INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng. **42**, 885–906 (1998)

SMOOTHNESS–RELAXATION STRATEGIES FOR SINGULAR AND HYPERSINGULAR INTEGRAL EQUATIONS

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ABSTRACT

Three stages are involved in the formulation of a typical direct boundary element method: derivation of an integral representation; taking a Limit To the Boundary (LTB) so as to obtain an integral equation; and discretization. We examine the second and third stages, focussing on strategies that are intended to permit the relaxation of standard smoothness assumptions. Two such strategies are indicated. The first is the introduction of various apparent or 'pseudo-LTBs'. The second is 'relaxed regularization', in which a regularized integral equation, derived rigorously under certain smoothness assumptions, is used when less smoothness is available. Both strategies are shown to be based on inconsistent reasoning. Nevertheless, reasons are offered for having some confidence in numerical results obtained with certain strategies. Our work is done in two physical contexts, namely two-dimensional potential theory (using functions of a complex variable) and three-dimensional elastostatics. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: boundary elements; Cauchy principal-value integrals; Hadamard finite part integrals; Hölder continuity; relaxed regularization

1. INTRODUCTION

In a recent paper,[1] the first two authors concluded (p. 702, summary-item (2)), that collocating 'at the junction between two standard conforming elements, with hypersingular integral equations, cannot be theoretically justified'. However, the third author has written several papers with Huang and Richardson[2–4] in which they do so collocate. In fact, they report good numerical computations (see also References 5 and 6), using regularized integral equations. In this paper, we shall attempt a constructive reconciliation between these reported good results and the theoretical stance reported in Reference 1.

First, we must reaffirm the work and statements in Reference 1 regarding theoretical smoothness requirements for existence of limits to the boundary (LTBs), which give rise to Cauchy-singular and hypersingular integral equations. So, where are the opportunities for relaxing these smoothness requirements? One possibility is to replace classical LTBs by something weaker, leading to various notions of 'pseudo-LTBs'. Another possibility is 'relaxed regularization', in which a regularized integral equation is derived rigorously using classical smoothness requirements, and then these

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CCC 0029–5981/98/050885–22$17.50
Received 3 June 1997
Revised 2 December 1997

requirements are relaxed. It turns out that these two possibilities are related. Moreover, they both require the use of some selective, even inconsistent, reasoning to obtain the final equations. Nevertheless, one can build a computational strategy on these equations using standard boundary elements; apparently, the effectiveness and reliability of this strategy can be considerable. Such success is notwithstanding the fact that classical smoothness demands for existence of relevant LTBs remain in place, and that these LTBs still do not exist without that smoothness.[1]

In this paper, we explore the above-mentioned computational strategy. We do this first in the simple context of Cauchy-singular and hypersingular integral equations derived from Cauchy's integral formula for analytic functions of a complex variable. These are closely related to two-dimensional Boundary-Value Problems (BVPs) for Laplace's equation. We use the complex Cauchy formula because both the Cauchy singularity and the hypersingularity at issue here appear, perhaps, in the cleanest, simplest, and most classical form. (This is not the case with the real-variable formula for potential theory; see Appendix I.) In Section 5, we consider comparable issues for the related but more complicated equations of linear elasticity in three dimensions. Modifications of our arguments for non-smooth boundaries are found in Appendix II.

Specifically, we review various integral representations in Section 2, and associated LTBs. In particular, we consider regularized formulations; these involve improper integrals only, provided the classical smoothness conditions are satisfied. Two related strategies for relaxing these conditions are then studied, namely pseudo-LTBs (Section 3) and 'relaxed regularization' (Section 4). Numerical aspects of these strategies are also discussed.

Throughout, we try to be as clear as possible regarding what smoothness demands are made on functions, and why they are needed. We also try to clarify what relaxation of these demands may be made, for whatever reason, and what the consequences of such relaxation might be. In the process, we pay particular attention to any departures from correct and consistent reasoning that might be used with various smoothness-relaxation strategies. Our goal is sufficient clarification of theoretical issues so that no doubt about them can remain. At the same time, we wish to emphasise that what exists, and/or might be true, or dictated on rigorous analytical grounds, is not necessarily the same thing as what might be possible or convenient in numerical computations with clever modifications, despite some analytical inconsistencies. Indeed, although doubt about what could happen numerically may exist, the evidence in References 2–4 suggests that one can have considerable confidence in the numbers obtained in this way.

2. SOME MODEL PROBLEMS: REGULARIZATION

2.1. Cauchy's integral formula

Let D be a bounded, simply connected, plane region with smooth boundary S. (Non-smooth boundaries are considered in Appendix II.) Suppose that $f(z)$ is an analytic function of the complex variable $z = x + \mathrm{i}y$ in D, and that $f(z)$ is continuous in $\overline{D} \equiv D \cup S$. (These conditions are sufficient for the validity of Cauchy's theorem.) Then, Cauchy's integral formula gives

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_S \frac{f(w)}{w - z}\,\mathrm{d}w, \quad z \in D \tag{1}$$

where S is traversed in the positive (anti-clockwise) sense. The connection between (1) and potential theory (Laplace's equation) is discussed in Appendix I. For present purposes, regard (1) as a representation integral for $f(z)$ in D in terms of its boundary values $f(w)$ on S. The question of how much of $f(w)$ may be prescribed in a well-posed BVP for $f(z)$ plays no role in this section and in Section 3. The focus here is on existence of LTBs of various representations for $f(z)$ and its first derivative. The matter of knowns and unknowns in (possibly discretized) integral equations is considered later.

Choose a point $Z \in S$. If we assume that f is Hölder-continuous at Z, we can let $z \to Z$ (LTB) in (1) to give

$$f(Z) = \frac{1}{\pi \mathrm{i}} \int_S \frac{f(w)}{w - Z}\, \mathrm{d}w, \quad Z \in S \tag{2}$$

where the integral must be interpreted as a Cauchy principal-value (CPV) integral (defined by (26) below) and we have used the jump conditions (Sokhotski–Plemelj formulas) for Cauchy integrals.

It should be noted, before going further, that the terminology LTB, as just used, may have two interpretations: it could mean

(i) the LTB as z goes to a single point Z only, or
(ii) the LTB as z goes to all points Z on S.

In sense (ii), if the LTB exists, that is it exists at all points Z, we call the result a 'boundary integral identity', or a 'boundary integral equation' (BIE). On the other hand, in sense (i) there is meaning and interest in whether a LTB does or does not exist, for a particular limit point Z, without reference to other points. Existence of such a LTB usually depends on the smoothness of f at Z.

A quite important issue arises then which involves the concept of a limit expression, like (2), for which the range of admissable Z on S may be restricted to exclude isolated points, Z_k, say. Such restriction may be made because LTBs in sense (i) may not exist at such Z_k, but the so-restricted limit-expressions are useful BIEs, nonetheless. Indeed, suppose we have a representation integral like (1) for which LTBs do not exist (for any reason) at a finite number of isolated points Z_k. Then, formulas like (2) obtained in a LTB, but with excluded Z_k as limit (collocation) points, form the basis for the familiar and well-understood boundary element methods used so confidently for more than three decades.

On the other hand, there is the strong desire nowadays to collocate at Z_k where well-defined LTBs do not exist. Finding ways to quantify and justify such collocation, if possible, for a variety of BIEs, is the motivation for much of what follows.

In the remainder of Section 2, we use the term LTB in the sense (ii), whereas in Section 3 we use the term LTB primarily, but not exclusively, in sense (i).

To continue, return to (1) and write it as

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_S \frac{f(w) - f(Z)}{w - z}\, \mathrm{d}w + f(Z), \quad z \in D \tag{3}$$

where $f(Z)$ is defined (as f is continuous on S) and we have used

$$\frac{1}{2\pi \mathrm{i}} \int_S \frac{1}{w - z}\, \mathrm{d}w = 1, \quad z \in D \tag{4}$$

which is obtained by taking $f = 1$ in (1). In order to take the LTB, $z \to Z$, we need more than mere continuity of f on S. First, as $f(z)$ is continuous for $z \in \overline{D}$, we have $f(z) \to f(Z)$ as

$z \to Z$ on the left-hand side of (3). Then, we deduce that

$$0 = \int_S \frac{f(w) - f(Z)}{w - Z} \, \mathrm{d}w, \quad Z \in S \tag{5}$$

provided the integral (5) exists: it will exist as an ordinary improper integral, in general, only if f satisfies the classical Hölder condition at Z. This condition also implies that the contour integral in (3) is continuous (no jumps) as z crosses S.

Equation (5) can be deduced directly from (1), of course. But the derivation here is simpler, because there are no jump conditions to worry about—the Cauchy-type (simple pole) singularity in (1) has been *regularized* in (3).

2.2. Generalizations for the first derivative

The derivations above generalize in various ways. For example, let us start with Cauchy's integral formula for the derivative of f:

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \int_S \frac{f(w)}{(w - z)^2} \, \mathrm{d}w, \quad z \in D \tag{6}$$

We can let $z \to Z$ in (6), assuming that $f'(z)$ is continuous in $\overline{D}$ and that f' is Hölder-continuous at Z. The result is

$$f'(Z) = \frac{1}{2\pi \mathrm{i}} \ \not\!\!\int_S \frac{f(w)}{(w - Z)^2} \, \mathrm{d}w, \quad Z \in S \tag{7}$$

where the integral must be interpreted as a Hadamard finite-part integral.

Alternatively, if we take $f(z) = a + c(z - b)$ in (6), where a, b and c are constants, we obtain

$$c = \frac{1}{2\pi \mathrm{i}} \int_S \frac{a + c(w - b)}{(w - z)^2} \, \mathrm{d}w, \quad z \in D$$

Choose $a = f(Z)$, $b = Z$ and $c = f'(Z)$, and subtract the result from (6) to give

$$f'(z) - f'(Z) = \frac{1}{2\pi \mathrm{i}} \int_S \frac{f(w) - f(Z) - (w - Z) f'(Z)}{(w - z)^2} \, \mathrm{d}w, \quad z \in D \tag{8}$$

Assume that $f'(z)$ is continuous in $\overline{D}$ and that f' is Hölder-continuous at Z, as before. Then, the two-term Taylor-series subtraction in the numerator ensures that the integrand has been regularized: we can let $z \to Z$ to give

$$0 = \int_S \frac{f(w) - f(Z) - (w - Z) f'(Z)}{(w - Z)^2} \, \mathrm{d}w, \quad Z \in S \tag{9}$$

The use of linear solutions to regularize the hypersingular integral equations of potential theory has been described by Rudolphi;[7] see Tanaka *et al.*[8] for a review.

The two formulas, (7) and (9), require the same smoothness conditions on $f(Z)$. This conclusion is consistent with those in Reference 1. However, in the next section, we consider other points of view.

2.3. *Smoothness-relaxation strategies*

Consider the regularized equation (9), which is derived by assuming that f' is Hölder-continuous on S. We consider two approaches to relaxing this smoothness condition, with numerical implementation in mind. In the first, several strategies for deriving LTBs are studied (Section 3). It is shown that what might be called 'pseudo-LTBs' can be defined under weaker smoothness conditions; they are not genuine LTBs.

In the second approach, we start from the regularized equation (9) (derived rigorously, via a valid LTB), and *then* we relax the smoothness condition under which it was derived. This strategy is called 'relaxed regularization';[4] it is described in Section 4.

It turns out that (in some cases) the final equations obtained by the above two approaches (pseudo-LTBs and relaxed regularization) are essentially the same. If one is prepared to accept these equations, the remaining issues concern their numerical treatment; these issues are also addressed in Section 4.

3. SOME MODEL PROBLEMS: PSEUDO-LTBS

In Section 2.1, we saw that $f(w)$ had to be Hölder continuous if the Cauchy integral on the right-hand side of (1) was to have a LTB as $z \to Z$. Moreover, this limiting value is seen to be $f(Z)$. Suppose now that we replace $f(w)$ by $g(w)$, where $g(w)$ is discontinuous at one point $Z \in S$ (and possibly at other points). Using g, we can define a new function h by

$$h(z) = \frac{1}{2\pi i} \int_S \frac{g(w)}{w - z} \, dw, \quad z \in D \tag{10}$$

If $g(w)$ approximates $f(w)$ for $w \in S$, in some sense, we can expect that $h(z)$ will approximate $f(z)$ for $z \in D$. With this as background, we shall consider several strategies for obtaining LTBs, given that g is not continuous. In fact, these are all 'pseudo-LTBs', not genuine LTBs, and, when they give a finite numerical value, it is because of some logical inconsistency.

Let us suppose, for simplicity, that $g(w)$ is Hölder-continuous for all $w \in S$, except at one point Z where g can have a discontinuity. To fix ideas, suppose that $S = S_1 \cup S_2$, where S_1 and S_2 are two pieces of S, joined together at Z and Z_0, say. By definition,

$$g(Z+) = \lim_{w \to Z} g(w) \quad \text{with } w \in S_1 \quad \text{and} \quad g(Z-) = \lim_{w \to Z} g(w) \quad \text{with } w \in S_2$$

Then, the discontinuity in g at Z is $g(Z+) - g(Z-)$. We also write $g_j(w)$ to mean $g(w)$ when $w \in S_j$, $j = 1, 2$.

It is clear that the LTB in (10) as $z \to Z$ does not exist, as a CPV or otherwise, since g is discontinuous at Z. In fact, $h(z)$ is logarithmically singular as $z \to Z$; this is a classical result, Reference 9, Section 33.

Let us now consider several plausible strategies for obtaining LTBs.

1. Split the integral in (10) into two parts, giving

$$h(z) = \frac{1}{2\pi i} \int_{S_1} \frac{g_1(w)}{w - z} \, dw + \frac{1}{2\pi i} \int_{S_2} \frac{g_2(w)}{w - z} \, dw, \quad z \in D \tag{11}$$

Now consider a possible LTB of (11) as $z \to Z$. It is known that separate LTBs for each integral in (11) do not exist: each integral is logarithmically singular as $z \to Z$, Reference 9, Equation (29.4).

2. Next, write

$$h(z) = \frac{1}{2\pi \mathrm{i}} \int_{S_1} \frac{g_1(w) - g(Z+)}{w - z} \, \mathrm{d}w + \frac{1}{2\pi \mathrm{i}} \int_{S_2} \frac{g_2(w) - g(Z-)}{w - z} \, \mathrm{d}w + \frac{g(Z+)}{2\pi \mathrm{i}} \int_{S_1} \frac{\mathrm{d}w}{w - z} + \frac{g(Z-)}{2\pi \mathrm{i}} \int_{S_2} \frac{\mathrm{d}w}{w - z}, \quad z \in D \tag{12}$$

Again, the LTBs of the last two integrals do not exist for the same reason as in (11), namely, the known singular behaviour of Cauchy integrals near the end points of their integration contours. However, the first two integrals are well behaved if g_1 and g_2 satisfy one-sided Hölder conditions on each side of Z; this means that

$$|g_1(w) - g(Z+)| \leqslant A|w - Z|^{\alpha} \quad \text{for all } w \in S_1 \tag{13}$$

$$|g_2(w) - g(Z-)| \leqslant B|w - Z|^{\beta} \quad \text{for all } w \in S_2 \tag{14}$$

where $A > 0$, $B > 0$, $0 < \alpha \leqslant 1$ and $0 < \beta \leqslant 1$. If these conditions hold, the LTBs of each of the first two integrals in (12) exist, regardless of the values of $g(Z+)$ and $g(Z-)$, and, moreover, they exist separately! Despite this, a LTB of the *entire* right-hand side of (12) does not exist at Z.

3. For a third approach, return to (10) and write it as

$$h(z) = \frac{1}{2\pi \mathrm{i}} \int_S \frac{g(w) - a}{w - z} \, \mathrm{d}w + a, \quad z \in D$$

where a is an arbitrary constant and we have used (4). Use this formula twice, once with $a = g(Z+)$ and once with $a = g(Z-)$, and then add them together to give

$$2h(z) = \frac{1}{2\pi \mathrm{i}} \int_S \frac{g(w) - g(Z+)}{w - z} \, \mathrm{d}w + \frac{1}{2\pi \mathrm{i}} \int_S \frac{g(w) - g(Z-)}{w - z} \, \mathrm{d}w + [g(Z+) + g(Z-)], \quad z \in D \tag{15}$$

where we note that both integrals are taken over all of S, despite the assumed discontinuity at Z. This formula looks attractive because one might argue that the last term could be replaced by $2f(Z)$. However, the two integrals in (15) will behave badly as $z \to Z$, unlike the first two integrals in (12). To see this, separate each of the two integrals in (15) into two, using $S = S_1 \cup S_2$ as in (12). Then, one of the integrands satisfies (13) and one satisfies (14); these two integrals are finite in the LTB. The other two integrals are unbounded as $z \to Z$.

4. For a fourth attempt at a finite LTB, consider the following strategy. Suppose that we are willing to make an assumption about the values $g(Z\pm)$, as they appear as multipliers of the third and fourth integrals in (12), namely that they are equal:

$$g(Z+) = g(Z-) = g_{\mathrm{a}} \tag{16}$$

say, where g_{a} might be the average of $g(Z+)$ and $g(Z-)$. However, we make no such assumption in the first two integrals in (12), so that $g(Z+)$ and $g(Z-)$ are to retain their different values, in general, in those two integrals. Under these albeit inconsistent assumptions, the right-hand side of (12) becomes

$$\frac{1}{2\pi i}\int_{S_1}\frac{g_1(w)-g(Z+)}{w-z}\,dw+\frac{1}{2\pi i}\int_{S_2}\frac{g_2(w)-g(Z-)}{w-z}\,dw+g_a,\quad z\in D \tag{17}$$

This expression is no longer equal to $h(z)$ (defined by the right-hand side of (10)). However, it does have a LTB (assuming that (13) and (14) hold): simply replace z by Z. But how this LTB is related to $f(Z)$ is quite unclear and needs further examination.

5. Perhaps the most useful of the possible LTBs, and its associated integral equation, comes by making the following two assumptions:

   (a) expression (17) is a formula for $f(z)$; and (b) $g_a = f(Z)$.

   With these assumptions, we obtain

$$0=\int_{S_1}\frac{g_1(w)-g(Z+)}{w-Z}\,dw+\int_{S_2}\frac{g_2(w)-g(Z-)}{w-Z}\,dw \tag{18}$$

   wherein $g(Z+)$ need not equal $g(Z-)$ once (18) is obtained as described! As noted previously, finite numbers may be obtained from (18) so long as $g(w)$ meets the conditions (13) and (14). However, (18) does not have meaning as a well-defined LTB like (5) does for $f(z)$. For (5) to be the LTB of (1), f must be Hölder-continuous on S.

Now, consider (18). This equation can be obtained in a different way, using 'relaxed regularization' (see Section 4); informally, this means 'assume sufficient smoothness, derive an integral equation (such as (5)) via a valid LTB (in the sense (ii) of Section 2.1) with no free terms, and then relax the smoothness requirements on f at selected points such as Z, requirements that are needed for a valid LTB at Z (sense (i))'. Regardless of its genesis, the result is something which looks like an integral equation derived via consistent reasoning, when, in fact, this is not the case. The process of relaxing the smoothness seems innocent enough, although it affects the numerical value (but not the finiteness of the value) of the weakly singular integrals which remain. However, there is the serious question now of how well (18) maintains contact with an underlying BVP. There is no doubt about (5) in this regard. These issues will be discussed later.

As a warning, though, note that the integral in (5) is zero for any Hölder continuous f, whereas the integrals in (18) do not necessarily sum to zero if $g(Z+)\neq g(Z-)$. In essence, the zero on the left-hand side of (18) depends on one set of assumptions about g, and the right-hand side depends on another set.

Contrast this inconsistency in (18) with an expression valid for discontinuous g arising from the following argument. Return to (10) and consider the LTB as z goes to any point $Z^*\in S$ except the point $Z^*=Z$ (at which we admit a discontinuity in g as before). The result is

$$h(Z^*)=\frac{1}{2}g(Z^*)+\frac{1}{2\pi i}\fint_S\frac{g(w)}{w-Z^*}\,dw,\quad Z^*\neq Z$$

Now, split the integral in two, and write it in a form similar to (12), giving

$$\begin{aligned}h(Z^*)=\ &\frac{1}{2}g(Z^*)+\frac{1}{2}\pi i\fint_{S_1}\frac{g_1(w)-g(Z+)}{w-Z^*}\,dw+\frac{1}{2\pi i}\int_{S_2}\frac{g_2(w)-g(Z-)}{w-Z^*}\,dw\\&+\frac{g(Z+)}{2\pi i}\fint_{S_1}\frac{dw}{w-Z^*}+\frac{g(Z-)}{2\pi i}\fint_{S_2}\frac{dw}{w-Z^*},\quad Z^*\in S^*\end{aligned} \tag{19}$$

wherein $S^* = S \backslash \{Z, Z_0\}$ and we have assumed that $Z^* \in S_1$, without loss of generality. (The first CPV integral can be regularized, if desired; the second is explicit.) We observe that, unlike (18), (19) is an exact equality, regardless of the discontinuity in g at Z, and (19) holds for all Z^*, except the isolated points $Z^* = Z$ and $Z^* = Z_0$. Thus (19) as a BIE (defined on S^*) would be a perfectly unambiguous vehicle for solving a BVP wherein a discontinuity in g were (a) important input data, or (b) an important aspect of the sought-for solution. With (19), we have the means to capture and maintain a discontinuity in g at Z, if desired. With (18), according to our own experience, and that reported in References 3 and 4, we fear that this is not so. In any case, (19) is a familiar BIE—the kind which is well understood and has been used with confidence for decades. Equation (18), and similar equations arising from 'relaxed regularization', are new by comparison.

3.1. Generalizations for the first derivative

The discussion above extends to integrals for $f'(z)$, as described in Section 2.2. Thus, we differentiate (10) and consider

$$h'(z) = \frac{1}{2\pi i} \int_S \frac{g(w)}{(w-z)^2} \, dw, \quad z \in D \tag{20}$$

Expanding (20) into a form similar to (12) gives

$$\begin{aligned} h'(z) = &\frac{1}{2\pi i} \int_{S_1} \frac{g_1(w) - g(Z+) - (w-Z)g'(Z+)}{(w-z)^2} \, dw \\ &+ \frac{1}{2\pi i} \int_{S_2} \frac{g_2(w) - g(Z-) - (w-Z)g'(Z-)}{(w-z)^2} \, dw \\ &+ \frac{g'(Z+)}{2\pi i} \int_{S_1} \frac{w-Z}{(w-z)^2} \, dw + \frac{g'(Z-)}{2\pi i} \int_{S_2} \frac{w-Z}{(w-z)^2} \, dw \\ &+ \frac{g(Z+)}{2\pi i} \int_{S_1} \frac{dw}{(w-z)^2} + \frac{g(Z-)}{2\pi i} \int_{S_2} \frac{dw}{(w-z)^2}, \quad z \in D \end{aligned} \tag{21}$$

Let us assume, as before, that (16) holds for the fifth and sixth terms; they can then be combined into

$$\frac{g_a}{2\pi i} \int_S \frac{dw}{(w-z)^2} = 0 \quad \text{for all } z \in D$$

using the calculus of residues. Similarly, if we assume that

$$g'(Z+) = g'(Z-) = g'_a \tag{22}$$

say, but only for the multipliers of the third and fourth integrals, they combine into

$$\frac{g'_a}{2\pi i} \int_S \frac{w-Z}{(w-z)^2} \, dw = g'_a \quad \text{for all } z \in D$$

Finally, if we assume that the right-hand side of (21) gives a formula for $f'(z)$ and that $g'_a = f'(Z)$, the LTB of (21) results in an equation similar to (9), namely

$$0 = \int_{S_1} \frac{g_1(w) - g(Z+) - (w-Z)g'(Z+)}{(w-Z)^2} \, dw + \int_{S_2} \frac{g_2(w) - g(Z-) - (w-Z)g'(Z-)}{(w-Z)^2} \, dw \tag{23}$$

In this formula, g and g' are allowed to be different on each side of Z in the weakly singular integrals in (23), now that the troublesome (free) terms have been discarded. This was brought about by the selective and inconsistent assumptions made in the various terms in (21).

Again, an equation like (23) can be obtained using 'relaxed regularization'; see (35) below. However, proceeding from (21) (and from (12) for $h(z)$), the inconsistencies in the selective use of assumptions, whereby all infinities in the LTBs are avoided, are more readily observed. Also, via (21), one is reminded that unique, well-defined LTBs at Z without adequate smoothness of g and g' simply do not exist. In any case, (23) does not have meaning via a well-defined LTB, whereas (9) does for $f'(z)$ expressed by (6). As was the case with (5) for continuous f versus (18) for discontinuous g, what to expect numerically from (9) for continuous f' as opposed to (23) for discontinuous g' is uncertain. This too is discussed further below. Again, as a warning, the zero on the left-hand side of (23) depends on one set of assumptions about g, and the right-hand side is based on another set.

As before, if we were really interested in modelling discontinuous g', in a consistent unambiguous fashion, it is possible to return to (20) and proceed as was done above in deriving (19). The resulting expression is

$$\begin{aligned} h'(Z^*) = {} & \frac{1}{2\pi \mathrm{i}} \displaystyle\not\int_{S_1} \frac{g_1(w) - g(Z+) - (w-Z)g'(Z+)}{(w-Z^*)^2}\,\mathrm{d}w \\ & + \frac{1}{2\pi \mathrm{i}} \int_{S_2} \frac{g_2(w) - g(Z-) - (w-Z)g'(Z-)}{(w-Z^*)^2}\,\mathrm{d}w \\ & + \frac{1}{2\pi \mathrm{i}} \{ g'(Z+)\mathscr{C}_1(Z^*) + g'(Z-)\mathscr{C}_2(Z^*) + g(Z+)\mathscr{C}_3(Z^*) + g(Z-)\mathscr{C}_4(Z^*) \} \end{aligned} \tag{24}$$

for $Z^* \in S^*$, where $\mathscr{C}_j(Z^*)$ $(j = 1, 2, 3, 4)$ are the LTBs at Z^* of the last four integrals in (21). We observe that, like (19), (24) is an exact equality, regardless of the discontinuity in g and/or g' at Z, and it holds for all $Z^* \in S$, except for the two isolated points $Z^* = Z$ and Z_0. Also (24) has features similar to (19) regarding proper modelling of discontinuous or smooth functions alike.

4. SOME MODEL PROBLEMS: RELAXED REGULARIZATION AND DISCRETIZATION

In this section, we suppose that we have derived a BIE, rigorously via a well-defined LTB, using classical smoothness requirements. Thus, concern about various LTBs is not an issue in this section. We then examine some consequences of relaxing those smoothness requirements.

Before discussing particular equations, we should keep in mind that f itself is not the unknown. For example, a typical problem might be to find u given v, where $f = u + \mathrm{i}v$, so that part of f is known; see Appendix I. However, this should not affect the following discussions.

Begin by partitioning S into N pieces (elements), S_j, $j = 1, 2, \ldots, N$, with end-points Z_{j-1} and Z_j; as S is closed, we have $Z_0 \equiv Z_N$. Let $\mathscr{E}$ denote the set of all the end-points. For the purposes of our discussion in Sections 4.1–4.3, we suppose that the partitioning of S into elements is exact, so that there is no approximation of the geometry of S. Nevertheless, we do consider other approximations, including various polynomial representations of certain functions defined on S; also, numerical quadrature, as needed, is implied throughout Section 4. However, nonexact element approximations of S are implicitly allowed in Section 4.4.

4.1. Unregularized equations

Consider the singular equation (2). Introducing the elements S_j, (2) becomes, exactly,

$$f(Z) = \frac{1}{\pi \mathrm{i}} \sum_{j=1}^{N} \int_{S_j} \frac{f(w)}{w - Z} \, \mathrm{d}w, \quad Z \in S \backslash \mathscr{E} \tag{25}$$

Note that we have to exclude the set of end-points because a Cauchy principal-value integral is essentially a two-sided integral; by definition,

$$⨍_S \frac{f(w)}{w - Z} \, \mathrm{d}w = \lim_{\varepsilon \to 0} \int_{S \backslash S_\varepsilon} \frac{f(w)}{w - Z} \, \mathrm{d}w, \quad Z \in S \tag{26}$$

where $S_\varepsilon = \{w \in S : |w - Z| < \varepsilon\}$ is a set of points on S close to, and on *both sides* of, Z. Thus, if we wanted to consider $Z \in \mathscr{E}$, so that $Z = Z_k$ say, we would have to consider the *sum* of the integrals over S_k and S_{k+1}—but we cannot consider these integrals separately, because they do not exist, even though their sum is well defined. This presents an obvious numerical difficulty if we want to collocate (evaluate) (2) at Z_j. For, in a typical boundary-element strategy, one approximates f by a quadratic function on each element S_j, collocates at Z_j (and at other points not in $\mathscr{E}$), and then evaluates the resulting integrals over each element without reference to neighbouring elements. This strategy cannot be justified for singular integral equations, involving CPV integrals. Similarly, such a strategy cannot be justified for hypersingular integral equations such as (7); this conclusion was reached in Reference 1.

The simplest method for avoiding this difficulty is to avoid $\mathscr{E}$. For example, let W_j be the mid-point of S_j, and then approximate f by a constant, f_j, on S_j; collocating at W_k then gives

$$f_k = \frac{1}{\pi \mathrm{i}} \sum_{j=1}^{N} f_j ⨍_{S_j} \frac{\mathrm{d}w}{w - W_k}, \quad k = 1, 2, \ldots, N$$

This method (the 'panel method') is known to be convergent.[10] An exactly similar method can be developed for the hypersingular equation (7), but we do not pursue this here.

4.2. Regularized equations

Consider the regularized equation (5). Introducing the elements S_j, (5) becomes, exactly,

$$0 = \sum_{j=1}^{N} \int_{S_j} \frac{f(w) - f(Z)}{w - Z} \, \mathrm{d}w, \quad Z \in S \tag{27}$$

We observe that this equation holds for *all* $Z \in S$, including those $Z \in \mathscr{E}$, because the integrals are all ordinary improper integrals. This means that we can integrate over each element without reference to neighbouring elements, even if $Z \in \mathscr{E}$.

Numerically, we can see that the regularized equation is attractive. For, suppose that we approximate f by a quadratic function g_j on each S_j. Then, we can collocate at Z_j (and at other points not in $\mathscr{E}$) and evaluate the resulting integrals over each element. Moreover, if we enforce continuity at Z_j so that

$$g_j(Z_j+) = g_{j+1}(Z_j-) \tag{28}$$

where $g_j(Z_j+) = \lim_{w \to Z_j} g_j(w)$ with $w \in S_j$, and $g_{j+1}(Z_j-) = \lim_{w \to Z_j} g_{j+1}(w)$ with $w \in S_{j+1}$, then we automatically get a Hölder-continuous approximation (because it is continuous and piecewise quadratic).

Note that if we do *not* impose (28), we still obtain finite integrals, even if we collocate at Z_j. This is an example of 'relaxed regularization',[4] in that the approximation to f is piecewise-continuous whereas (27) was derived under the assumption that f is Hölder-continuous.

Next, consider the regularized equation (9). Introducing elements as before, (9) becomes, exactly,

$$0 = \sum_{j=1}^{N} \int_{S_j} \frac{f(w) - f(Z) - (w-Z)f'(Z)}{(w-Z)^2} \, \mathrm{d}w, \quad Z \in S \tag{29}$$

The same observation can be made: this equation holds for all $Z \in S$, and we can integrate over each element without reference to neighbouring elements.

Now, to evaluate (29) numerically, suppose we approximate f by quadratics g_j on each S_j, as before, and collocate at Z_j; all the integrals involve bounded integrands. Enforcing continuity at Z_j is easily done. However, in general, we have

$$g_j'(Z_j+) \neq g_{j+1}'(Z_j-) \tag{30}$$

so that the approximation is not differentiable at Z_j. On the other hand, the exact f is required to be differentiable at Z_j. This is another example of 'relaxed regularization'.[4]

4.3. Relaxed regularization: general ideas

We have seen two examples of 'relaxed regularization' above. For a third example, see Reference 11 and the discussion in Reference 1, Section 8.1. The ideas behind 'relaxed regularization' can be exposed in a general way; they will be made quite explicit later. Thus, we begin with a BIE, which is derived rigorously under certain smoothness assumptions. Let us write such an equation as

$$(Au)(Z) = d(Z), \quad Z \in S \tag{31}$$

where A is an operator, u is the unknown function and d is a known forcing function. To be precise, we must specify that $u \in X$, $d \in Y$ and $A : X \to Y$, where X and Y are function spaces. Then, assuming that our problem is uniquely solvable, we can always find the unique $u \in X$ for which $Au = d$, for any given $d \in Y$.

For a specific example, consider the regularized equation (9). Then, we can take $X = C^{1,\alpha}$ and $Y = \text{range}\{A\}$. The precise formula for A can be extracted from (9); if $f = u + \mathrm{i}v$ and v is known, A is defined by taking the real part of (9). If S is partitioned exactly, A can also be defined using (29). Note that (31) holds for all $Z \in S$; discretizations will be discussed later. We note that (19) and (24) provide additional examples of (31), despite the assumed discontinuities in g and/or g'. This is true since the point of discontinuity Z and the other junction point Z_0 between intervals are excluded from S to define S^*, so that, for instance, we may take $X = C^{1,\alpha}(S^*)$ for (24).

On the other hand, consider another BIE

$$(A'\tilde{u})(Z) = d(Z), \quad Z \in S \tag{32}$$

with $\tilde{u} \in X'$ and $A' : X' \to Y'$, where $X \subset X'$ and $Y \subseteq Y'$; we require that $A'\tilde{u} = A\tilde{u}$ whenever $\tilde{u} \in X$. Thus, the operator A' acts on a larger space X', but it gives the same result as A if it is restricted to act on the smaller space X.

For a specific example of (32), consider (29). Take X' to be the space of piecewise-$C^{1,\alpha}$ functions on S, where discontinuities of slope and/or function values are permitted when $w \in \mathscr{E}$. Thus, the right-hand side of (29) is defined for $f \in X'$ and $Z \notin \mathscr{E}$. For $Z \in \mathscr{E}$, we proceed as follows. Consider $Z_k \in \mathscr{E}$, where Z_k is the junction between S_k and S_{k+1}. Then, as $f(Z_k)$ and $f'(Z_k)$ may not be defined for $f \in X'$, we suppose that (29) at Z_k is replaced by

$$0 = \sum_{j=1}^{N} I_j(Z_k) \tag{33}$$

where

$$I_k(Z_k) = \int_{S_k} \frac{f(w) - f(Z_k+) - (w - Z_k) f'(Z_k+)}{(w - Z_k)^2} \, \mathrm{d}w$$

$$I_{k+1}(Z_k) = \int_{S_{k+1}} \frac{f(w) - f(Z_k-) - (w - Z_k) f'(Z_k-)}{(w - Z_k)^2} \, \mathrm{d}w$$

$$I_j(Z_k) = \int_{S_j} \frac{f(w) - f_{\mathrm{a}}(Z_k) - (w - Z_k) f'_{\mathrm{a}}(Z_k)}{(w - Z_k)^2} \, \mathrm{d}w$$

for $j \neq k, k+1$, and $f_{\mathrm{a}}(Z_k)$ and $f'_{\mathrm{a}}(Z_k)$ are 'approximations' to (the possibly undefined) $f(Z_k)$ and $f'(Z_k)$, respectively; we could take the average values,

$$f_{\mathrm{a}}(Z_k) = \tfrac{1}{2}\{f(Z_k+) + f(Z_k-)\} \quad \text{and} \quad f'_{\mathrm{a}}(Z_k) = \tfrac{1}{2}\{f'(Z_k+) + f'(Z_k-)\}$$

but any other finite quantities may be used without affecting the existence of the integrals over those elements S_j which do not have Z_k as an end-point. (However, our choices for f_{a} and f'_{a} imply that we recover (29) if f' *is* continuous at Z_k.) Thus, we have defined $(A'\tilde{u})(Z)$ for all $Z \in S$.

Formally, the idea of 'relaxed regularization' amounts to solving (32) instead of (31). The consequences of doing this are unclear, but the above simple framework highlights some features. First, existence is not a problem: assuming that the forcing function d is unchanged, the sought solution u will satisfy $A'u = d$. However, uniqueness may be lost: we have enlarged the solution space (from X to X'), so there may be more than one solution of (32). It seems to be difficult to answer this uniqueness question, in general, for the following reason. Typically, properties of BIEs are deduced by exploiting the link with the associated BVP. Here, this link has been severed explicitly by relaxing the smoothness assumptions, so that one has to face the BIE directly.

Informally, the idea of 'relaxed regularization' amounts to 'assume sufficient smoothness, derive an integral equation (such as (9)) via a valid LTB (in the sense (ii) of Section 2.1), and then relax the smoothness requirements on f at chosen points Z_k, requirements that are needed for a valid LTB at such Z_k (sense (i))'.

Note that if $N = 2$, (33) is the same as (23), the latter having been obtained as a pseudo-LTB. (When $N = 2$, there are no integrals I_j involving f_{a} and f'_{a}.) Similar remarks could be made for (27), as an example of (32); with $N = 2$, the relaxed-regularization process gives an equation which is the same as (18). Thus (23) and (18), are relaxed-regularized versions of (9) and (5), respectively, previously derived in Section 3 as pseudo-LTBs. This shows that relaxed regularization and pseudo-LTBs are related ideas.

4.4. Relaxed regularization: numerical implementation

Let us consider numerical implementations next. Thus, we consider matrix approximants $\mathbf{A}$ and $\mathbf{A}'$ to the operators A and A', respectively. In both cases, we suppose that f is approximated by a low-order polynomial g_j on each element S_j, collocate at a finite number of points on S and then evaluate some integrals (perhaps numerically) over the elements.

For (31) in the form (29), our approximate BIE is

$$0 = \sum_{j=1}^{N} \int_{S_j} \frac{g_j(w) - g_j(Z) - (w-Z)g_j'(Z)}{(w-Z)^2}\,\mathrm{d}w, \quad Z \in S \backslash \mathscr{E} \tag{34}$$

We cannot collocate at $Z \in \mathscr{E}$ and remain on firm theoretical ground, because our approximation to f is not smooth at such Z, in general. This leads naturally to the use of non-conforming elements. With such elements, every entry in the corresponding $\mathbf{A}$ is well defined, and no inconsistent reasoning is needed anywhere. Furthermore, if one wishes to go back to the representation integral from which (31) is derived, the LTBs associated with the collocation points, leading to the individual entries in $\mathbf{A}$, exist and are well defined.

Next, consider (32). Again, we use g_j on S_j, and permit discontinuities only at the element junctions $Z_j \in \mathscr{E}$, $j = 1, 2, \ldots, N$. Assume further that any such discontinuities, having 'physical' or 'real' origin, are modelled exactly with the element representation (so that modelling-induced and 'other' discontinuities, if any, are indistinguishable at this stage). Then, if only the same collocation points previously used for $\mathbf{A}$ are chosen when finding the matrix approximant $\mathbf{A}'$, all other representations, integrations, etc., being identical, it must be true that $\mathbf{A}'=\mathbf{A}$. This means that if we collocate away from points of discontinuity, with boundary element representations, we can have, as has been known for many years, a rational, approximate, numerical scheme. Errors are only those associated with finite approximation of continuous operators, piecewise polynomial representations of smooth functions, quadrature errors, and the like. But there is (usually) no ambiguity in the governing integral equation itself.

On the other hand, suppose we insist on collocating at the element junctions. This gives the following equations (amongst others, as needed, obtained by collocation at other points):

$$0 = \sum_{j=1}^{N} \tilde{I}_j(Z_k), \quad k = 1, 2, \ldots, N \tag{35}$$

where

$$\tilde{I}_k(Z_k) = \int_{S_k} \frac{g_k(w) - g_k(Z_k+) - (w-Z_k)g_k'(Z_k+)}{(w-Z_k)^2}\,\mathrm{d}w$$

$$\tilde{I}_{k+1}(Z_k) = \int_{S_{k+1}} \frac{g_{k+1}(w) - g_{k+1}(Z_k-) - (w-Z_k)g_{k+1}'(Z_k-)}{(w-Z_k)^2}\,\mathrm{d}w$$

$$\tilde{I}_j(Z_k) = \int_{S_j} \frac{g_j(w) - g_{\mathrm{a}}(Z_k) - (w-Z_k)g_{\mathrm{a}}'(Z_k)}{(w-Z_k)^2}\,\mathrm{d}w$$

for $j \neq k, k+1$, and we may make any desired definitions for $g_{\mathrm{a}}(Z_k)$ and $g_{\mathrm{a}}'(Z_k)$, such as

$$g_{\mathrm{a}}(Z_k) = \tfrac{1}{2}\{g_k(Z_k+) + g_{k+1}(Z_k-)\} \quad \text{and} \quad g_{\mathrm{a}}'(Z_k) = \tfrac{1}{2}\{g_k'(Z_k+) + g_{k+1}'(Z_k-)\} \tag{36}$$

It is interesting to note that if g_j is a *quadratic* function, then

$$\tilde{I}_k(Z_k) = \tfrac{1}{2}(Z_k - Z_{k-1})\,g_k'' \quad \text{and} \quad \tilde{I}_{k+1}(Z_k) = \tfrac{1}{2}(Z_{k+1} - Z_k)\,g_{k+1}''$$

whereas if g_j is a *linear* function, then $\tilde{I}_k(Z_k) = \tilde{I}_{k+1}(Z_k) = 0$. Also, if the approximation to f is continuous everywhere (conforming elements), the expressions for $\tilde{I}_j(Z_k)$ simplify somewhat, as we can take

$$g_k(Z_k+) = g_{k+1}(Z_k-) = g_a(Z_k) \tag{37}$$

Furthermore, note that collocating at element junctions, as above, implies that the assumed discontinuous behaviour at the collocation point Z_k contributes to every entry in the k-th row of the matrix $\mathbf{A}'$. This feature of $\mathbf{A}'$ seems to be quite new in boundary-element modelling. Despite this, under assumptions like (36) and (37) (or similar ones), it is known that good numerical results may be obtained from equations like (32).[3,4] It is even possible that convergence proofs (as $N \to \infty$) for specific classes of BVPs might be found in the future. In a sense, (33) allows more computational possibilities than (9) does—some more useful than others, no doubt—despite the questionable link that (33) has with the BVP to be solved, and the shortcomings noted in References 5 and 6.

In summary, then, we can choose to use non-conforming or conforming boundary elements. If we use non-conforming elements, our theoretical arguments are sound, but such elements have some undesirable features when compared to conforming elements; for example, they lead to a much larger system matrix. On the other hand, if we use conforming elements, we have to make an intuitive step, relaxing the assumed smoothness at the collocation points in order to obtain a numerical algorithm which, despite theoretical shortcomings, seems to perform well.

5. ELASTICITY

Consider a bounded, three-dimensional domain D with smooth boundary S. (Non-smooth boundaries are discussed in Appendix II.) We suppose that D is filled with a homogeneous elastic material. Let a typical interior point $p \in D$ have Cartesian co-ordinates (x_1, x_2, x_3); we also denote these by $x_i(p)$, $i = 1, 2, 3$. In the absence of body forces, the components of the displacement at p, $u_i(p)$, satisfy

$$\partial_i \sigma_{ij}(p) = 0, \quad p \in D, \ j = 1, 2, 3 \tag{38}$$

where $\partial_i \equiv \partial/\partial x_i$, the usual summation convention has been adopted, the stresses σ_{ij} are given by Hooke's law as $\sigma_{ij}(p) = c_{ijkl}\partial_k u_l$, and c_{ijkl} are the elastic constants; for an isotropic material, $c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are the Lamé moduli and δ_{ij} is the Kronecker delta.

The Somigliana representation for the displacement at $p \in D$ can be written as

$$u_j(p) = \int_S \{G_{jk}(p, Q)\, t_k(Q) - T_{jk}(p, Q)\, u_k(Q)\}\, \mathrm{d}s_Q, \quad p \in D \tag{39}$$

Here, G_{ij} is the usual fundamental solution for a point load acting in an unbounded solid,

$$t_i(Q) = \sigma_{ij}(Q)\, n_j(Q) \tag{40}$$

are the traction components; $\mathbf{n}(Q)$ is the outward unit normal vector at Q,

$$T_{ji}(p, Q) = n_k(Q) \left[c_{iklm}(\partial/\partial y_l) G_{jm}(p, q)\right]_{q \to Q}$$

are the traction components at Q corresponding to G_{ij}; and $q \in D$ has co-ordinates (y_1, y_2, y_3).

Now, let $P \in S$. Assume that u_i is Hölder-continuous at P (for $i = 1, 2, 3$). Then, letting $p \to P$ in (39), we obtain

$$\tfrac{1}{2} u_j(P) = \int_S G_{jk}(P,Q)\, t_k(Q)\, \mathrm{d}s_Q - \fint_S T_{jk}(P,Q)\, u_k(Q)\, \mathrm{d}s_Q, \quad P \in S \tag{41}$$

This gives the standard direct BIEs for elastostatics.

We can differentiate (39) to calculate the stresses at $p \in D$; the result can be written as

$$\sigma_{ij}(p) = \int_S \{t_k(Q)\, D_{kij}(p,Q) - u_k(Q)\, S_{kij}(p,Q)\}\ \mathrm{d}s_Q, \quad p \in D \tag{42}$$

where $D_{kij} = c_{ijlm}\partial_l G_{mk}$ and $S_{kij} = c_{ijlm}\partial_l T_{mk}$. Equation (42) is known as the *Somigliana stress identity*. If we assume that u_i has Hölder-continuous tangential derivatives at P, we can let $p \to P$ in (42); the result can be written in terms of finite-part integrals. Usually, of course, we compute the tractions on S, using (40).

5.1. The regularized Somigliana displacement identity

Take $\mathbf{u}=\mathbf{a}$ in (39), where $\mathbf{a}$ is an arbitrary constant vector, giving

$$a_j = -a_k \int_S T_{jk}(p,Q)\, \mathrm{d}s_Q, \quad p \in D \tag{43}$$

hence

$$\int_S T_{jk}(p,Q)\, \mathrm{d}s_Q = -\delta_{jk}, \quad p \in D$$

Subtract (43) from (39) to give

$$u_j(p) - a_j = \int_S \{G_{jk}(p,Q)\, t_k(Q) - T_{jk}(p,Q)\, [u_k(Q) - a_k]\}\ \mathrm{d}s_Q, \quad p \in D \tag{44}$$

So, choosing $a_j = u_j(P)$, where $P \in S$, we obtain

$$u_j(p) - u_j(P) = \int_S \{G_{jk}(p,Q)\, t_k(Q) - T_{jk}(p,Q)\, [u_k(Q) - u_k(P)]\}\ \mathrm{d}s_Q, \quad p \in D \tag{45}$$

This is called the *regularized Somigliana displacement identity*. Letting $p \to P$, we find that

$$0 = \int_S \{G_{jk}(P,Q)\, t_k(Q) - T_{jk}(P,Q)\, [u_k(Q) - u_k(P)]\}\ \mathrm{d}s_Q, \quad P \in S \tag{46}$$

This equation holds provided that $\mathbf{u}(p)$ is continuous in $\overline{D}$ and $\mathbf{u}(Q)$ is Hölder-continuous at P.

The choice $a_j = u_j(p)$ in (44) may also be made; see Reference 12.

5.2. The regularized Somigliana stress identity

Next, consider the Somigliana stress identity (42). The most general linear displacement field is

$$u_i^L(p) = a_i + C_{ij}(x_j - b_j)$$

where a_i, b_i and C_{ij} are constants. The corresponding stresses are constant, and are given by

$$\sigma^L_{ij}(p) = c_{ijkl}\partial_k\{a_l + C_{lm}(x_m - b_m)\} = c_{ijkl}C_{lk}$$

Hence, substituting these fields into (42), we obtain

$$\sigma^L_{ij} = \sigma^L_{kl}\int_S n_l(Q)\,D_{kij}(p,Q)\,\mathrm{d}s_Q - \int_S u^L_k(Q)\,S_{kij}(p,Q)\,\mathrm{d}s_Q, \quad p \in D \tag{47}$$

In particular, taking $C_{ij} \equiv 0$ implies that

$$\int_S S_{kij}(p,Q)\,\mathrm{d}s_Q = 0, \quad p \in D$$

whence (47) simplifies somewhat.

Now, let us make the choices $a_i = u_i(P)$, $b_i = x_i(P)$ and $C_{ij} = \partial_j u_i(P)$, whence $\sigma^L_{ij} = \sigma_{ij}(P)$. Emphasising the dependence on $P \in S$, we write

$$u^{\mathrm{L}}_i(q;P) = u_i(P) + [x_j(q) - x_j(P)]\,\partial_j u_i(P) \quad \text{and} \quad \sigma^{\mathrm{L}}_{ij}(q;P) = \sigma_{ij}(P)$$

Note that $\mathbf{u}^{\mathrm{L}}(q;P) - \mathbf{u}(P)$ is the directional derivative of $\mathbf{u}$ at P in the direction from P to q. In particular, $\mathbf{u}^{\mathrm{L}}(Q;P) - \mathbf{u}(P)$ is the tangential derivative of $\mathbf{u}$ at P in the direction from P to Q when Q is close to P. Furthermore, note that $(\partial_j\mathbf{u})(P)$ can be expressed in terms of the tangential derivatives of $\mathbf{u}$ at P and the traction at P.

Subtracting (47) from (42), we obtain

$$\sigma_{ij}(p) - \sigma_{ij}(P) = \int_S \left\{ \left[t_k(Q) - t^{\mathrm{L}}_k(Q;P)\right] D_{kij}(p,Q) - \left[u_k(Q) - u^{\mathrm{L}}_k(Q;P)\right] S_{kij}(p,Q)\right\} \mathrm{d}s_Q \tag{48}$$

for $p \in D$, where $t^{\mathrm{L}}k(Q;P) = n_j(Q)\sigma_{jk}(P)$. Equation (48) is known as the *regularized Somigliana stress identity*. It is regularized provided that

$$|\mathbf{u}(Q) - \mathbf{u}^{\mathrm{L}}(Q;P)| = O(R^{1+\alpha}) \quad \text{and} \quad |\mathbf{t}(Q) - \mathbf{t}^{\mathrm{L}}(Q;P)| = O(R^{\alpha}) \quad \text{as } R = |\mathbf{x}(Q) - \mathbf{x}(P)| \to 0$$

where $\alpha > 0$. As we have assumed that S is smooth at P, these conditions will be met if $\mathbf{u}$ has Hölder continuous tangential derivatives at P, and the tractions are Hölder continuous at P.

With the above assumptions, together with the assumption that the stresses are continuous in $\overline{D}$, we can let $p \to P$ in (48) to give

$$0 = \int_S \left\{ \left[t_k(Q) - t^{\mathrm{L}}_k(Q;P)\right] D_{kij}(P,Q) - \left[u_k(Q) - u^{\mathrm{L}}_k(Q;P)\right] S_{kij}(P,Q)\right\} \mathrm{d}s_Q, \quad P \in S \tag{49}$$

Equations (48) and (49) have been used extensively by Cruse and his co-workers; see References 2–4 and references therein. Closely related equations can also be found in the literature; see Reference 8 for a review.

The integrals in (49) are all ordinary improper integrals. This means that, just as in Section 4.2, we can partition S into elements exactly, approximate the unknowns on each element, and then integrate over each element. Such schemes lead to well-defined integrals, with bounded integrands, even when collocating along element boundaries; the use of conforming elements will lead to a Hölder-continuous displacement field on S, but this field will have discontinuous tangential derivatives across element boundaries.

In Section 5.3, we shall consider some relaxed-regularization strategies for Somigliana's identities, much as was done in Sections 4.2–4.4, but first let us review the underlying assumptions for the validity of (49), supposing here that S is smooth at P. Then, we require the following:

(i) $\mathbf{u}$ satisfies the equilibrium equations (38) in D;
(ii) the stresses σ_{ij} are continuous in $\overline{D}$;
(iii) $\mathbf{u}$ has Hölder-continuous tangential derivatives at P; and
(iv) $\mathbf{t}$ is Hölder-continuous at P.

Two further observations can be made. First, if $\mathbf{t}$ is discontinuous at P (as is often the case in applications), then we should expect that (at least one component of) $\mathbf{u}$ will have a logarithmically-singular tangential derivative at P; this was shown by Heise, Reference 13, p. 310.

Second, the conditions (iii) and (iv) are *not* equivalent to

(v) the stresses σ_{ij} are Hölder-continuous at P.

Indeed, (iii) and (iv) imply that all components of the displacement-gradient tensor $\mathbf{G}$ must be Hölder-continuous at P whereas (v) implies that all components of the strain tensor $\mathbf{E}$ must be Holder-continuous. Since $\mathbf{E}$ is only the symmetric part of $\mathbf{G}$, (v) represents a weaker condition than (iii) and (iv).

5.3. The regularized Somigliana identities: relaxed regularization

We begin by noting that the regularized Somigliana displacement identity (45) for the elastic displacement $\mathbf{u}(p)$ is the (vector) analogue of (3) for the scalar function $f(z)$. Regarding possible pseudo-LTBs, vector analogues of all of the expressions (10)–(18) are obtainable for $\mathbf{u}$. In particular, the analogue of (18), perhaps the most useful of the pseudo-LTBs, is

$$\begin{aligned} 0 = &\int_{S_1} \left\{ G_{jk}(P,Q)\, t_k^1(Q) - T_{jk}(P,Q)\, [u_k^1(Q) - u_k(P+)] \right\} \mathrm{d}s_Q \\ &+ \int_{S_2} \left\{ G_{jk}(P,Q)\, t_k^2(Q) - T_{jk}(P,Q)\, [u_k^2(Q) - u_k(P-)] \right\} \mathrm{d}s_Q \end{aligned} \tag{50}$$

where $S = S_1 \cup S_2$, u_k^j is u_k evaluated on S_j, t_k^j is t_k evaluated on S_j, and P is a point on the frontier between S_1 and S_2. All of the stated differences and concerns between (18) and (5) pertain to (50) and (45). The main point is that (50) does not have meaning as a well-defined LTB, like (45) does, unless both (a) the displacement is continuous in $\overline{D}$, and (b) the boundary displacement is Hölder continuous at P.

We do not pursue here the ambiguities associated with computing with (50) if (a) and (b) are not satisfied (cf. Section 4.4). Rather we consider the comparable issues surrounding equation (51) below. These issues are the more important ones in applications.

Toward this end, note that the regularized Somigliana stress identity (48) is the vector analogue of (8). If we now allow relaxation of the stated smoothness required for the well-defined LTB (48), that is, only (i) and (ii) in Section 5.2 above are satisfied but (iii) and (iv) are not, we may write

$$\begin{aligned} 0 = &\int_{S_1} \left\{ \left[t_k^1(Q) - t_k^{\mathrm{L}} k(Q;P+) \right] D_{kij}(P,Q) - \left[u_k^1(Q) - u_k^{\mathrm{L}}(Q;P+) \right] S_{kij}(P,Q) \right\} \mathrm{d}s_Q \\ &+ \int_{S_2} \left\{ \left[t_k^2(Q) - t_k^{\mathrm{L}}(Q;P-) \right] D_{kij}(P,Q) - \left[u_k^2(Q) - u_k^{\mathrm{L}}(Q;P-) \right] S_{kij}(P,Q) \right\} \mathrm{d}s_Q \end{aligned} \tag{51}$$

as the vector analogue of (23). Again, all of the stated differences and concerns about (23) and (9) pertain to (51) and (49).

Note especially that the 'zero' on the left-hand sides of (49) and (51) presumes that 'the stresses are continuous at P'. However, as noted above, allowing t_k^{L} and/or u_k^{L} to have discontinuities at P in (51) is generally inconsistent with the assumption of continuous stresses on which that zero is based. Indeed (51) does not have meaning as a well-defined LTB, like (49) does, unless both (iii) and (iv) as well as (i) and (ii) in Section 5.2 are satisfied.

5.4. Discussion

When Cruse and Richardson[3] speak of an existing LTB for continuous stresses, but discontinuous tractions and/or displacement gradients, they are speaking of specific versions of (51) (for various prescriptions of known boundary data). Equation (51) represents a pseudo-LTB; it is obtained under assumptions and reasoning which are inconsistent, so that it is not a well-defined genuine LTB—contrary to the claims in.[3] With this inconsistency goes ambiguity and doubt, theoretically, about what (51) actually means as a legitimate model for the BVP to be solved, and what one might expect in computations with (51).

It is tempting to blur this matter of the meaning of (51), for discontinuous tractions or displacement gradients, by perhaps arguing as follows. Forget (51), and simply derive (49) assuming all the smoothness necessary to do so. Next, introduce element approximations; relax the smoothness assumptions accordingly, collocate at nodes, and introduce average values of discontinuous functions as best you can, at nodes, according as the element approximations introduce discontinuities. Then, as has been done with BIEs of a less controversial nature for decades, go ahead and compute. This process seems innocent enough, and reasonable, in that most numerical approximations, not only in BIE analysis, involve representation functions which are not as smooth, perhaps, as the function to be approximated. When challenged on this, you can respond that the 'real' problem is the one of interest, and we know its solution is (often) smooth. Of course, we must allow some inaccuracies in making approximations.

The subtle difficulty with the preceding quite-plausible argument, when applied to (49), is this. If you insert functions such as $\mathbf{u}$ and $\mathbf{t}$ with relaxed smoothness characteristics into an equation like (49), and then you wish to collocate at points of discontinuity, (49) becomes (51), whether the 'insufficient smoothness' comes from an element representation or from a problem wherein these characteristics have some 'reality'. The equation cannot tell the difference! However, if you collocate with (49) only at points P where (iii) and (iv) are satisfied, you may insert a host of functions with a variety of relaxed smoothness characteristics, as long as these characteristics are consistent with the 'well-definedness' conditions (iii) and (iv) at P. It makes no difference whether such functions are element-based or 'real' in the sense just mentioned. The equality in such equations, even with 'well-defined' collocation points, may not be satisfied exactly because of the approximate representations. Nevertheless, such equations reflect rational and well-understood approximations associated with piecewise polynomial approximations of smooth functions. There is no ambiguity of meaning in the equation itself as to how it is related to a well-defined BIE with good representation of the underlying BVP. There is no inconsistency in reasoning regarding terms which appear in such an equation and terms which have been discarded. With (51) though, none of the things in the last three sentences are true.

Having said all this, we wish to emphasize that we have no doubt whatsoever about the integrity of the numerical data reported in the literature.[2–6] In particular, the numerical data obtained by

Cruse and his co-workers,[2–4] using equations like (51), suggest that the effects of the inconsistencies based on smoothness–relaxation can be small or even negligible, depending on the type of BVP. Further, in References 5 and 6 collocation was done under similar smoothness–relaxation inconsistencies, for some scalar hypersingular integral equations. Considerable caution was expressed in Reference 5 regarding data obtained there for a plane fluid-flow problem and similarly in Reference 6 for a three-dimensional acoustic scattering problem, because the logical inconsistencies of smoothness relaxation were recognized and acknowledged. On reading[5,6] again, however, we notice that good data were, in fact, obtained and convergence was observed, once average values of first derivatives were introduced to remove scale dependence. Convergence seemed to be somewhat problem dependent, and it was at a slower rate than with more logically consistent collocation practices, but good data were obtained nonetheless.

We suspect now that reasonable discontinuities, modelled in violation of theoretical requirements for a well-defined LTB, which contribute to the 'known-data column' in computations, will usually have a small detrimental effect, if any. Further, if modelling discontinuities in unknown functions are limited to discontinuities in first derivatives, rather than the functions themselves, equation (51) apparently works rather well, especially if one uses assumed averages for first derivatives, as suggested in Section 4. This equation probably 'tries very hard' to yield a function, as smooth as possible, in order to be 'faithful to the zero' on the left-hand side of (51). Even though (51) ostensibly allows discontinuities at the collocation points, this is inconsistent with that 'zero', as we have argued extensively in this paper.

More specifically, consider the key matrix $\mathbf{A}'$ (Section 4.4) which governs a discretized version of (51). Without free terms, however, inconsistent the reasoning to discard them may be, equation (51) gives to the diagonal terms of $\mathbf{A}'$ the same character as the diagonal terms of $\mathbf{A}$. The off-diagonal terms, in both $\mathbf{A}'$ and $\mathbf{A}$, have similar character, as well. Thus, if discontinuities are replaced by averages of neighbouring values, one would expect to obtain reasonable results from using $\mathbf{A}'$, compared with $\mathbf{A}$. Differences in these matrices, if the averages are introduced, are due to little, if anything, more than differences in quadrature results from different collocation-point-with-respect-to-element geometries. Moreover, one would also expect convergence of $\tilde{\mathbf{u}}$ to $\mathbf{u}$, with finer and finer discretizations, since with any reasonable piecewise (polynomial) representation over elements, neighbouring slopes will approach each other. Ultimately, since the elements used in $\mathbf{A}'$ are more desirable than elements needed for $\mathbf{A}$, even though the convergence rates of $\tilde{\mathbf{u}}$ to u and $\tilde{\mathbf{u}}$ to u may be different, $\mathbf{A}'$ looks like a good modelling choice, indeed! In turn then, (51), and its cousins (23) and (18) from which respective $\mathbf{A}'$ are derived, all look like acceptable BIEs for computational purposes, despite the disparaging remarks we have made about them on logical grounds.

We remark in closing that the bulk of boundary-element work over the years, including the more recent work with hypersingular equations, to our knowledge, has been based on BIEs derived from well-defined LTBs. Most element modelling with those BIEs has not been in violation of the needs of a well-defined LTB. None of the ambiguities of meaning considered in this paper have thus been present. Questions of accuracy and convergence in numerical computations have therefore been of a rather familiar nature. But possible loss of contact with the underlying BVP, with equations like (51) is relatively new, and this idea is more than a matter of numerical accuracy—notwithstanding the fact that some workers are interested in the idea, if at all, only insofar as numerical matters are concerned.

Understanding all this in such detail now, perhaps we have been overly conservative regarding the numerical dangers of this matter of smoothness for many, even most, problems. However, we are not aware of any theorems to quantify numerical accuracy and convergence issues with equations like (51). Perhaps BIEs, derivable from ill-defined LTBs, are more robust and forgiving of inconsistencies than we think. For the sake of those in the boundary-element community, including ourselves on occasion, who wish to use boundary elements in violation of theoretical demands, we genuinely hope that this is the case.

APPENDIX I: POTENTIAL THEORY

If we write $f = u + \mathrm{i}v$, where u and v are real, and take the real part of (1), we obtain

$$2u(p) = \int_S \left\{ u(Q) \frac{\partial}{\partial n_Q} G(p, Q) - \frac{\partial u}{\partial n_Q} G(p, Q) \right\} \mathrm{d}s_Q, \quad p \in D \tag{52}$$

where $G(p, Q) = (1/\pi) \log |p - Q|$ and $\partial/\partial n_Q$ denotes normal differentiation at $Q \in S$ out of D; the term involving $\partial u/\partial n_Q$ arises from the Cauchy–Riemann equations and an integration by parts. (Note that we have identified the points $p \in D$ and $Q \in S$ with the complex variables $z \in D$ and $w \in S$, respectively.) Equation (52) is the familiar integral representation for a harmonic function in terms of its boundary values and its normal derivative on S. This representation is usually obtained by applying Green's theorem in D to $u(q)$ and $G(p, q)$.

If we write $f = u + \mathrm{i}v$ and take the real part of (2), we obtain the standard direct BIE of potential theory, connecting u and $\partial u/\partial n$ on S, namely

$$u(P) = \int_S \left\{ u(Q) \frac{\partial}{\partial n_Q} G(P, Q) - \frac{\partial u}{\partial n_Q} G(P, Q) \right\} \mathrm{d}s_Q, \quad P \in S$$

This equation is usually derived by letting $p \to P$ in (52).

APPENDIX II: NON-SMOOTH S

In this Appendix, we discuss the modifications required to treat non-smooth S.

II.1. Contour integrals

We assume that S is a simple Jordan contour, so that S can have corners. Then, Cauchy's integral formula, (1), is valid. However, (2) is not valid: if Z is at a corner of S, the left-hand side of (2) must be multiplied by a factor of (β/π), where β is the (interior) angle at Z, Reference 9, Appendix II. Nevertheless, it turns out that the regularized equation (5) *is* valid at corners. This interesting property can be established using the following 'extension argument'.

II.1.1. An 'extension argument'. Suppose that S has a corner at Z. Partition S into three pieces, $S = S_1 \cup S_2 \cup S'$, where Z is at the junction of S_1 and S_2, which are themselves smooth, and S' includes any other corners. We can write (3) as

$$2\pi\mathrm{i}\{f(z) - f(Z)\} = \int_{S_1} \frac{f_1(w; Z)}{w - z} \,\mathrm{d}w + \int_{S_2} \frac{f_2(w; Z)}{w - z} \,\mathrm{d}w + \int_{S'} \frac{f(w) - f(Z)}{w - z} \,\mathrm{d}w, \quad z \in D \tag{53}$$

where $f_j(w; Z) = f(w) - f(Z)$, $w \in S_j$, $j = 1, 2$. Clearly, the third integral is continuous as $z \to Z$, since $Z \notin S'$. Now, consider the first integral. Extend S_1 smoothly beyond Z, giving a longer curved piece $T_1 = S_1 \cup E_1$, where E_1 is the curved extension. Define g_1 on T_1 by

$$g_1(w;Z) = \begin{cases} f_1(w;Z), & w \in S_1 \\ 0, & w \in E_1 \end{cases}$$

Hence,

$$\int_{S_1} \frac{f_1(w;Z)}{w-z}\,\mathrm{d}w = \int_{T_1} \frac{g_1(w;Z)}{w-z}\,\mathrm{d}w, \quad z \in D$$

Moreover, as f_1 is Hölder continuous on S_1 and vanishes as $w \to Z$ (with $w \in S_1$), the extension by zero ensures that g_1 is itself Hölder continuous at Z. Hence, as T_1 is smooth at Z, we can use standard results to let $z \to Z$. A similar extension argument succeeds for the integral over S_2 in (53), whence we can let $z \to Z$ to obtain (5) for non-smooth S.

II.1.2. Further comments. Let us begin by noting that the regularized form of Cauchy's integral formula for f', namely (9), is valid when S has corners, provided that $f'(Z)$ is Hölder-continuous for all $Z \in S$; this is a stringent condition at the corners. On the other hand, the hypersingular equation (7) must be modified at corners.

The discussion of relaxed regularization and discretization in Section 4 is largely independent of whether S has corners or not. Thus, if S does have corners, we merely arrange that they are in $\mathscr{E}$, so that each element S_j is smooth.

II.2. Elasticity

Suppose that S is not a smooth surface, so that it may have corners and edges. Then, the Somigliana representation (39) is still valid. However, the left-hand side of (41) must be modified if P is at a non-smooth point of S; see Hartmann.[14]

If we assume that $\mathbf{u}(p)$ is continuous in $\overline{D}$ and $\mathbf{u}(Q)$ is Hölder-continuous at P, the extension argument of Section II.2.1 can be adapted to show that (46) is valid when S is a non-smooth surface. The extension argument can also be used to show that (49) is valid when S is a non-smooth surface. However, the underlying assumptions are stringent if one wants to use (49) at a corner or edge. For example, one cannot use (49) along an edge where the stresses are infinite, as occurs typically along the edges of a cubical cavity.

Next, let us review the discussion in Section 5.2 when S is not smooth. Let S_m $(m = 1, 2, \ldots, M)$ be smooth pieces of S meeting at P, where there is an edge or a corner. Then, we need conditions (i) and (ii). We need $\partial_j u_k$ to be defined at P. This implies that the tangential derivatives of $\mathbf{u}$ at $Q_m \in S_m$ must have a limit as $Q_m \to P$, for each m, and, moreover, these M limits must be connected through the unique values of $\partial_j u_k$ at P. These conditions replace (iii), and ensure that

$$|\mathbf{u}(Q_m) - \mathbf{u}^{\mathrm{L}}(Q_m;P)| = O(R_m^{1+\alpha}) \quad \text{as } R_m \to 0$$

where $R_m = |\mathbf{x}(Q_m) - \mathbf{x}(P)|$ and $Q_m \in S_m$. Similarly, condition (iv) should be replaced by the condition that

$$n_j(Q_m)\,[\sigma_{jk}(Q_m) - \sigma_{jk}(P)] = O(R_m^{\alpha}) \quad \text{as } R_m \to 0$$

for each m (no sum). Note that this condition does not require that $\mathbf{n}(P)$ be defined.

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