On the derivation of boundary integral equations for scattering by an infinite one-dimensional rough surface

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A crucial ingredient in the formulation of boundary-value problems for acoustic scattering of time-harmonic waves is the radiation condition. This is well understood when the scatterer is a bounded obstacle. For plane-wave scattering by an infinite, rough, impenetrable surface S, the physics of the problem suggests that all scattered waves must travel away from (or along) the surface. This condition is used, together with Green's theorem and the free-space Green's function, to derive boundary integral equations over S. This requires careful consideration of certain integrals over a large semicircle of radius r; it is known that these integrals vanish as $r\rightarrow\infty$ if the scattered field satisfies the Sommerfeld radiation condition, but that is not the case here—reflected plane waves must be present. The integral equations obtained are Helmholtz integral equations; they must be modified for grazing incident waves. As such integral equations are often claimed to be exact, and are often used to generate benchmark numerical solutions, it seems worthwhile to establish their validity or otherwise. [S0001-4966(97)01406-9]

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INTRODUCTION

Consider the scattering of a plane time-harmonic acoustic wave by a bounded obstacle. To fix ideas here, we consider a two-dimensional obstacle, with a smooth, sound-hard surface S. Mathematically, this is an exterior Neu mann problem for the Helmholtz equation. In order to have a well-posed problem, we impose the Sommerfeld radiation condition,

$$\sqrt{r}\left(\frac{\partial u}{\partial r}-iku\right)\rightarrow 0 \quad \text{as } r\rightarrow\infty, \tag{1}$$

uniformly in all directions. Here, u is the scattered field, r is a plane polar coordinate, k is the wave number, and we have assumed a time dependence of $e^{-i\omega t}$. Physically, the radiation condition ensures that the scattered waves propagate outwards, away from the obstacle. Mathematically, the radiation condition also yields uniqueness and existence for the boundary-value problem.

A familiar method for solving the above problem is to derive a boundary integral equation for the boundary values of u on S. In the derivation, Green's theorem is applied to u and a fundamental solution G, in the region bounded internally by S and externally by C_r, a large circle of radius r. It turns out that the radiation condition implies that the integral

$$I(u;C_r)\equiv\int_{C_r}\left(u\frac{\partial G}{\partial r}-G\frac{\partial u}{\partial r}\right)ds\rightarrow 0 \quad \text{as } r\rightarrow\infty, \tag{2}$$

and so only boundary integrals over S remain. Thus, the radiation condition is a crucial ingredient for two results: a well-posed boundary-value problem; and the vanishing of a standard integral over a large circle. See Colton and Kress (1983) for more information.

For a sound-hard surface S, the procedure described above leads to the following boundary integral equation:

$$u(p)-\int_S u(q)\frac{\partial G}{\partial n_q}(p,q)\,ds_q=\int_S\frac{\partial u_{\text{inc}}}{\partial n_q}G(p,q)\,ds_q, \quad p\in S, \tag{3}$$

here, u_{inc} is the given incident wave. One can also derive an equation for the boundary values of the total field $u_{\text{tot}}=u_{\text{inc}}+u$; this boundary integral equation is

$$u_{\text{tot}}(p)-\int_S u_{\text{tot}}(q)\frac{\partial G}{\partial n_q}(p,q)\,ds_q=2u_{\text{inc}}(p), \quad p\in S. \tag{4}$$

We shall refer to (3) and (4) as *standard Helmholtz integral equations*. Similar equations can be derived for sound-soft surfaces (exterior Dirichlet problem, $u_{\text{tot}}=0$ on S).

Now, it is known that the waves scattered by a bounded two-dimensional obstacle have the form

$$u(r,\theta)=\frac{e^{ikr}}{\sqrt{r}}f(\theta)+O(r^{-3/2}) \quad \text{as } r\rightarrow\infty, \tag{5}$$

where (r,θ) are plane polar coordinates, and f is called the far-field pattern. Thus, apart from being outgoing (e^{ikr}), the waves decay with distance from the obstacle ($1/\sqrt{r}$). Indeed, the radiation condition implies that $u=O(r^{-1/2})$ as $r\rightarrow\infty$. As G also satisfies the radiation condition, we find that the integrand in (2) is

$$u\left(\frac{\partial G}{\partial r}-ikG\right)-G\left(\frac{\partial u}{\partial r}-iku\right)=o(r^{-1}) \quad \text{as } r\rightarrow\infty,$$

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whence $I(u;\mathscr{C})\to 0$ as $r\to\infty$, where $\mathscr{C}$ is any piece of C_r. This shows that the result (2) is due, essentially, to the decay of u (and G), not to any cancellation effects induced by the integration.

The description given above changes completely when the obstacle is *unbounded*. For example, suppose that S is an infinite, flat plane. Then, an incident plane wave will be scattered (reflected) specularly as a single propagating plane wave. More generally, suppose that S is an infinite corrugated surface; then, an incident plane wave will be scattered into a spectrum of plane waves. The specification of a ''radiation condition'' for such problems continues to attract attention (see, for example, Ramm, 1986); clearly, the Sommerfeld radiation condition is not appropriate, as it is not satisfied by a propagating plane wave. Nevertheless, it is customary to proceed, *assuming* that the scattered field can be represented in terms of plane waves, at least at some distance from S. Typically, this requires the discarding of an integral such as (2), but with the large circle C_r replaced by a large *semicircle* H_r. Can this step be justified? This paper began as an attempt to do this.

Another possible approach is to assume that u can be written as a surface distribution of sources or dipoles; see Sec. VI E. One might also invoke the limiting absorption principle, in which the wave number k is replaced by $k+i\varepsilon$, where ε is small and positive; the corresponding u is required to decay as $r\to\infty$. However, this is delicate (compared to scattering by a bounded obstacle) as the limits $\varepsilon\to 0$ and $z\to\infty$ for $\exp\{iz(k+i\varepsilon)\}$ do not commute.

The motivation behind the present work is the pervasive view that solving a boundary integral equation gives a rigorous, exact way (apart from numerical errors) of solving problems involving the scattering of plane waves by infinite rough surfaces. Indeed, one can find many books and papers setting out this view. (References to the literature will be given later.) However, very little attention has been given to the *derivation* of the boundary integral equations themselves, most writers being content to start by writing down a standard Helmholtz integral equation, (3) or (4). We will show that (3) is valid for plane-wave scattering by an infinite, one-dimensional, rough surface. We will also show that (4) is valid, except for grazing incident waves (in which case the right-hand side should be replaced by u_{inc}).

The paper is organized as follows. Section I is devoted to formulating the problem, with some discussion on radiation conditions and some background on angular-spectrum representations and integral representations (using G). Estimation of integrals over the semicircle H_r is carried out in Secs. II, III, and IV. Thus, the method of stationary phase and an expansion method are used in Secs. II and III, respectively, but only for a single plane wave. Results for $I(u;H_r)$ are obtained in Sec. IV, and are then used in Sec. V to derive various boundary integral equations of Helmholtz type. Extensive discussion of the results is given in Sec. VI. For example, it is shown that the standard Helmholtz integral equations are valid for a finite patch of roughness and for finite incident beams.

I. FORMULATION

Consider the scattering of a plane wave by an infinite rough surface, S. In this paper, we assume that the surface is one-dimensional, so that it can be described by

$$z=s(x), \quad -\infty<x<\infty$$

with $-h<s(x)\leqslant 0$ and some constant $h\geqslant 0$. The acoustic medium occupies $z>s$ and, for definiteness, we assume that S is a smooth, sound-hard surface. Thus we can write the total field as

$$u_{\mathrm{tot}}=u_{\mathrm{inc}}+u,$$

where u is the scattered field and

$$u_{\mathrm{inc}}(r,\theta)=e^{-ikr\cos(\theta+\theta_i)}, \quad |\theta_i|\leqslant \tfrac{1}{2}\pi, \tag{6}$$

is the incident plane wave; θ_i is the angle of incidence (it is the angle between the direction of propagation and the negative z axis), and (r,θ) are plane polar coordinates: $x=r\sin\theta$ and $z=r\cos\theta$. All the fields u_{tot}, u_{inc}, and u satisfy the Helmholtz equation

$$(\nabla^2+k^2)u=0, \tag{7}$$

for $z>s$. The boundary condition is

$$\frac{\partial u_{\mathrm{tot}}}{\partial n}=0 \quad \text{on } S, \tag{8}$$

where $\partial/\partial n$ denotes normal differentiation *out* of the acoustic medium.

A. Reflection by a flat surface

If S is flat $(s=0)$, the problem is trivial. Nevertheless, this problem can still teach us something. It is well known that the scattered field is

$$u(r,\theta)=e^{ikr\cos(\theta-\theta_i)} \quad \text{for } |\theta_i|<\tfrac{1}{2}\pi. \tag{9}$$

When $|\theta_i|=\frac{1}{2}\pi$ (''grazing incidence''), we have $u\equiv 0$: The incident wave satisfies the boundary condition on S.

Thus, for $|\theta_i|<\frac{1}{2}\pi$,

$$u_{\mathrm{tot}}=2\,e^{ikx\sin\theta_i}\cos(kz\cos\theta_i)$$

and

$$2\,e^{ikx\sin\theta_i}\cos(kz\cos\theta_i)+A_+e^{ikx}+A_-e^{-ikx}$$

both ''solve'' the problem, where A_+ and A_- are arbitrary constants. Of course, we disallow this second solution, unless $A_+=A_-=0$: but why? The answer is because of the radiation condition (which we have yet to specify). Physically, we want to exclude all ''incoming'' plane waves, apart from the incident wave. We will be more precise in Sec. I B.

B. Angular-spectrum representations

For any S, the scattered field above the corrugations, $z>0$, may be written using an angular-spectrum representation,

$$u(x,z)=\int_{-\infty}^{\infty}F(\mu)e^{ik(\mu x+mz)}\frac{d\mu}{m(\mu)} \tag{10}$$

$$=\int_{-\pi/2}^{\pi/2}A(\alpha)e^{ikr\cos(\theta-\alpha)}d\alpha$$

$$+\text{ evanescent terms.} \tag{11}$$

Here, $F(\mu)$ is the *spectral amplitude*, $A(\alpha)=F(\sin\alpha)$ and

$$m(\mu)=\begin{cases}\sqrt{1-\mu^2}, & |\mu|<1,\\ i\sqrt{\mu^2-1}, & |\mu|>1.\end{cases}$$

The integrals are superpositions of plane waves; they are propagating, homogeneous plane waves when $|\mu|<1$ and they are evanescent, inhomogeneous plane waves when $|\mu|>1$ In (11), we see the propagating plane waves explicitly: They propagate at an angle α to the positive z axis, with an (unknown) complex amplitude, $A(\alpha)$; the ''evanescent terms'' decay exponentially with z. For more information on angular-spectrum representations, see Clemmow (1966) and DeSanto and Martin (1996).

In general, the spectral amplitude must be considered as a generalized function, and not as a continuous or analytic function. This simple observation is motivated by known results for particular surfaces. Thus, for a flat surface we have

$$F(\mu)=\delta(\mu-\sin\theta_i)\cos\theta_i,\quad |\theta_i|<\tfrac{1}{2}\pi,$$

where δ is the Dirac delta function, whereas for a periodic surface F is a discrete sum of delta functions. So, we split the scattered field into three parts as

$$u=u_{\mathrm{pr}}+u_{\mathrm{ev}}+u_{\mathrm{con}}, \tag{12}$$

where

$$u_{\mathrm{pr}}(r,\theta)=\sum_{n=1}^{N}A_n v(r,\theta;\alpha_n),$$

$$u_{\mathrm{ev}}(r,\theta)=\sum_{m=1}^{M}B_m w(r,\theta;\mu_m), \tag{13}$$

$$u_{\mathrm{con}}(r,\theta)=\int_{-\infty}^{\infty}C(\mu)e^{ikr(\mu\sin\theta+m\cos\theta)}\frac{d\mu}{m(\mu)}, \tag{14}$$

$$v(r,\theta;\alpha)=e^{ikr\cos(\theta-\alpha)},\quad \text{with } |\alpha|\leqslant\tfrac{1}{2}\pi, \tag{15}$$

and

$$w(r,\theta;\mu)=e^{ikr\mu\sin\theta}e^{-kr\cos\theta\sqrt{\mu^2-1}}\quad \text{with } |\mu|>1. \tag{16}$$

The first term in (12) is a sum of propagating plane waves; the coefficients A_n and the angles α_n are unknown in general. The second term in (12) is a sum of evanescent waves; the coefficients B_m and μ_m are unknown in general. The third term in (12) is a continuous spectrum of plane waves; the unknown function C is continuous. Properties and consequences of the general representation (12) were investigated by DeSanto and Martin (1996).

Let us now return to the radiation condition. Having chosen an origin O, arbitrarily, we consider a large semicircle H_r, with radius r and center at O. We then require that all propagating plane-wave components $v(r,\theta;\alpha_n)$ in u propagate *outwards* through H_r, away from O. This is almost built into the decomposition (12): we have to be careful with grazing waves ($|\alpha_n|=\frac{1}{2}\pi$). For example, if $\alpha_n=\frac{1}{2}\pi$, $v=e^{ikx}$; this wave leaves the semicircle at $\theta=\frac{1}{2}\pi$ but enters at $\theta=-\frac{1}{2}\pi$. A simple way to impose our radiation condition without excluding grazing waves is to split the half-space $z>0$ and the semicircle H_r into two parts. Thus with

$$H_r^{\pm}=\{(r,\theta):0\leqslant\pm\theta\leqslant\tfrac{1}{2}\pi\} \tag{17}$$

being quarter-circles, we require that in the region $x\geqslant0$, $z>0$, we use $0\leqslant\alpha_n\leqslant\frac{1}{2}\pi$, so that all plane waves propagate out through H_r^{+}. Similarly, in the region $x\leqslant0$, $z>0$, we use $-\frac{1}{2}\pi\leqslant\alpha_n\leqslant0$, so that all plane waves propagate out through H_r^{-}. This form of the radiation condition will be used to derive boundary integral equations.

C. Boundary integral equations

One way to determine the scattered field is to derive a boundary integral equation over the rough surface S. The appropriate fundamental solution is

$$G(P,Q)=G(\mathbf{y},\mathbf{x})=-\tfrac{1}{2}iH_0^{(1)}(k|\mathbf{x}-\mathbf{y}|),$$

where $\mathbf{x}$ and $\mathbf{y}$ are the position vectors of Q and P, respectively, with respect to the origin O, and $H_n^{(1)}$ is a Hankel function. Apply Green's theorem to u and G in the region D_r with boundary $\partial D_r=H_r\cup S_r\cup T_r$, where H_r is a large semicircle of radius r and center O,

$$S_r=\{(x,z):z=s(x),-r<x<r\}$$

is a truncated rough surface, and T_r consists of two line segments joining the ends of H_r and S_r. Then, as both u and G satisfy the Helmholtz equation (7) in D_r [apart from the singularity in $G(P,Q)$ at $P=Q$], we obtain

$$2u(P)=\int_{\partial D_r}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)-\frac{\partial u}{\partial n_q}G(P,q)\right\}ds_q,$$

where $P\in D_r$, $q\in\partial D_r$ and $\partial/\partial n_q$ denotes normal differentiation at q. Use of the boundary condition (8) yields

$$2u(P)=\int_{S_r}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\mathrm{inc}}}{\partial n_q}G(P,q)\right\}ds_q$$

$$+I(u;H_r)+I(u;T_r), \tag{18}$$

where

$$I(u;\mathscr{S})=\int_{\mathscr{S}}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)-\frac{\partial u}{\partial n_q}G(P,q)\right\}ds_q$$

and normal differentiation is taken in a direction away from the origin [so that $\partial/\partial n=\partial/\partial r$ on H_r, consistent with (2)].

The scattered field u and its derivative $\partial u/\partial x$ are bounded in the neighborhood of S. This assumption together with simple bounds and the large-argument asymptotic behavior of Hankel functions show that $I(u;T_r)=O(r^{-1/2})$ as $r\to\infty$, whence

$$I(u;T_r)\to0\quad \text{as } r\to\infty. \tag{19}$$

Before estimating $I(u;H_r)$ for large r, using (12), we consider a single propagating plane-wave component in (12). Thus, we shall evaluate $I(v;H_r)$ as $r\to\infty$, where v is defined by (15). Indeed, we shall evaluate the limit using two different methods: the method of stationary phase (Sec. II) and an expansion method (Sec. III). We shall discuss the evaluation of $I(u;H_r)$ itself for large r in Sec. IV. Boundary integral equations will then be derived from (18) in Sec. V.

II. THE METHOD OF STATIONARY PHASE

We use the method of stationary phase to estimate $I(v;H_r)$. There are two cases, depending on the value of α, which can be smoothed together using a uniform approximation.

A. The method of stationary phase: $|\alpha|<\frac{1}{2}\pi$

We are interested in large values of $r=|\mathbf{x}|$ for fixed values of $\mathbf{y}$ and k. We have

$$G(P,q)\simeq\frac{B}{\sqrt{kr}}\exp\{ik(r-\mathbf{y}\cdot\hat{\mathbf{x}})\}$$

$$=\frac{B}{\sqrt{kr}}e^{ikr}e^{-ik\rho\cos(\theta-\varphi)}$$

as $r\to\infty$, where $\hat{\mathbf{x}}=\mathbf{x}/r$, $\mathbf{y}=(\rho\sin\varphi,\rho\cos\varphi)$ and

$$B=-\tfrac{1}{2}i\sqrt{(2/\pi)}e^{-i\pi/4}. \tag{20}$$

Hence, for large r,

$$v\frac{\partial G}{\partial r}-G\frac{\partial v}{\partial r}\simeq ik\frac{B}{\sqrt{kr}}[1-\cos(\theta-\alpha)]\times e^{ikr(1+\cos(\theta-\alpha))}e^{-ik\rho\cos(\theta-\varphi)}$$

and then

$$I(v;H_r)\simeq iBe^{ikr}L(kr), \tag{21}$$

where

$$L(\lambda)=\sqrt{\lambda}\int_{-\pi/2}^{\pi/2}g(\theta)e^{i\lambda F(\theta)}d\theta,$$

$$g(\theta)=[1-\cos(\theta-\alpha)]e^{-ik\rho\cos(\theta-\varphi)}$$

and

$$F(\theta)=\cos(\theta-\alpha).$$

For large $\lambda\equiv kr$, the dominant contribution to $L(\lambda)$ comes from those points c in the range of integration at which the phase F is stationary: $F'(c)=0$. As $F'(\theta)=\sin(\alpha-\theta)$ and $|\alpha|<\frac{1}{2}\pi$, the only stationary-phase point is at $\theta=\alpha$. Then (Bleistein and Handelsman, 1986, p. 220)

$$L(\lambda)\sim\mathscr{B}g(\alpha)e^{i\lambda F(\alpha)}\quad\text{as }\lambda\to\infty, \tag{22}$$

where

$$\mathscr{B}=\sqrt{\frac{2\pi}{|F''(\alpha)|}}\exp\left\{\frac{1}{4}i\pi\operatorname{sgn}F''(\alpha)\right\}=\sqrt{2\pi}e^{-i\pi/4}.$$

But $g(\alpha)=0$, and so $L(\lambda)=o(1)$ as $\lambda\to\infty$. In fact, an integration by parts gives

$$L(\lambda)=\frac{i}{\sqrt{\lambda}\cos\alpha}[(1-\sin\alpha)e^{i\Phi}+(1+\sin\alpha)e^{-i\Phi}]+O(\lambda^{-1})$$

as $\lambda\to\infty$, where $\Phi=\lambda\sin\alpha-k\rho\sin\varphi$. Hence, from (21), we obtain

$$I(v;H_r)=O((kr)^{-1/2})\quad\text{as }kr\to\infty,\text{ for }|\alpha|<\tfrac{1}{2}\pi. \tag{23}$$

B. The method of stationary phase: $|\alpha|=\frac{1}{2}\pi$

Suppose that $\alpha=\frac{1}{2}\pi$. In this case, $F(\theta)=\sin\theta$ is stationary at $\theta=\pm\frac{1}{2}\pi$, which are end points of the range of integration. We have

$$g(\tfrac{1}{2}\pi)=0\text{ and }g(-\tfrac{1}{2}\pi)=2e^{ik\rho\sin\varphi}.$$

It follows that

$$L(\lambda)\sim\sqrt{2\pi}e^{i\pi/4}e^{-i\lambda}e^{ik\rho\sin\varphi}\quad\text{as }\lambda\to\infty, \tag{24}$$

whence

$$I(v;H_r)=e^{ik\rho\sin\varphi}+O((kr)^{-1/2})\quad\text{as }kr\to\infty,\text{ for }\alpha=\tfrac{1}{2}\pi. \tag{25}$$

When $\alpha=-\frac{1}{2}\pi$, we obtain the same result except that φ is replaced by $-\varphi$. In this case, the relevant stationary-phase point is at $\theta=\frac{1}{2}\pi$.

When $\rho=0$, we can give an independent check of the result (24). In this case, we have

$$L(\lambda)=\sqrt{\lambda}\int_{-\pi/2}^{\pi/2}(1-\sin\theta)e^{i\lambda\sin\theta}d\theta$$

$$=\pi\sqrt{\lambda}\{J_0(\lambda)-iJ_1(\lambda)\},$$

where J_m is a Bessel function. The result follows from the well-known asymptotic approximation,

$$J_m(\lambda)\sim\sqrt{\frac{2}{\pi\lambda}}\cos\left(\lambda-\frac{1}{2}m\pi-\frac{1}{4}\pi\right)\quad\text{as }\lambda\to\infty. \tag{26}$$

C. Uniform asymptotics

We have seen that the results for $|\alpha|<\frac{1}{2}\pi$ and $|\alpha|=\frac{1}{2}\pi$ are different, that is, the asymptotic estimate of $I(v;H_r)$ is not uniform in the parameter α. However, we can obtain a uniform approximation (see Appendix A); for example, if α is near $\frac{1}{2}\pi$, we find that

$$I(v;H_r)\simeq\cos(\tfrac{1}{2}\delta)e^{ik\rho\sin\varphi}\operatorname{erfc}(\mu), \tag{27}$$

where

$$\operatorname{erfc}(\mu)=\frac{2}{\sqrt{\pi}}\int_{\mu}^{\infty}e^{-x^2}dx \tag{28}$$

is the complementary error function,

$$\mu=\sqrt{2\lambda}e^{-i\pi/4}\sin\tfrac{1}{2}\delta\text{ and }\delta=\tfrac{1}{2}\pi-\alpha.$$

Note that if $\alpha=\frac{1}{2}\pi$ ($\delta=0$), we recover (25). On the other hand, if $\alpha<\frac{1}{2}\pi$ ($\delta>0$), we recover (23), since $\mathrm{erfc}(\mu)\sim\pi^{-1/2}\mu^{-1}\exp(-\mu^2)$ as $\mu\to\infty$.

III. AN EXPANSION METHOD

The integral $I(v;H_r)$ is over a semicircle of radius r, between $\theta=-\frac{1}{2}\pi$ and $\theta=+\frac{1}{2}\pi$. We can evaluate this integral explicitly, using appropriate expansions of v and G. Thus

$$v(r,\theta;\alpha)=e^{ikr\cos(\theta-\alpha)}=\sum_{m=-\infty}^{\infty} i^m J_m(kr)e^{-im(\theta-\alpha)}, \tag{29}$$

henceforth, we suppress the limits when the summation is over all integers. Similarly,

$$G(P,q)=-\frac{1}{2}i\sum_n H_n(kr)J_n(k\rho)e^{in(\theta-\varphi)} \tag{30}$$

for $r>\rho$, where $H_n\equiv H_n^{(1)}$. Hence

$$I(v;H_r)=\sum_m\sum_n i^m J_n(k\rho)W_{mn}(kr)A_{mn}e^{i(m\alpha-n\varphi)},$$

where

$$W_{mn}(w)=-\tfrac{1}{2}i\pi w\{J_m(w)H_n'(w)-J_m'(w)H_n(w)\}$$

and

$$A_{mn}=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}e^{i(n-m)\theta}d\theta$$

$$=\begin{cases}1 & \text{if } m=n,\\ \dfrac{2(-1)^j}{(2j+1)\pi} & \text{if } m=n+2j+1,\\ 0 & \text{otherwise,}\end{cases}$$

here, j is an arbitrary integer. It follows that

$$I(v;H_r)=\sum_n i^n J_n(k\rho)\{W_{nn}(kr)-F_n(kr,\alpha)\}e^{in(\alpha-\varphi)},$$

where

$$F_n(kr,\alpha)=\frac{-2i}{\pi}\sum_j\frac{1}{2j+1}W_{n+2j+1,n}e^{i(2j+1)\alpha}. \tag{31}$$

We want to estimate $I(v;H_r)$ for large r. We can evaluate the first term in the braces exactly: W_{nn} is essentially a Wronskian, given by $W_{nn}=1$. For F_n, we have

$$W_{mn}(w)\sim\exp\{i(m-n)\pi/2\}\quad\text{as } w\to\infty,\text{ for fixed } m \text{ and } n. \tag{32}$$

We proceed formally, and use this approximation in (31). (This procedure can be justified; see Appendix B.) The result is

$$F_n(kr,\alpha)\sim\mathscr{G}(\alpha+\tfrac{1}{2}\pi)\quad\text{as } kr\to\infty,$$

independently of n, where

$$\mathscr{G}(\theta)=\frac{-2i}{\pi}\sum_j\frac{e^{i(2j+1)\theta}}{2j+1}=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin(2j+1)\theta}{2j+1}, \tag{33}$$

this is a familiar Fourier series:

$$\mathscr{G}(\theta)=\begin{cases}1, & 0<\theta<\pi,\\ -1, & -\pi<\theta<0,\\ 0, & \theta=0,\pm\pi,\end{cases} \tag{34}$$

and is defined by periodicity for other values of θ. Hence, for large kr, $F_n\sim1$ for $0\leqslant|\alpha|<\frac{1}{2}\pi$ but $F_n=o(1)$ for $|\alpha|=\frac{1}{2}\pi$. Thus we obtain the same (nonuniform) results as derived in Secs. II A and II B. The drawbacks with this method are that it does not yield results that are uniform in α for α near $\pm\frac{1}{2}\pi$, and it is very complicated to use for three-dimensional problems.

IV. ASYMPTOTIC BEHAVIOR OF $I(u;H_r)$

When a plane wave is reflected by a rough surface S, we can use the angular-spectrum representation (12) for the reflected field above the corrugations. Thus we have

$$I(u;H_r)=I(u_{\mathrm{pr}};H_r)+I(u_{\mathrm{ev}};H_r)+I(u_{\mathrm{con}};H_r).$$

For $I(u_{\mathrm{ev}};H_r)$, with u_{ev} defined by (13), we have

$$I(w;H_r)\simeq iB\sqrt{kr}e^{ikr}\int_{-\pi/2}^{\pi/2}g(\theta)e^{ikrF(\theta)}d\theta,$$

where w is defined by (16),

$$F(\theta)=\mu\sin\theta+i\sqrt{\mu^2-1}\cos\theta$$

and

$$g(\theta)=[1-F(\theta)]e^{-ik\rho\cos(\theta-\varphi)},$$

as $|\mu|>1$, integration by parts shows that

$$I(w;H_r)=O((kr)^{-1/2})\quad\text{as } kr\to\infty.$$

Hence, from (13),

$$I(u_{\mathrm{ev}};H_r)\to0\quad\text{as } r\to\infty.$$

For u_{con}, we have

$$u_{\mathrm{con}}(r,\theta)\sim\sqrt{\frac{2\pi}{kr}}e^{i(kr-\pi/4)}C(\sin\theta)\quad\text{as } kr\to\infty.$$

This result makes essential use of the continuity of $C(\mu)$ (see Clemmow, 1966, Sec. 3.2). Thus, u_{con} satisfies the Sommerfeld radiation condition (1), whence

$$I(u_{\mathrm{con}};H_r)\to0\quad\text{as } r\to\infty.$$

[A direct derivation of this result, based on (14) and the method of stationary phase, is given in Appendix C.]

Finally, consider $I(u_{\mathrm{pr}};H_r)$. If $|\alpha_n|<\frac{1}{2}\pi$, the results of the previous sections are immediately applicable, and show that $I(u_{\mathrm{pr}};H_r)\to0$ as $r\to\infty$. Next, consider grazing waves, $|\alpha_n|=\frac{1}{2}\pi$, and write

$$v_{\pm}=v(r,\theta;\pm\tfrac{1}{2}\pi)=e^{\pm ikx}.$$

We have

$$I(u_{\mathrm{pr}};H_r)=I(u_{\mathrm{pr}};H_r^+)+I(u_{\mathrm{pr}};H_r^-),$$

where $H_r^{\pm}$ are quarter-circles defined by (17); the radiation condition (all plane waves must propagate outwards through H_r, away from O) implies that we limit our attention to $I(v_{\pm};H_r^{\pm})$, as $v_{\pm}$ propagates inwards through $H_r^{\mp}$. So, from Sec. II A, we have

$$I(v_+;H_r^+)\simeq iBe^{i\lambda}\sqrt{\lambda}\int_0^{\pi/2}(1-\sin\theta)\times e^{-ik\rho\cos(\theta-\varphi)}e^{i\lambda\sin\theta}d\theta$$

for large $\lambda\equiv kr$. There is one point of stationary phase (cf. Sec. II B) at $\theta=\frac{1}{2}\pi$, but the integrand vanishes there whence $I(v_+;H_r^+)\to 0$ as $r\to\infty$. A similar argument succeeds for $I(v_-;H_r^-)$.

In summary, we find that our radiation condition ensures that

$$I(u;H_r)\to 0 \quad \text{as } r\to\infty. \tag{35}$$

V. BOUNDARY INTEGRAL EQUATIONS

In Sec. I, we used Green's theorem to obtain the integral representation

$$2u(P)=\int_{S_r}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\text{inc}}}{\partial n_q}G(P,q)\right\}ds_q + I(u;H_r)+I(u;T_r)$$

when $P\in D_r$, the region bounded by the semicircle H_r, the truncated rough surface S_r, and the two line segments T_r. Note that the left-hand side of this equation does not depend on r, so that the right-hand side of the equation must have a limit as $r\to\infty$. Taking this limit, using (19) and (35), we obtain

$$2u(P)=\int_S\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\text{inc}}}{\partial n_q}G(P,q)\right\}ds_q, \quad P\in D_\infty, \tag{36}$$

where D_∞ is the unbounded region $z>s$ and

$$\int_S=\lim_{r\to\infty}\int_{S_r},$$

which is the standard definition of a principal-value integral at infinity. In fact, the integral over S exists as an ordinary improper integral; to see this, we note that the integrand behaves like

$$U(x)e^{ik|x|}|x|^{-1/2} \quad \text{as } |x|\to\infty,$$

where $U(x)$ is only required to be bounded.

We remark that Beckmann and Spizzichino (1963, p. 180) and Ogilvy (1991, p. 75) discard $I(u;H_r)$ by assuming erroneously that $u=0$ on H_r.

Letting $P\to p\in S$ in (36) gives

$$u(p)-\int_S u(q)\frac{\partial G}{\partial n_q}(p,q)ds_q=\int_S\frac{\partial u_{\text{inc}}}{\partial n_q}G(p,q)ds_q, \quad p\in S. \tag{37}$$

This is (formally) the standard Helmholtz integral equation for the boundary values of u on S.

It is common to not work with (37) but with an integral equation for the total field, u_{tot}. To obtain such an equation, we start by defining a region $\widetilde{D}_r$ with boundary $\partial\widetilde{D}_r$. Given r, let $\widetilde{H}_r$ denote the semicircle in $z\leqslant 0$, with radius r and center O; then, $\widetilde{D}_r$ is the bounded region in $z<s$ enclosed by $\widetilde{H}_r$ and the rough surface. The boundary $\partial\widetilde{D}_r$ consists of a piece of S_r, namely S_t with $t(r)<r$, and a piece of $\widetilde{H}_r$, namely $\widetilde{H}_r\setminus\widetilde{T}_r$ where $\widetilde{T}_r$ consists of two circular arcs joining the ends of $\widetilde{H}_r$ and the ends of S_t. Now, apply Green's theorem to u_{inc} and G in $\widetilde{D}_r$. As both fields satisfy (7) in $\widetilde{D}_r$, the result is

$$0=\int_{S_t}\left\{u_{\text{inc}}(q)\frac{\partial G}{\partial n_q}(P,q)-\frac{\partial u_{\text{inc}}}{\partial n_q}G(P,q)\right\}ds_q - I(u_{\text{inc}};\widetilde{H}_r)+I(u_{\text{inc}};\widetilde{T}_r) \tag{38}$$

when $P\in D_r$, taking into account the direction of the normal vector on S. As before, simple bounds show that $I(u_{\text{inc}};\widetilde{T}_r)\to 0$ as $r\to\infty$. If $|\theta_i|<\frac{1}{2}\pi$, u_{inc} [given by (6)] is a plane wave propagating outwards through $\widetilde{H}_r$, whence

$$I(u_{\text{inc}};\widetilde{H}_r)\to 0 \quad \text{as } r\to\infty(|\theta_i|<\tfrac{1}{2}\pi).$$

For grazing incidence, we have

$$I(u_{\text{inc}};\widetilde{H}_r)=u_{\text{inc}}(P)(|\theta_i|=\tfrac{1}{2}\pi).$$

We combine these formulas and write

$$I(u_{\text{inc}};\widetilde{H}_r)\to\mathscr{U}_i(P) \quad \text{as } r\to\infty(|\theta_i|\leqslant\tfrac{1}{2}\pi). \tag{39}$$

Letting $r\to\infty$ in (38), and adding the result to (36), we obtain

$$2u(P)=\int_S u_{\text{tot}}(q)\frac{\partial G}{\partial n_q}(P,q)ds_q-\mathscr{U}_i(P), \quad P\in D_\infty. \tag{40}$$

Then, letting $P\to p\in S$ gives

$$u_{\text{tot}}(p)-\int_S u_{\text{tot}}(q)\frac{\partial G}{\partial n_q}(p,q)ds_q=2u_{\text{inc}}(p)-\mathscr{U}_i(p), \quad p\in S. \tag{41}$$

Now, the standard Helmholtz integral equation for the total field is

$$w(p)-\int_S w(q)\frac{\partial G}{\partial n_q}(p,q)ds_q=2u_{\text{inc}}(p), \quad p\in S. \tag{42}$$

Thus, for nongrazing incident waves $(\mathscr{U}_i=0)$, we see that u_{tot} does satisfy the standard Helmholtz integral equation. However, for grazing incident waves, u_{tot} does not satisfy the Helmholtz integral equation (42), but $2u_{\text{tot}}$ does.

VI. DISCUSSION

A. Previous work: Helmholtz integral equations

The idea that a boundary integral equation may be used to solve the problem of plane-wave scattering by an infinite rough surface is familiar. It is discussed in books on such problems; see for example Maystre and Dainty (1991), Ogilvy (1991, Secs. 4.1.1 and 6.3) and Voronovich (1994, Sec. 3.1). In particular, the standard Helmholtz integral equation (4) is equation (6.53) in Ogilvy's book and equation (3.1.37) in Voronovich's book.

Many recent authors refer to the paper by Holford (1981) on the scattering of a plane wave by a periodic sound-soft surface. He obtains the integral representation [his equation (20)]

$$u_{\mathrm{tot}}(P)=u_{\mathrm{inc}}(P)+\frac{1}{2}\int_S u_{\mathrm{tot}}(q)\frac{\partial G}{\partial n_q}(P,q)\,ds_q,\qquad P\in D_\infty, \tag{43}$$

as in Sec. I C, by applying Green's theorem to u_{tot} and G in the semicircular region D_r [using our notation and the boundary condition (8)]. He claims that "the term $u_{\mathrm{inc}}(P)$ is the contribution from the large semicircle" H_r as $r\to\infty$. He does not prove this statement and, moreover, it is not true for grazing incident waves. To see this, we note that applying Green's theorem to u_{tot} and G in D_r gives

$$2u_{\mathrm{tot}}(P)=\int_{S_r}u_{\mathrm{tot}}(q)\frac{\partial G}{\partial n_q}(P,q)\,ds_q+I(u_{\mathrm{tot}};H_r)+I(u_{\mathrm{tot}};T_r).$$

Now, $I(u_{\mathrm{tot}};T_r)\to 0$ as $r\to\infty$ and

$$I(u_{\mathrm{tot}};H_r)=I(u;H_r)+I(u_{\mathrm{inc}};H_r)=I(u;H_r)+I(u_{\mathrm{inc}};C_r)-I(u_{\mathrm{inc}};\widetilde{H}_r),$$

where $C_r=H_r\cup\widetilde{H}_r$ is a large circle. But, for an incident plane wave,

$$I(u_{\mathrm{inc}};C_r)=2u_{\mathrm{inc}},$$

exactly, whence

$$I(u_{\mathrm{tot}};H_r)\to 2u_{\mathrm{inc}}-\mathscr{U}_i\quad \text{as } r\to\infty,$$

where $\mathscr{U}_i$ is defined by (39). Thus (43) is correct whenever $\mathscr{U}_i\equiv 0$.

Holford himself refers to earlier papers by Urosovskii (1960), who in turn refers to Lysanov (1956). For more recent work, we can cite Thorsos (1988), Bishop and Smith (1992) and McSharry *et al.* (1995). All these papers start from the Helmholtz integral equation for u_{tot}, (4), or the analogous equation for a sound-soft surface. Moreover, all but one of these papers are concerned with plane-wave incidence, the exception being the paper by Thorsos (1988). He considers a 'tapered' plane wave; we will discuss beams of finite extent in Sec. VI D.

B. Far-field asymptotics

Care is needed when approximating the scattered field at large distances from an infinite surface. To illustrate this, consider the integral representation (40). Let $P\equiv(x,z)\in D_\infty$ and $q\equiv(\xi,s(\xi))\in S$ have position vectors $\mathbf{y}$ and $\mathbf{q}$, respectively. In the far field, $kR\gg 1$, where $R=|\mathbf{y}-\mathbf{q}|$, so that we can use the large-argument approximation for $H_1^{(1)}(kR)$. Thus, assuming for simplicity that $\mathscr{U}=\mathscr{U}_i=0$, (40) gives

$$2u(x,z)=\frac{ik}{2}\int_S u_{\mathrm{tot}}(q)\frac{H_1^{(1)}(kR)}{R}\mathbf{n}(q)\cdot(\mathbf{q}-\mathbf{y})\,ds_q \sim iB\sqrt{k}\int_S u_{\mathrm{tot}}(q)\frac{e^{ikR}}{R^{3/2}}\mathbf{n}(q)\cdot(\mathbf{q}-\mathbf{y})\,ds_q \tag{44}$$

as $kz\to\infty$, where B is defined by (20) and $\mathbf{n}(q)$ is the unit normal vector at q pointing out of D_∞.

If $u_{\mathrm{tot}}(q)$ has a compact support (so that it vanishes for $|\mathbf{q}|>L$, say), or if S is finite (bounded scatterer), we can make a second approximation:

$$R=\{r^2-2\mathbf{y}\cdot\mathbf{q}+q^2\}^{1/2}=r-\hat{\mathbf{y}}\cdot\mathbf{q}+O(q^2/r)\quad \text{as } r\to\infty, \tag{45}$$

where $r=|\mathbf{y}|$, $\hat{\mathbf{y}}=\mathbf{y}/r$, and $q=|\mathbf{q}|$. (The notation used here differs from that used in Secs. I C and II A.) To the same order, we can also replace $(\mathbf{q}-\mathbf{y})$ in (44) by $(-\mathbf{y})$. Hence, we find that u is given by (5), where the far-field pattern is

$$f(\theta)=-\frac{iB\sqrt{k}}{2}\int_S u_{\mathrm{tot}}(q)\mathbf{n}(q)\cdot\hat{\mathbf{y}}\exp\{-ik\hat{\mathbf{y}}\cdot\mathbf{q}\}\,ds_q.$$

However, one cannot justify the use of the approximation (45) for plane-wave incidence and unbounded surfaces (see the discussion by Ogilvy, 1991, p. 78). This is immediately clear, because there must be a reflected plane wave, whereas (45) leads to a cylindrical wave. For explicit confirmation, consider the reflection of a plane wave by a flat surface, so that $u_{\mathrm{tot}}(q)=2\exp\{ik\xi\sin\theta_i\}$; the integral in (44) can then be estimated using the method of stationary phase [and yields the correct u, given by (9)], whereas the integral for $f(\theta)$ diverges.

C. A finite patch of roughness

Suppose that the infinite surface S is flat, apart from a finite patch of roughness, S_{patch}, confined to $|x|<L$, say. A plane-wave incident on such a patch will generate a specular plane wave and a cylindrical wave. Thus, for nongrazing incidence, the standard Helmholtz integral equation for the total field, (4), is valid.

To see that the decomposition itself is valid, write

$$u_{\mathrm{tot}}=u_{\mathrm{flat}}+u_{\mathrm{cyl}},$$

where

$$u_{\mathrm{flat}}=e^{-ikr\cos(\theta+\theta_i)}+e^{ikr\cos(\theta-\theta_i)}\quad(|\theta_i|<\tfrac{1}{2}\pi)$$

is the total field for reflection by an infinite flat sound-hard surface. Thus

$$\frac{\partial u_{\mathrm{cyl}}}{\partial n}=-\frac{\partial u_{\mathrm{flat}}}{\partial n}\quad \text{on } S_{\mathrm{patch}}. \tag{46}$$

There are now three cases to consider, namely, ridges, grooves, and a combination thereof.

1. Ridges

Suppose that S_{patch} consists of a finite number of ridges (''bosses''), so that $s(x) \geqslant 0$. Then, the scattering problem is equivalent to the scattering of *two* plane waves, u_{flat}, by a finite bounded obstacle (a ''double-body'') with boundary $S_{\text{patch}} \cup S'_{\text{patch}}$, where S'_{patch} is the reflection of S_{patch} in the line $z=0$. Thus u_{cyl} is a cylindrical wave, satisfying the Sommerfeld radiation condition (1).

The use of bosses to model rough surfaces is well known (Ogilvy, 1991, Sec. 6.1). The use of images to treat scatterers near flat impenetrable boundaries is also well known; for a recent application, see Chao *et al.* (1996). Indeed, if we introduce the exact Green's function for the half-plane $z \geqslant 0$,

$$\begin{aligned} G^E(P,Q) &\equiv G^E(x,z;\xi,\zeta) \\ &= -\tfrac{1}{2} i \{ H_0^{(1)}(k\sqrt{(x-\xi)^2+(z-\zeta)^2}) \\ &\quad + H_0^{(1)}(k\sqrt{(x-\xi)^2+(z+\zeta)^2}) \}, \end{aligned}$$

we find that

$$\begin{aligned} 2u_{\text{cyl}}(P) = \int_{S_{\text{ridge}}} \bigg\{ & u_{\text{cyl}}(q) \frac{\partial G^E}{\partial n_q}(P,q) \\ & + \frac{\partial u_{\text{flat}}}{\partial n_q} G^E(P,q) \bigg\} ds_q, \quad P \in D_\infty, \end{aligned}$$

where S_{ridge} is the union of all the ridge surfaces ($S_{\text{patch}} \setminus S_{\text{ridge}}$ is part of $z=0$). Letting $P \to p \in S_{\text{ridge}}$ yields a boundary integral equation for $u_{\text{cyl}}(p)$.

Alternatively, we can write

$$u_{\text{cyl}}(P) = \int_{S_{\text{ridge}}} \nu_1(q) G^E(P,q) ds_q, \quad P \in D_\infty, \tag{47}$$

where the boundary condition (46) implies that the source density ν_1 solves a Fredholm integral equation of the second kind over S_{ridge}.

2. Grooves

Suppose that S_{patch} consists of a finite number of grooves, so that $s(x) \leqslant 0$. Then, $\partial u_{\text{cyl}}/\partial z$ is known, in principle, for all x on $z=0$: It is zero except across the mouth of each groove. As there is a finite number of grooves, it follows that u_{cyl} is a cylindrical wave; it has an angular-spectrum representation with a continuous spectral amplitude.

Let S_{mouth} be the union of all the groove mouths; it is part of $z=0$. We can write

$$u_{\text{cyl}}(P) = \int_{S_{\text{mouth}}} \nu_2(q) G^E(P,q) ds_q, \quad P \equiv (x,z) \text{ and } z>0. \tag{48}$$

To find the source density ν_2, we can apply Green's theorem inside each groove to u_{cyl} and G; we have the boundary condition (46) on the surface of each groove, and we have (transmission) conditions enforcing the continuity of u_{cyl} and $\partial u_{\text{cyl}}/\partial z$ across the mouth of each groove.

Note that we cannot use (48) inside the grooves because of the image singularities in G^E. This extra complication with grooves (compared to ridges) has given rise to more sophisticated methods for solving such problems, involving more complicated integral representations; see Willers (1987) and Asvestas and Kleinman (1994). Applications of G^E to rough-surface scattering were made by Berman and Perkins (1985) and by Shaw and Dougan (1995).

3. Ridges and grooves

From the discussion above, we see that if S_{patch} consists of a finite number of ridges and grooves, then u_{cyl} can be represented using

$$u_{\text{cyl}}(P) = \int_{S_+} \nu(q) G^E(P,q) ds_q,$$

where $S_+ = S_{\text{ridge}} \cup S_{\text{mouth}}$, $P \equiv (x,z)$ and $z > \max\{s(x),0\}$. The determination of ν on S_+ is complicated, although S_+ is a finite surface. If we use the Helmholtz integral equation for u_{tot}, which we know is legitimate, we have to solve an integral equation over an infinite surface. However, this can be reduced to an integral equation over S_{patch} as follows. Since $\partial G(p,q)/\partial n_q = 0$ when both p and q are on the flat part of S, $S_{\text{flat}} = S \setminus S_{\text{patch}}$, (4) gives

$$u_{\text{tot}}(p) = 2u_{\text{inc}}(p) + \int_{S_{\text{patch}}} u_{\text{tot}}(q) \frac{\partial G}{\partial n_q}(p,q) ds_q, \quad p \in S_{\text{flat}}.$$

This means that u_{tot} on the (infinite) flat part of S is known in terms of u_{tot} on the rough part of S. Hence, we can write (4) for $p \in S_{\text{patch}}$ as

$$u_{\text{tot}}(p) - \int_{S_{\text{patch}}} u_{\text{tot}}(q) K(p,q) ds_q = 2f(p), \quad p \in S_{\text{patch}},$$

where

$$K(p,q) = \frac{\partial G}{\partial n_q}(p,q) + \int_{S_{\text{flat}}} \frac{\partial G}{\partial n_l}(p,l) \frac{\partial G}{\partial n_q}(q,l) ds_l,$$

$$f(p) = u_{\text{inc}}(p) + \int_{S_{\text{flat}}} u_{\text{inc}}(q) \frac{\partial G}{\partial n_q}(p,q) ds_q.$$

D. Finite beams

So far we have taken the incident field to be a plane wave. However, for many applications, the incident field is a finite beam. To construct such a beam, we start by considering a single line-source at Q,

$$u_{\text{inc}}(P) = G(P,Q).$$

As u_{inc} satisfies the Sommerfeld radiation condition, an energy argument (DeSanto and Martin, 1996) shows that the scattered field cannot include any reflected plane waves. Thus, the standard Helmholtz integral equations, (3) and (4), are valid; see DeSanto and Brown (1986, Sec. 4.1).

Next, we distribute the line-sources over a finite curve $\mathcal{Q}$ (or a finite region), to give

$$u_{\text{inc}}(P) = \int_{\mathcal{Q}} \nu_{\text{inc}}(Q) G(P,Q) ds_Q,$$

where ν_{inc} is prescribed and can be adjusted to make u_{inc} beamlike. (If $\mathcal{Q}$ is far from S, the asymptotic approximations described in Sec. VI B can be used.) It follows that the standard Helmholtz integral equations remain valid.

Another way to generate a beam is to use an angular-spectrum representation,

$$u_{\mathrm{inc}}(r,\theta)=\int_{-\pi/2}^{\pi/2} A_{\mathrm{inc}}(\alpha)e^{ikr\cos(\theta+\alpha)}d\alpha, \qquad (49)$$

where A_{inc} is a prescribed continuous function, and is typically taken as a Gaussian (Saillard and Maystre, 1990). The standard Helmholtz integral equations are valid for incident fields of this type.

Thorsos (1988) uses a ''tapered'' plane wave. This incident field does not satisfy the Helmholtz equation, and so the derivation of the Helmholtz integral equation for u_{tot} fails. [The Helmholtz integral equation for u, (3), *is* valid without this qualification.] Nevertheless, as Thorsos points out, the tapered plane wave is an approximation to an actual wave field, constructed using (49); see Thorsos (1988, Sec. I B).

E. Other integral equations

An alternative way of solving scattering problems, touched on above, is to assume that the scattered field can be written as

$$u(P)=\int_S \gamma(q)G(P,q)ds_q, \quad P\in D,$$

where the source density γ is unknown. For sound-hard surfaces, the boundary condition (8) yields an integral equation for γ,

$$\gamma(p)-\int_S \gamma(q)\frac{\partial G}{\partial n_p}(p,q)ds_q=\frac{\partial u_{\mathrm{inc}}}{\partial n_p}, \quad p\in S.$$

For sound-soft surfaces, the corresponding integral equation is

$$\int_S \gamma(q)G(p,q)ds_q=-u_{\mathrm{inc}}(p), \quad p\in S;$$

this has been solved numerically by Lentz (1974), Rodríguez *et al.* (1992), and others.

Chandler-Wilde and Ross (1995, 1996) use a double-layer potential for sound-soft surfaces,

$$u(P)=\int_S \gamma(q)\frac{\partial G_1}{\partial n_q}(P,q)ds_q, \quad P\in D,$$

where G_1 satisfies an impedance condition on a line $z=-h_0$ ($h_0>h$).

All of the formulations mentioned in this section are *indirect*, in that they assume that $u(P)$ can be represented in a specified form. Thus the radiation condition is implicit in the representation.

VII. CONCLUSIONS

We have seen that the use of standard Helmholtz integral equations for the scattering of a plane wave by an infinite, sound-hard, one-dimensional, rough surface is justified in most circumstances. In particular, the equation for the boundary values of the scattered field is always valid, whereas the integral equation for the boundary values of the total field is valid for nongrazing incident waves (a simple modification is required for grazing incident waves). These results underpin the use of these integral equations for numerical computations.

The standard Helmholtz integral equations are valid if the roughness is confined to a finite portion of an otherwise flat but infinite surface. They are also valid for incident beams of finite width.

Similar results may be obtained for sound-soft surfaces. Extension to electromagnetic and elastodynamic problems, and to penetrable interfaces, should be straightforward.

Finally, extension of these ideas to two-dimensional rough surfaces can also be made, although the analysis is more difficult and the results are different. Some of these aspects are currently under investigation.

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APPENDIX A: UNIFORM ASYMPTOTICS

We derive a uniform approximation for $I(v;H_r)$, using a method discussed by Bleistein and Handelsman (1986, Sec 9.4). We start by focussing on the non-uniformity at $\alpha=\frac{1}{2}\pi$; the nonuniformity at $\alpha=-\frac{1}{2}\pi$ can be treated similarly. Write $L=L_1+L_2$ where

$$L_1(\lambda)=\sqrt{\lambda}\int_{-\pi/2}^{0} g(\theta)e^{i\lambda F(\theta)}d\theta$$

and $L_2=L-L_1$. We have $L_2=o(1)$ as $\lambda\to\infty$, uniformly in α, for α bounded away from $-\frac{1}{2}\pi$. For L_1 there is a point of stationary phase at $\theta=\alpha-\pi$ (outside the range of integration) which approaches the end point at $\theta=-\frac{1}{2}\pi$ as $\alpha\to\frac{1}{2}\pi$. Let us make a preliminary change of variables, mapping the end point to the origin: Put $\theta=x-\frac{1}{2}\pi$ and $\alpha=\frac{1}{2}\pi-\delta$ giving

$$L_1(\lambda)=\sqrt{\lambda}\int_0^{\pi/2} h(x)e^{i\lambda f(x;\delta)}dx$$

with

$$h(x)=[1+\cos(x+\delta)]e^{ik\rho\sin(\varphi-x)}$$

and

$$f(x;\delta)=-\cos(x+\delta).$$

Thus L_1 has a stationary-phase point at $x=-\delta$, which approaches the end-point $x=0$ as $\delta\to 0$. The prototype special function with this property is the complementary error function, defined by (28). In order to relate this function to L_1, we change the integration variable from x to t, using

$$f(x;\delta)-f(0;\delta)=\tfrac{1}{2}t^2+\gamma t,$$

requiring that $x=-\delta$ is mapped to $t=-\gamma$ gives $\gamma=2\times\sin\frac{1}{2}\delta$. Hence,

$$L_1(\lambda)=\sqrt{\lambda}e^{i\lambda f(0;\delta)}\int_0^T \tilde{h}(t)e^{i\lambda(1/2\,t^2+\gamma t)}dt,$$

where

$$\tilde{h}(t)=h(x(t))\frac{dx}{dt} \text{ and } \frac{dx}{dt}=\frac{t+\gamma}{f'(x;\delta)}.$$

For the dominant contribution, we can set the upper limit $T=\infty$ and replace $\tilde{h}(t)$ by $\tilde{h}(0)$; this gives

$$L_1(\lambda)\simeq\sqrt{\lambda}e^{i\lambda f(0;\delta)}\frac{\gamma h(0)}{f'(0;\delta)}\int_0^\infty e^{i\lambda[(1/2)\,t^2+\gamma t]}dt.$$

Standard manipulations show that the integral can be expressed as

$$\sqrt{\frac{\pi}{2\lambda}}e^{i\pi/4}e^{-1/2i\lambda\gamma^2}\operatorname{erfc}(\mu) \quad \text{with} \quad \mu=\gamma\sqrt{\frac{1}{2}\lambda}e^{-i\pi/4}.$$

Substituting back, we obtain

$$L_1(\lambda)\simeq\sqrt{2\pi}\cos\tfrac{1}{2}\delta e^{i\pi/4}e^{-i\lambda}e^{ik\rho\sin\varphi}\operatorname{erfc}(\mu)$$

whence the final result (27) follows.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF F_n

In order to estimate $F_n(kr,\alpha)$ for large kr, we substituted the asymptotic approximation (32) into (31). However, this requires some justification, as (32) presupposes that m and n are fixed. [A hint that nonuniform behavior might be expected comes from (29): The left-hand side is a plane wave whereas every term on the right-hand side is $O((kr)^{-1/2})$ as $kr\to\infty$.] We start by writing

$$F_n(w,\alpha)=w\{H_n'(w)\mathscr{F}_n(w,\alpha)-H_n(w)(\partial/\partial w)\mathscr{F}_n(w,\alpha)\},$$

where

$$\mathscr{F}_n(w,\alpha)=\sum_j\frac{1}{2j+1}J_{n+2j+1}(w)e^{i(2j+1)\alpha}. \tag{B1}$$

Next, we use a standard integral representation for Bessel functions,

$$J_m(w)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(m\theta-w\sin\theta)}d\theta,$$

whence

$$\mathscr{F}_n(w,\alpha)=\frac{i}{4}\int_{-\pi}^{\pi}\mathscr{G}(\theta+\alpha)e^{i(n\theta-w\sin\theta)}d\theta \tag{B2}$$

where $\mathscr{G}(\theta)$ is defined by (33). Finally, we use the method of stationary phase to estimate (B2) for large w; there are stationary-phase points at $\theta=\pm\frac{1}{2}\pi$, whence (22) gives

$$\mathscr{F}_n(w,\alpha)=\sqrt{(\tfrac{1}{2}\pi/w)}\sin(w-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+O(w^{-1})$$

as $w\to\infty$, for $0\leqslant|\alpha|<\frac{1}{2}\pi$, whereas

$$\mathscr{F}_n(w,\pm\tfrac{1}{2}\pi)=O(w^{-1}) \quad \text{as } w\to\infty.$$

These results are *exactly* the same as we would have obtained if we had replaced the Bessel function in (B1) by the leading term in its large-argument asymptotic expansion, namely (26).

APPENDIX C: ASYMPTOTIC BEHAVIOR OF $I(U_{\mathrm{con}};H_r)$

Disregarding the evanescent terms in (14), we can write u_{con} as the integral in (11), whence

$$I(u_{\mathrm{con}};H_r)=\int_{-\pi/2}^{\pi/2}A(\alpha)I(v;H_r)d\alpha\simeq iBe^{i\lambda}\sqrt{\lambda}\mathscr{D}(\lambda)$$

as $\lambda\equiv kr\to\infty$, where we have used (21), and

$$\begin{aligned}\mathscr{D}(\lambda)&=\int_{-\pi/2}^{\pi/2}A(\alpha)\int_{-\pi/2}^{\pi/2}[1-\cos(\theta-\alpha)]\\&\quad\times e^{-ik\rho\cos(\theta-\varphi)}e^{i\lambda\cos(\theta-\alpha)}d\theta d\alpha\\&=\int_{-\pi/2}^{\pi/2}A(\alpha)\int_{-\pi/2-\alpha}^{\pi/2-\alpha}(1-\cos\psi)\\&\quad\times e^{-ik\rho\cos(\psi+\alpha-\varphi)}e^{i\lambda\cos\psi}d\psi d\alpha.\end{aligned}$$

As A is a continuous function of α, we can change the order of integration to give

$$\mathscr{D}(\lambda)=\int_0^{\pi}(1-\cos\psi)Q(\psi)e^{i\lambda\cos\psi}d\psi, \tag{C1}$$

where

$$Q(\psi)=\int_{-\pi/2}^{\pi/2-\psi}\{A(\alpha)e^{-ik\rho\cos(\psi+\alpha-\varphi)}+A(-\alpha)e^{-ik\rho\cos(\psi+\alpha+\varphi)}\}d\alpha.$$

Now we can estimate $\mathscr{D}(\lambda)$ for large λ using the method of stationary phase. The stationary-phase points are $\psi=0$ and $\psi=\pi$. At $\psi=0$, $1-\cos\psi=0$, whereas $Q(\pi)=0$. Hence, the integrand in (C1) vanishes at both stationary-phase points, whence $\mathscr{D}(\lambda)=O(\lambda^{-1})$ as $\lambda\to\infty$. Thus, we deduce that $I(u_{\mathrm{con}};H_r)\to 0$ as $r\to\infty$.

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