F(a_2),$$

$$a_1 = \sqrt{2k\rho}\,\sin\tfrac{1}{2}(\theta - \varphi), \qquad a_2 = \sqrt{2k\rho}\,\cos\tfrac{1}{2}(\theta + \varphi),$$

and (ρ, φ) are plane polar coordinates: $x = \rho\cos\varphi$, and $y = \rho\sin\varphi$. This solution is valid in the whole of the xy-plane, with the breakwater on the negative y-axis. In $x > 0$, we have $|\varphi| < \tfrac{1}{2}\pi$; since $|\theta| < \tfrac{1}{2}\pi$, we have $a_2 > 0$. Now, since $F(v) \sim \tfrac{1}{2}\mathrm{i}v^{-1}\mathrm{e}^{\mathrm{i}v^2}$ as $v \to \infty$, we see that

$$\exp\{\mathrm{i}[\lambda y - kx\cos\theta]\}\,F(a_2) \sim \tfrac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}k\rho}\,(2k\rho)^{-1/2}\sec\tfrac{1}{2}(\theta + \varphi)$$

as $\rho \to \infty$ in $x > 0$; thus, $A_{\text{ex}}^{(2)}$ decays like $\rho^{-1/2}$ in $x > 0$. Now, write

$$A_{\text{ex}}^{(1)}(x, y) = A_{\text{ex}}^{\text{inc}}(x, y) \left\{ 1 + \frac{\text{i}(1 + \text{i})}{\sqrt{2\pi}} F(-a_1) \right\}, \tag{2.37}$$

where
$$A_{\text{ex}}^{\text{inc}}(x, y) = \text{e}^{\text{i}\lambda y} \text{e}^{\text{i}kx(\cos\theta - 1)}$$

is the exact incident field for a plane wave. $A_{\text{ex}}^{(1)}$ gives the transmitted waves in $x > 0$, and should be compared with the parabolic approximation A_{diff}, given by (2.32). First, we compare $A_{\text{ex}}^{\text{inc}}$ with A_{inc}, defined by (2.2); we have

$$A_{\text{ex}}^{\text{inc}}(x, y) = A_{\text{inc}}(x, y) \exp\{\text{i}kx[\cos\theta - 1 + \tfrac{1}{2}\sin^2\theta]\},$$

and the exponent is seen to be $O(\theta^4)$ as $\theta \to 0$. Second, we compare the arguments of the two Fresnel integrals: we have

$$\nu_- = \sqrt{\tfrac{1}{2}k\rho} \, \frac{\sin\varphi - \cos\varphi\sin\theta}{\sqrt{\cos\varphi}} \quad \text{and} \quad -a_1 = 2\sqrt{\tfrac{1}{2}k\rho} \sin\tfrac{1}{2}(\varphi - \theta).$$

If we suppose that both φ and θ are small (so that we are close to normal incidence and we observe close to the x-axis), we find that

$$-a_1 = \nu_- + \text{third-order terms in } \theta \text{ and } \varphi.$$

Thus, the parabolic approximation and the exact solution are seen to be analytically close, provided θ and φ are both small and $k\rho$ is large. Numerical comparisons with the results of Penney & Price (1952) (based on Sommerfeld's solution) have also been made by Dalrymple *et al.* (1984).

(c) *Numerical results*

An example of the full solution (2.31), with (2.34) and (2.35), is shown in dimensionless coordinates in figure 1, for $\epsilon = 0.45$, $s = 1$, $f = 0.01$, $\omega = \tfrac{1}{7}\pi$, and $h = 8$m. These values, with the exception of f, correspond to the values used by Melo & Guza (1991a). Eight wavelengths (each of 120.6m) are shown for $\theta = 0$. The small value of f permits a significant amount of wave energy to diffract into the porous medium.

In figure 2, the frictional parameter f has been increased ten-fold to a more realistic value, and the dissipation takes place in a very narrow region near the interface between Q_- and Q_+. Furthermore, there is a greater penetration of the wave energy near the origin as the diffraction (due to a discontinuity in wave amplitude) is stronger.

In figure 3, the angle of incidence is $-30°$, showing the partial reflection of the wave from the interface.

3. A perfect boundary condition for porous walls

In practice, the water depth h will vary with position, and so it will usually be necessary to solve the corresponding parabolic equation numerically. In order to minimize the size of the computational domain, it is necessary to have efficient lateral boundary conditions, which allow waves to leave the computational domain regardless of the wave direction, crest curvature or strength of scattering. Dalrymple & Martin (1992) have derived such a *perfect boundary condition*,
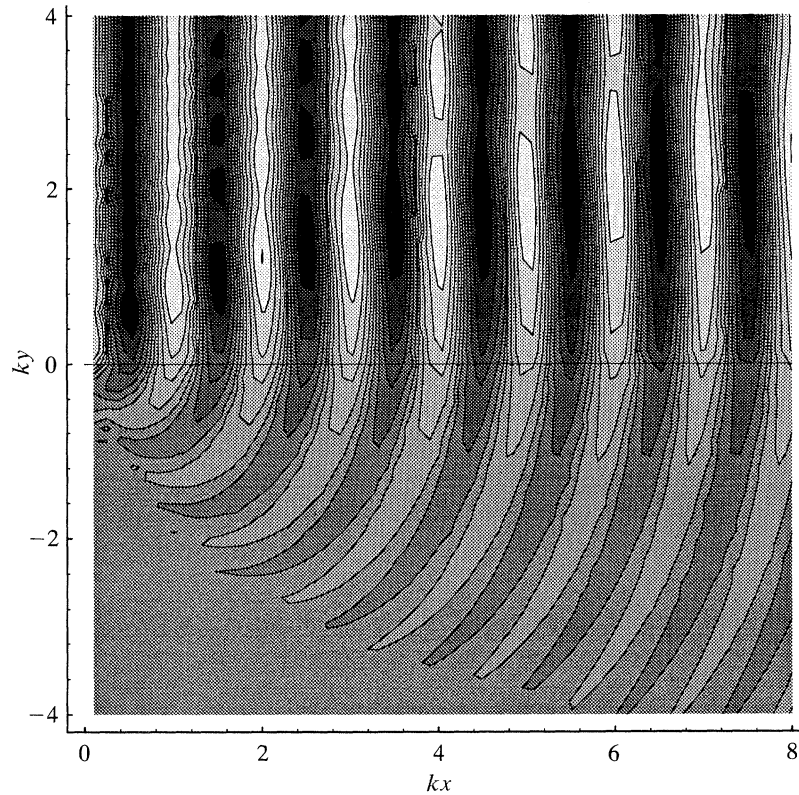
Figure 1. Waves are incident from the left with $\theta = 0$ and $\omega = \frac{1}{7}\pi$. The other parameters are $h = 8$m, $\epsilon = 0.45$, $s = 1$ and $f = 0.01$. Note the diffraction into the porous medium, $y < 0$.

which, in principle, gives an exact representation for the motion in a quadrant of constant-depth water. It is non-local, but has the form of a generalized impedance boundary condition after discretization.

We now develop a similar boundary condition for a porous medium. It could be used for propagation down a porous-walled channel. Specifically, we consider a channel $\mathcal{C} = \{(x, y) : x > 0 \text{ and } -b < y < b\}$ of width $2b$, filled with water. The two quadrants $Q_{\pm}^{b} = \{(x, y) : x > 0 \text{ and } \pm y > b\}$ are filled with a porous medium, and bounded by semi-infinite breakwaters along $x = 0$ for $|y| > b$. A parabolic equation is solved in $\mathcal{C}$ by marching in x, subject to certain boundary conditions on the walls $y = \pm b$, which we shall now derive.

Denote the exact solution in $Q_{\pm}^{b}$ by $A_{\pm}^{b}(x, y)$. We have $A_{\pm}^{b}(x, y) = A_{-}(x, b \mp y)$, where $A_{-}(x, y)$ is the function obtained in § 2a; its Laplace transform was found in § 2a(ii). From (2.16) (with $C_{-}(p) \equiv 0$), we have

$$\partial \overline{A}_{-}/\partial y = -\mathrm{i}\Omega_1 \sqrt{\pi p}\, \overline{A}_{-}(p, y). \tag{3.1}$$

Denote the solution in the channel by $A_{\mathrm{c}}(x, y)$. On $y = \pm b$, A_{c} is related to A_{-} by the interface conditions, (2.10). When these are used in (3.1), we obtain

$$\partial \overline{A}_{\mathrm{c}}(p, b)/\partial y = \mathrm{i}m\Omega\sqrt{\pi/p}\,\{p\overline{A}_{\mathrm{c}}(p, b)\},$$

on $y = b$, where $m = \epsilon \Lambda/(s + \mathrm{i}f)$. We can invert, using the convolution theorem,
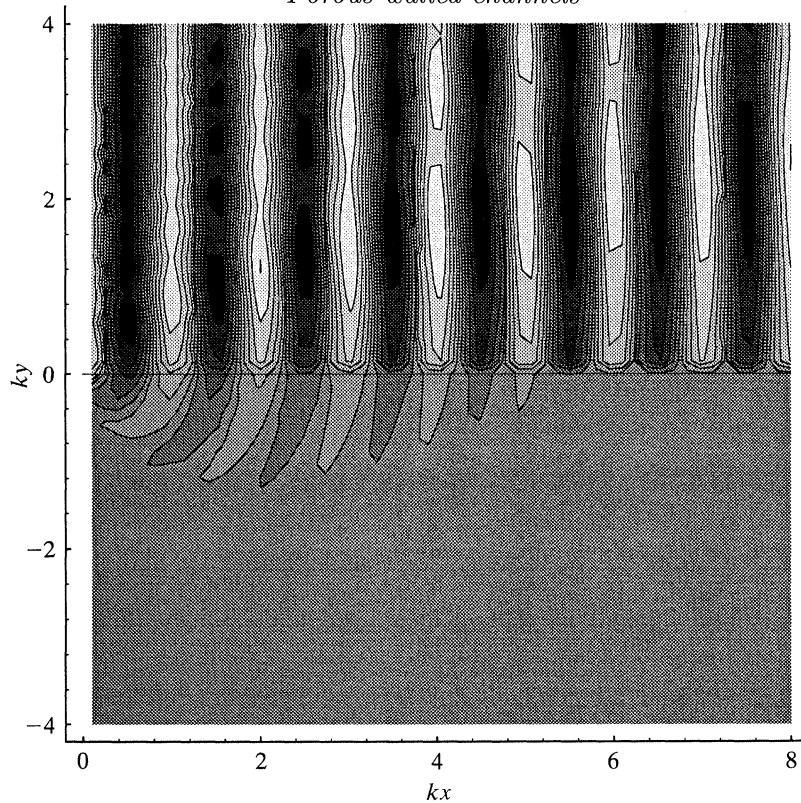
Figure 2. As for figure 1, except the friction factor has been increased to $f = 0.1$.

to give

$$\frac{\partial A_{\mathrm{c}}(x,b)}{\partial y} = \mathrm{i}m\Omega \int_0^x \frac{\partial A_{\mathrm{c}}(\xi,b)}{\partial \xi} \frac{\mathrm{d}\xi}{\sqrt{x-\xi}}, \tag{3.2}$$

which is just equation (A 9) in (Dalrymple & Martin 1992), apart from the factor m. This is the exact boundary condition to be imposed on $A_{\mathrm{c}}(x,y)$ at the porous wall $y = b$. It can be discretized exactly as done previously by Dalrymple & Martin (1992): for simplicity, we use a uniform discretization in x, with stations at $x_r = r\Delta x$, and then approximate $A_{\mathrm{c}}(\xi,b)$ by a continuous piecewise-linear function, so that the integration in (3.2) can be done analytically; this gives

$$\frac{\partial A_{\mathrm{c}}^r}{\partial y} + ma A_{\mathrm{c}}^r = m\sum_{l=0}^{r-1} b_l^r A_{\mathrm{c}}^l \tag{3.3}$$

as our generalized impedance boundary condition on $y = b$, where

$$a = -2\mathrm{i}\Omega/\sqrt{\Delta x}, \quad b_0^r = a\left(\sqrt{r} - \sqrt{r-1}\right),$$

$$b_l^r = a\left(2\sqrt{r-l} - \sqrt{r-l-1} - \sqrt{r-l+1}\right), \quad \text{for } l = 1, 2, \ldots, r-1,$$

and $A_{\mathrm{c}}^r = A_{\mathrm{c}}(x_r, b)$. The analogous boundary condition on $y = -b$ is obtained by changing the sign of the first term on the left-hand side of (3.3); note that the two porous quadrants can have different properties, leading to different values
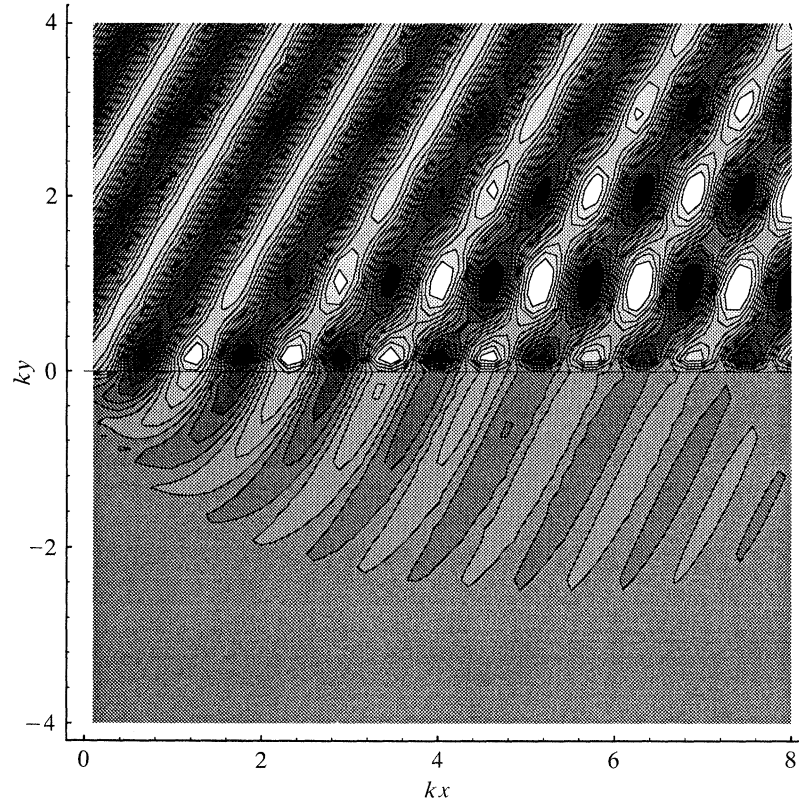
Figure 3. As for figure 2, except the waves are incident at $\theta = -30°$. Note the increase in wave penetration into the porous medium.

of m on the two walls. Examples of the efficacy of (3.3) (with $m = 1$) have been given by Dalrymple & Martin (1992).

4. Analytical solution for a porous-walled channel

Consider the symmetric problem of waves at normal incidence to the entrance of a channel with identical porous media in $Q_{\pm}^{b}$. By symmetry, we can write

$$\overline{A}_c(p, y) = p^{-1} + C(p)\cos\{y\Omega\sqrt{\pi p}\} \quad \text{for } |y| < b, \tag{4.1}$$

where $C(p)$ is to be determined from the conditions on the walls. From § 3, we know that the porous media are modelled exactly by the boundary conditions

$$\pm\partial\overline{A}_c/\partial y = im\Omega\sqrt{\pi p}\,\overline{A}_c \quad \text{on } y = \pm b.$$

When these are applied to (4.1), we obtain

$$C(p) = m/(pD(\mu)), \tag{4.2}$$

where

$$D(\mu) = i\sin\mu - m\cos\mu \quad \text{and} \quad \mu = b\Omega\sqrt{\pi p}.$$

(a) Solution for the rigid-walled channel

If the walls are rigid, we have $\epsilon = m = 0$, whence $C(p) = 0$ and so

$$A_\mathrm{c}(x, y) = 1,$$

which is the expected result.

(b) Solution for a gap in a breakwater

If the quadrants $Q_\pm^b$ are filled with water, we have $m = 1$, whence (4.2) gives

$$C(p) = -\mathrm{e}^{\mathrm{i}\mu}/p.$$

Denoting the corresponding solution in the 'channel' $\mathcal{C}$ by $A_\mathrm{gap}(x, y)$, we obtain

$$\overline{A}_\mathrm{gap}(p, y) = p^{-1} - \tfrac{1}{2}p^{-1}\left[\exp\{\mathrm{i}(b + y)\Omega\sqrt{\pi p}\} + \exp\{\mathrm{i}(b - y)\Omega\sqrt{\pi p}\}\right]$$

from (4.1). To invert this expression, we integrate (2.20) with respect to α, from $\mathrm{i}\infty$ to α to give

$$\frac{\exp\{\mathrm{i}\alpha\sqrt{\pi p}\}}{p} = \mathcal{L}\left\{\frac{\mathrm{i}}{x^{1/2}}\int_{\mathrm{i}\infty}^{\alpha}\exp\left(\frac{\pi t^2}{4x}\right)\mathrm{d}t\right\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-\mathrm{i}\pi/4}\,F(U)\right\},$$

where $U = \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/4}\alpha\sqrt{\pi/x}$; here, we have made the substitution $t = 2u\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{x/\pi}$ and then rotated the path of integration in the u-plane (which is readily justified). Hence

$$A_\mathrm{gap}(x, y) = 1 + \frac{\mathrm{i}(1 + \mathrm{i})}{\sqrt{2\pi}}\left\{F(\nu_+^b) + F(\nu_-^b)\right\}, \tag{4.3}$$

where $\nu_\pm^b = \sqrt{k/(2x)}(b \pm y)$. This solution agrees with that obtained by Kirby & Dalrymple (1986). Similar solutions are also described by Stamnes (1987, §20.1).

Note that we could have obtained the solution (4.3) by simple combinations of the solution for scattering by a single semi-infinite breakwater. This method is not available for problems involving two (or more) different differential equations, as for the porous-walled channel.

(c) Solution for the porous-walled channel

Let us write

$$A_\mathrm{c}(x, y) = A_\mathrm{gap}(x, y) + A_\mathrm{wall}(x, y). \tag{4.4}$$

We find that

$$A_\mathrm{wall}(x, y) = \tfrac{1}{2}(1 - m)\{A_\mathrm{w}(x, b + y) + A_\mathrm{w}(x, b - y)\}, \tag{4.5}$$

where

$$\overline{A}_\mathrm{w}(p, Y) = \frac{\mathrm{i}\sin\mu}{pD(\mu)}\exp\{\mathrm{i}Y\Omega\sqrt{\pi p}\}; \tag{4.6}$$

here, $Y = b \pm y$, so that we have $Y > 0$. We have been unable to invert this expression by inspection. So, we resort to the inversion theorem (Bromwich integral); this calculation is described in Appendix C. We find that

$$A_\mathrm{w}(x, Y) = A_\mathrm{res}(x, Y) + A_\mathrm{cut}(x, Y), \tag{4.7}$$

where A_{res} comes from the residues at the poles of $\overline{A}_{\mathrm{w}}(p, Y)$ in the complex p-plane, and A_{cut} comes from the branch point at $p = 0$. Specifically, we obtain

$$A_{\mathrm{res}}(x, Y) = \frac{2kb^2 m}{1 - m^2} \sum_{n=0}^{\infty} \frac{\exp\{\mathrm{i}(Y/b)(\mu_0 + n\pi)\}}{(\mu_0 + n\pi)^2} \exp\left\{-\frac{\mathrm{i}x}{2kb^2}(\mu_0 + n\pi)^2\right\} \quad (4.8)$$

and

$$A_{\mathrm{cut}}(x, Y) = \frac{1}{2\pi} \int_0^{\infty} \mathrm{e}^{-xt} \{G_+(t) - G_-(t)\} \frac{\mathrm{d}t}{t}. \quad (4.9)$$

Here, μ_0 is the root of $D(\mu_0) = 0$ that is smallest in magnitude and lies in the sector $-\frac{1}{4}\pi < \arg \mu_0 < 0$,

$$G_\pm(t) = \frac{\sin \mu_\pm}{D(\mu_\pm)} \exp\left\{\mathrm{i}Y\Omega\mathrm{e}^{\pm \mathrm{i}\pi/2}\sqrt{\pi t}\right\} \quad \text{and} \quad \mu_\pm(t) = b\Omega\mathrm{e}^{\pm \mathrm{i}\pi/2}\sqrt{\pi t}.$$

(d) Solution for large x

We can examine the decay of the solutions downwave as $x \to \infty$. For the problem of a gap in a breakwater, the exact solution of the parabolic model is (4.3). Using $F(v) \sim F(0) - v$ as $v \to 0$, we find that

$$A_{\mathrm{gap}}(x, y) \sim b\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k/(\pi x)} \quad \text{as } x \to \infty. \quad (4.10)$$

Note that there is no dependence on y at this order. If we want a better approximation, we can use $F(v) \sim F(0) - v - \frac{1}{3}\mathrm{i}v^3$ as $v \to 0$; this gives

$$A_{\mathrm{gap}}(x, y) \sim b\sqrt{\frac{2k}{\pi x}}\left\{\mathrm{e}^{-\mathrm{i}\pi/4} + \frac{k}{6x}(3y^2 + b^2)\mathrm{e}^{\mathrm{i}\pi/4}\right\} \quad \text{as } x \to \infty. \quad (4.11)$$

This formula agrees *precisely* with equation (47) in (Penney & Price 1952). Note that the *exact* (closed-form) solution of (1.2) for scattering by a gap between two semi-infinite rigid barriers does not seem to be known: the 'solution' used by Penney & Price (1952) is just a linear combination of Sommerfeld solutions for a single semi-infinite barrier, and is not exact.

Now, consider the porous-walled channel. First of all, we note that, due to (C 2), every term in the series for (4.8) decays exponentially with x. For the branch-cut contribution, we can find an asymptotic approximation to the integral in (4.9) using Watson's lemma. We have $G_\pm(t) \sim -\mu_\pm/m$ as $t \to 0$, whence

$$A_{\mathrm{cut}}(x, Y) \sim \frac{b\Omega}{\mathrm{i}m\sqrt{\pi}} \int_0^{\infty} t^{-1/2}\mathrm{e}^{-xt}\,\mathrm{d}t = \frac{b}{m}\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2k}{\pi x}}$$

as $x \to \infty$. Thus, the branch-cut integral gives algebraic decay with x, as opposed to the exponential decay from the residues. It follows from (4.5) and (4.7) that

$$A_{\mathrm{wall}}(x, y) \sim \left(\frac{1}{m} - 1\right)b\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2k}{\pi x}}$$

as $x \to \infty$. Finally, combining this result with (4.4) and (4.10), we obtain

$$A_{\mathrm{c}}(x, y) \sim \frac{b}{m}\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{\frac{2k}{\pi x}} \quad \text{as } x \to \infty. \quad (4.12)$$

When we compare this result with (4.10), we find that

$$\lim_{x \to \infty} \frac{A_{\mathrm{c}}(x,y)}{A_{\mathrm{gap}}(x,y)} = \frac{1}{m} = \frac{s + \mathrm{i}f}{\epsilon} \sqrt{\frac{k}{K_1}}.$$

This gives an analytical confirmation of Peregrine's suggestion: the wave-height decay for the porous-walled channel is similar to that for a gap in a thin breakwater; in fact, we find that they only differ by a multiplicative factor of $1/m$, which typically has a magnitude that is greater than unity.

Appendix A. Evaluation of $I_{\pm}$

Consider I_{+}, defined by (2.27) for $y > 0$. The exponent in I_{+} is $\gamma\xi + \frac{1}{2}\mathrm{i}ky^2/\xi = \frac{1}{2}\mathrm{i}\mu(t + t^{-1})$ with $t = \xi|\sin\theta|/y$ and $\mu = |\lambda|y$. So, changing the variable of integration to t gives

$$I_{+} = \sqrt{\frac{\mu}{ky^2}} \int_0^T L(t) \, \exp\{\tfrac{1}{2}\mathrm{i}\mu(t + t^{-1})\} \, \mathrm{d}t, \tag{A 1}$$

where $L(t) = mt^{-3/2} + t^{-1/2}\operatorname{sgn}\theta$, $T = x|\sin\theta|/y$ and, for the moment, we assume $|\sin\theta| > 0$. The integral simplifies by making the substitution $t = \mathrm{e}^{-2\varphi}$ to give

$$I_{+} = 2\sqrt{\frac{\mu}{ky^2}} \int_{\varphi_0}^{\infty} \{A \cosh\varphi + B \sinh\varphi\} \, \exp\{\mathrm{i}\mu\cosh 2\varphi\} \, \mathrm{d}\varphi, \tag{A 2}$$

where $A = m + \operatorname{sgn}\theta$, $B = m - \operatorname{sgn}\theta$ and $\varphi_0 = -\frac{1}{2}\log T$. But

$$\zeta \int_{\varphi_0}^{\infty} \cosh\varphi \, \exp\{\tfrac{1}{2}\mathrm{i}\zeta^2\cosh 2\varphi\} \, \mathrm{d}\varphi = \zeta\mathrm{e}^{\frac{1}{2}\mathrm{i}\zeta^2} \int_{\varphi_0}^{\infty} \cosh\varphi \, \mathrm{e}^{\mathrm{i}\zeta^2\sinh^2\varphi} \, \mathrm{d}\varphi$$

$$= \mathrm{e}^{\frac{1}{2}\mathrm{i}\zeta^2} F(\zeta\sinh\varphi_0),$$

where F is the Fresnel integral defined by (2.28), and

$$\zeta \int_{\varphi_0}^{\infty} \sinh\varphi \, \exp\{\tfrac{1}{2}\mathrm{i}\zeta^2\cosh 2\varphi\} \, \mathrm{d}\varphi = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\zeta^2} F(\zeta\cosh\varphi_0),$$

whence

$$I_{+} = \sqrt{2/(ky^2)} \left\{ A\,\mathrm{e}^{\mathrm{i}\mu} F(\sqrt{2\mu}\,\sinh\varphi_0) + B\,\mathrm{e}^{-\mathrm{i}\mu} F(\sqrt{2\mu}\,\cosh\varphi_0) \right\}. \tag{A 3}$$

If we now return to the original variables, we find that we have the same formula for $\sin\theta > 0$ as for $\sin\theta < 0$; moreover, the formula is also valid when $\sin\theta = 0$. The result is (2.29).

A similar calculation succeeds for I_{-}, defined by (2.27) for $y < 0$. The exponent is now $\gamma\xi + \frac{1}{2}\mathrm{i}K_1 y^2/\xi = \frac{1}{2}\mathrm{i}\mu(t + t^{-1})$ with $t = \xi|\sin\theta|/(\Lambda|y|)$ and $\mu = \Lambda|\lambda||y|$. So, changing the variable of integration to t shows that I_{-} is given by (A 1), with $L(t) = t^{-3/2} - t^{-1/2}\operatorname{sgn}\theta$ and $T = x|\sin\theta|/(\Lambda|y|)$. Putting $t = \mathrm{e}^{-2\varphi}$ then gives I_{-} as (A 2), but now $A = 1 - \operatorname{sgn}\theta$ and $B = 1 + \operatorname{sgn}\theta$. Hence, I_{-} is given by (A 3), with the current values for A, B, μ and $\varphi_0 = -\frac{1}{2}\log T$. Again, we obtain the same formula for $\sin\theta > 0$ as for $\sin\theta < 0$; however, since $A = 0$ for $\theta > 0$, one term inside the braces in (A 3) is absent. Thus, the result is (2.30).

Appendix B. Comparison with Kirby & Dalrymple (1986)

Kirby & Dalrymple (1986) have solved (2.3) in $x > 0$. Thus, their equation (30b) gives

$$A(x,y) = A_{\mathrm{inc}}(x,y) - \pi^{-1/2}\mathrm{e}^{-\mathrm{i}\pi/4}\, I(x,y),$$

where I is given by an integral over the breakwater. So, for a breakwater along $x = 0$, $y < 0$, we obtain

$$I(x,y) = \sqrt{\tfrac{1}{2}k/x} \int_{-\infty}^{0} \exp\{\mathrm{i}\lambda\zeta + \tfrac{1}{2}\mathrm{i}k(y-\zeta)^2/x\}\,\mathrm{d}\zeta.$$

The exponent is a quadratic in ζ; completing the square, it can be written as $\tfrac{1}{2}\mathrm{i}(k/x)[\zeta - (y - x\sin\theta)]^2 + \mathrm{i}\lambda y - \gamma x$. Then, the change of variable $u = -\sqrt{(\tfrac{1}{2}k/x)}\,[\zeta - (y - x\sin\theta)]$ shows that $I(x,y) = A_{\mathrm{inc}}(x,y)\,F(\sigma_-)$, whence $A = A_{\mathrm{diff}}$, as given by (2.32).

Appendix C. Inversion of $\overline{A}_{\mathrm{w}}(p,Y)$

$\overline{A}_{\mathrm{w}}(p,Y)$ is defined by (4.6). To apply the inversion theorem, we start by determining the singularities of $\overline{A}_{\mathrm{w}}(p,Y)$ in the complex p-plane. We exclude the cases $m = 0$ (§4a) and $m = 1$ (§4b).

Clearly, $\overline{A}_{\mathrm{w}}(p,Y)$ has a branch point at $p = 0$; cut the p-plane along the negative real axis, and choose the branch with $-\pi < \arg p < \pi$. We have $\mu = b\Omega\sqrt{\pi p} = b\sqrt{2kp}\,\mathrm{e}^{\mathrm{i}\pi/4}$ whence

$$-\tfrac{1}{4}\pi < \arg\mu < \tfrac{3}{4}\pi. \tag{C 1}$$

Now, suppose that μ_0 satisfies $D(\mu_0) = 0$; we have $\mathrm{e}^{-2\mathrm{i}\mu_0} = (1 - m)/(1 + m)$. Then, periodicity implies that $D(\mu_0 + n\pi) = 0$ for any integer n. However, we are only interested in those roots that satisfy (C 1).

Because $s > 0$ and $f \geqslant 0$, we have $0 \leqslant \arg(s + \mathrm{i}f) < \tfrac{1}{2}\pi$; $\arg K_1$ lies in the same interval. Hence, from (2.19), we obtain $-\tfrac{1}{2}\pi < \arg m < \tfrac{1}{4}\pi$; thus, $\mathrm{Re}\,m > 0$. It follows that $|(1 - m)/(1 + m)| < 1$ whence $\mathrm{Im}\,\mu_0 < 0$. Thus, all the roots of $D(\mu) = 0$ lie along a line parallel to, and below the real axis in the complex μ-plane. Let μ_0 be that root satisfying

$$-\tfrac{1}{4}\pi < \arg\mu_0 < 0 \quad \text{with} \quad |\mu_0| = \min. \tag{C 2}$$

Then, the corresponding singularities in the p-plane are at

$$p_n = (2kb^2)^{-1}\mathrm{e}^{-\mathrm{i}\pi/2}(\mu_0 + n\pi)^2, \quad n = 0, 1, 2, \dots;$$

note that $\mathrm{Re}\,p_n < 0$ and $\mathrm{Im}\,p_n < 0$, with $\mathrm{Re}\,p_n \uparrow 0$ and $\mathrm{Im}\,p_n \downarrow -\infty$ as $n \to \infty$.

Summarising, $\overline{A}_{\mathrm{w}}(p,Y)$ has simple poles at $p = p_n$ and a branch point at $p = 0$. Let us now use the inversion formula; this gives

$$A_{\mathrm{w}}(x,Y) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \overline{A}_{\mathrm{w}}(p,Y)\,\mathrm{e}^{px}\,\mathrm{d}p,$$

for $x > 0$, where $c > 0$. It follows that

$$A_{\mathrm{w}}(x,Y) = A_{\mathrm{res}}(x,Y) + A_{\mathrm{cut}}(x,Y),$$

where A_{cut} arises from deforming the Bromwich contour around the cut and A_{res} is the residue contribution from all the poles. We find that

$$A_{\text{res}}(x, Y) = \sum_{n=0}^{\infty} R_n \, \exp\{p_n x + iY\Omega\sqrt{\pi p_n}\},$$

where

$$R_n = \lim_{\mu \to \mu_0 + n\pi} \left\{ \frac{(\mu - \mu_0 - n\pi)i \sin \mu}{pD(\mu)} \right\} = \frac{im}{p_n(m^2 - 1)}.$$

Hence, we obtain (4.8). Wrapping the inversion contour around the cut, we obtain the formula (4.9) in a straightforward manner. We observe that

$$G_{\pm}(t) \sim -i(1 \pm m)^{-1} \exp\{\mp(1 + i)Y\sqrt{kt}\} \quad \text{as } t \to \infty,$$

and so the integral in (4.9) is certainly convergent for all $x > 0$.

References

Booij, N. 1981 Gravity waves on water with non-uniform depth and current. Report 81–1, Department of Civil Engineering, Delft University of Technology, The Netherlands.

Born, M. & Wolf, E. 1980 *Principles of optics*, 6th edn. Oxford: Pergamon Press

Dalrymple, R. A. 1992 Water wave propagation in jettied channels. In *Proc. 23rd Int. Coastal Engng Conf.*, pp. 3040–3053.

Dalrymple, R. A., Kirby, J. T. & Hwang, P. A. 1984 Wave diffraction due to areas of energy dissipation. *J. Waterway Port Coastal Ocean Engng* **110**, 67–79.

Dalrymple, R. A., Losada, M. A. & Martin, P. A. 1991 Reflection and transmission from porous structures under oblique wave attack. *J. Fluid Mech.* **224**, 625–644.

Dalrymple, R. A. & Martin, P. A. 1992 Perfect boundary conditions for parabolic water-wave models. *Proc. R. Soc. Lond.* A**437**, 41–54.

Gradshteyn, I. S. & Ryzhik, I. M. 1980 *Table of integrals, series and products*. New York: Academic Press.

Kirby, J. T. 1986 Higher order approximations in the parabolic equation method for water waves. *J. geophys. Res.* **91**, 933–952.

Kirby, J. T. & Dalrymple, R. A. 1983 A parabolic equation for the combined refraction–diffraction of Stokes waves by mildly varying topography. *J. Fluid Mech.* **136**, 453–466.

Kirby, J. T. & Dalrymple, R. A. 1986 Modeling waves in surfzones and around islands. *J. Waterway Port Coastal Ocean Engng* **112**, 78–93.

Mei, C. C. 1983 *The applied dynamics of ocean surface waves*. New York: Wiley.

Mei, C. C. & Tuck, E. O. 1980 Forward scattering by thin bodies. *SIAM Jl appl. Math.* **39**, 178–191.

Meister, E. 1987 Some solved and unsolved canonical problems of diffraction theory. *Lecture Notes in Mathematics*, vol. 1285, pp. 320–336. Berlin: Springer.

Meister, E., Penzel, F., Speck, F. O. & Teixeira, F. S. 1992 Two-media scattering problems in a half-space. In *Partial differential equations with real analysis* (ed. H. Begehr & A. Jeffrey), pp. 122–146. Harlow: Longman.

Melo, E. & Guza, R. T. 1991a Wave propagation in jettied entrance channels. I: Models. *J. Waterway Port Coastal Ocean Engng* **117**, 471–492.

Melo, E. & Guza, R. T. 1991b Wave propagation in jettied entrance channels. II: Observations. *J. Waterway Port Coastal Ocean Engng* **117**, 493–510.

Noble, B. 1988 *Methods based on the Wiener–Hopf technique*. New York: Chelsea.

Panchang, V. G., Cushman-Roisin, B. & Pearce, B. R. 1988 Combined refraction–diffraction of short-waves in large coastal regions. *Coastal Engng* **12**, 133–156.

Penney, W. G. & Price, A. T. 1952 The diffraction theory of sea waves and the shelter afforded by breakwaters. *Phil. Trans. R. Soc. Lond.* A**244**, 236–253.

Radder, A. C. 1979 On the parabolic equation method for water-wave propagation. *J. Fluid Mech.* **95**, 159–176.

Sollitt, C. K. & Cross, R. H. 1972 Wave transmission through permeable breakwaters. In *Proc. 13th Int. Coastal Engng Conf.*, pp. 1827–1846.

Stamnes, J. J. 1986 *Waves in focal regions.* Bristol: Adam Hilger.

Tsay, T.-K. & Liu, P. L.-F. 1982 Numerical solution of water-wave refraction and diffraction problems in the parabolic approximation. *J. geophys. Res.* **87**, 7932–7940.