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On the derivation of boundary integral equations for scattering by an infinite two-dimensional rough surface

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A plane acoustic wave is incident upon an infinite, rough, impenetrable surface S. The aim is to find the scattered field by deriving a boundary integral equation over S, using Green's theorem and the free-space Green's function. This requires careful consideration of certain integrals over a large hemisphere of radius r; it is known that these integrals vanish as $r\to\infty$ if the scattered field satisfies the Sommerfeld radiation condition, but that is not the case here—reflected plane waves must be present. It is shown that the well-known Helmholtz integral equation is not valid in all circumstances. For example, it is not valid when the scattered field includes plane waves propagating away from S along the axis of the hemisphere. In particular, it is not valid for the simplest possible problem of a plane wave at normal incidence to an infinite flat plane. Some suggestions for modified integral equations are discussed. [S0022-2488(98)01302-4]

I. INTRODUCTION

A bounded three-dimensional obstacle with a smooth surface S is surrounded by a compressible fluid. A plane time-harmonic sound wave is incident on the obstacle; the problem is to calculate the scattered field u. In order to have a well-posed boundary-value problem (with existence and uniqueness), one imposes the Sommerfeld radiation condition,

$$r\left(\frac{\partial u}{\partial r}-iku\right)\to 0 \quad \text{as } r\to\infty, \tag{1}$$

uniformly in all directions. Here r is a spherical polar coordinate, k is the wave number, and the time-dependence is $e^{-i\omega t}$. Physically, the radiation condition ensures that the scattered waves propagate outwards, away from the obstacle.

A well-known method for solving the above problem is to derive a boundary integral equation for the boundary values of u on S. In the derivation, Green's theorem is applied to u and a fundamental solution G, in the region bounded internally by S and externally by C_r, a large sphere of radius r. It turns out that the radiation condition implies that the integral

$$I(u;C_r)\equiv\int_{C_r}\left(u\frac{\partial G}{\partial r}-G\frac{\partial u}{\partial r}\right)\,ds\to 0 \quad \text{as } r\to\infty, \tag{2}$$

and so only boundary integrals over S remain. For more information, see Colton and Kress.[1]

Assume that S is a sound-hard surface (Neumann condition). Then, the method described above leads to the following boundary integral equation:

$$u(p)-\int_S u(q)\frac{\partial G}{\partial n_q}(p,q)\,ds_q=\int_S\frac{\partial u_{\text{inc}}}{\partial n_q}G(p,q)\,ds_q,\quad p\in S, \tag{3}$$

here, u_{inc} is the given incident wave. One can also derive an equation for the boundary values of the total field $u_{\mathrm{tot}}=u_{\mathrm{inc}}+u$; this boundary integral equation is

$$u_{\mathrm{tot}}(p)-\int_S u_{\mathrm{tot}}(q)\frac{\partial G}{\partial n_q}(p,q)ds_q=2u_{\mathrm{inc}}(p),\quad p\in S. \tag{4}$$

We shall refer to Eqs. (3) and (4) as *standard Helmholtz integral equations*. Similar equations can be derived for sound-soft surfaces ($u_{\mathrm{tot}}=0$ on S).

Suppose now that the obstacle is *unbounded*. The prototypical problem is scattering (reflection) of a plane wave by an infinite flat plane, S. As is well known, the incident wave is reflected specularly as a single propagating plane wave. More generally, if S is an infinite rough surface, an incident plane wave will be scattered into a spectrum of plane waves. For such problems, the Sommerfeld radiation condition is definitely inappropriate as it is not satisfied by a plane wave. Nevertheless, it is customary to proceed, *assuming* that the scattered field can be represented in terms of plane waves, at least at some distance from S. Typically, this requires the discarding of an integral such as (2), but with the large sphere C_r replaced by a large *hemisphere* H_r. This paper began as an attempt to justify this step.

In a previous paper,[2] we derived boundary integral equations of Helmholtz type for *one-dimensional* rough surfaces. We found that the standard Helmholtz integral equations are valid, except that the right-hand side of Eq. (4) must be replaced by $u_{\mathrm{inc}}(p)$ for grazing incidence.

It is perhaps surprising that an analysis of this kind has not been given before: most authors have been content to write down an integral equation such as Eq. (4), prior to extensive numerical computations. However, it turns out that the necessary analysis for scattering by a *two-dimensional* rough surface is not straightforward and, moreover, it yields some surprises. For example, the simplest problem, namely reflection of a plane wave at *normal* incidence upon a *flat* surface, leads to divergent integrals: the standard Helmholtz integral equation (3) is not valid for this problem.

The paper is organized as follows. Section II is devoted to formulating the problem, with some background on angular-spectrum representations and integral representations (using G). Green's theorem is applied inside a volume whose closed boundary is made up of three pieces: the large hemisphere H_r; a large circular piece, S_r, of the rough surface; and a cylindrical surface T_r, joining S_r and H_r. Estimation of integrals over H_r is carried out in Secs. III–V. Thus the method of stationary phase for multiple integrals and an expansion method are used in Secs. III and IV, respectively, but only for a single plane wave. Results for $I(u;H_r)$ are obtained in Sec. V. The contribution from integrating over T_r is considered in Sec. VI. Unlike in two dimensions (one-dimensional rough surface), this contribution may not be negligible; it is evaluated under additional, but reasonable, *a priori* assumptions on the form of the scattered field near S. This is a weakness of the present analysis. Finally, boundary integral equations of the Helmholtz type are derived in Sec. VII. Further work is needed to tighten up the analysis and to investigate the numerical consequences.

II. FORMULATION

Consider the scattering of a plane wave by a two-dimensional rough surface, S, described by

$$z=s(x,y),\quad -\infty<x<\infty,\quad -\infty<y<\infty$$

with $-h<s(x,y)\leqslant 0$ and some constant $h\geqslant 0$. The acoustic medium occupies $z>s$ and, for definiteness, we assume that S is a smooth, sound-hard surface. Thus we can write the total field as

$$u_{\mathrm{tot}}=u_{\mathrm{inc}}+u,$$

where u is the scattered field. The incident plane wave is

$$u_{\mathrm{inc}}(r,\theta,\phi)=\exp\{i\mathbf{k}_i\cdot\mathbf{x}\},\quad 0\leqslant\theta_i\leqslant\tfrac{1}{2}\pi, \tag{5}$$

where $\mathbf{k}_i = k(\sin\theta_i, 0, -\cos\theta_i)$, θ_i is the angle of incidence (it is the angle between the direction of propagation and the negative z-axis),

$$\mathbf{x} = r\hat{\mathbf{x}} = r(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta),$$

and (r,θ,ϕ) are spherical polar coordinates: $x = r\,\sin\theta\cos\phi$, $y = r\,\sin\theta\sin\phi$ and $z = r\cos\theta$. All the fields u_{tot}, u_{inc}, and u satisfy the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0$$

for $z > s$. The boundary condition is

$$\frac{\partial u_{\text{tot}}}{\partial n} = 0 \quad \text{on } S, \tag{6}$$

where $\partial/\partial n$ denotes normal differentiation *out* of the acoustic medium.

A. Reflection by a flat surface

It is instructive to consider the very simple problem of reflection by a flat surface, so that $s = 0$. The textbook solution for the scattered field is

$$u(r,\theta,\phi) = \exp\{i\mathbf{k}_s \cdot \mathbf{x}\} \quad \text{for } 0 \leqslant \theta_i < \tfrac{1}{2}\pi, \tag{7}$$

where $\mathbf{k}_s = k(\sin\theta_i, 0, \cos\theta_i)$. When $\theta_i = \frac{1}{2}\pi$ (''grazing incidence''), we have $u \equiv 0$: the incident wave satisfies the boundary condition on S.

So, for $0 \leqslant \theta_i < \frac{1}{2}\pi$,

$$u_{\text{tot}} = 2\,e^{ikx\sin\theta_i}\cos(kz\cos\theta_i)$$

solves the problem. But consider

$$u'_{\text{tot}} \equiv u_{\text{tot}} + u_g \tag{8}$$

with

$$u_g = V(\beta)e^{ik(x\cos\beta + y\sin\beta)},$$

where β and $V(\beta)$ are arbitrary, with $-\pi < \beta \leqslant \pi$. u'_{tot} also ''solves'' the problem, in that it satisfies the Helmholtz equation and the boundary condition. Of course, we disallow this second solution, unless $V \equiv 0$: but why? The answer is because of the radiation condition (which we have yet to specify). For example, take $\beta = 0$ and $V(0) = 1$, so that $u_g = e^{ikx}$; this gives an ''outgoing'' grazing wave at $x = +\infty$ but it is an ''incoming'' grazing wave at $x = -\infty$, we must therefore exclude it. Indeed, we must exclude *all* contributions u_g, for any β and V.

A similar condition is imposed on the two-dimensional problem.[3] However, the three-dimensional problem has another feature, for we could consider replacing u_g in Eq. (8) by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} V(\beta)e^{ik(x\cos\beta + y\sin\beta)}d\beta,$$

where V is a continuous function; but, as u_g has been excluded, we must also exclude all linear combinations of such plane grazing waves. In particular, by taking $V(\beta) = (-i)^n e^{in\beta}$, we see that we must exclude the cylindrical standing waves

$$J_n(kR)e^{in\phi}, \tag{9}$$

where J_ν is a Bessel function,[4] $R = r\sin\theta$, and (R,ϕ,z) are cylindrical polar coordinates of the point at $\mathbf{x}$. On the other hand, the exact scattered field, given by Eq. (7), when evaluated on any plane $z =$ constant, has an azimuthal Fourier component proportional to

$$J_n(k_i R)e^{in\phi}, \tag{10}$$

where $k_i = k\sin\theta_i < k$. Thus if one wants to formulate a radiation condition, mathematically, it must be such that fields (9) are excluded but fields (10) are permitted.

This discussion suggests that the specification of a *mathematical* radiation condition for the present class of problems (plane-wave scattering by an infinite two-dimensional rough surface) will not be straightforward. However, the *physical* purpose of a radiation condition is clear: it is to exclude all ''incoming'' waves apart from the incident wave. We shall return to radiation conditions in Sec. II B.

B. Angular-spectrum representations

For any rough surface S, the scattered field in the half-space $z>0$ may be written using an angular-spectrum representation,

$$\begin{aligned} u(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{ik(\mu x+\nu y+mz)}\frac{d\mu\, d\nu}{m(\kappa)} \\ &= \int_0^{\pi/2}\int_{-\pi}^{\pi} A(\alpha,\beta)v(r,\theta,\phi;\alpha,\beta)d\alpha\, d\beta + \text{evanescent terms.} \end{aligned} \tag{11}$$

Here F is the *spectral amplitude*, $A(\alpha,\beta)=F(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta)$, $\kappa=\sqrt{\mu^2+\nu^2}$, and

$$m(\kappa)=\begin{cases}\sqrt{1-\kappa^2}, & 0\leqslant\kappa<1,\\ i\sqrt{\kappa^2-1}, & \kappa>1,\end{cases}$$

the function v is defined by

$$v(r,\theta,\phi;\alpha,\beta)=\exp\{i\mathbf{k}\cdot\mathbf{x}\},\ \ 0\leqslant\alpha\leqslant\frac{1}{2}\pi,\ \ |\beta|\leqslant\pi, \tag{12}$$

where

$$\mathbf{k}=k\hat{\mathbf{k}}=k(\sin\alpha\ \cos\beta,\sin\alpha\ \sin\beta,\cos\alpha).$$

The integrals are superpositions of plane waves; they are propagating, homogeneous plane waves when $0\leqslant\kappa<1$, and they are evanescent, inhomogeneous plane waves when $\kappa>1$. In Eq. (11), we see the propagating plane waves explicitly: they propagate in the direction of $\hat{\mathbf{k}}$, with an (unknown) complex amplitude, $A(\alpha,\beta)$; the ''evanescent terms'' decay exponentially with z. For more information on angular-spectrum representations, see Clemmow[5] and DeSanto and Martin.[6]

In general, the spectral amplitude must be considered as a generalized function. Thus it is convenient to extract a continuous component from F, writing the scattered field as

$$u=u_{\mathrm{pr}}+u_{\mathrm{ev}}+u_{\mathrm{con}}, \tag{13}$$

where

$$u_{\mathrm{pr}}(r,\theta,\phi)=\sum_{n=0}^{N} A_n v(r,\theta,\phi;\alpha_n,\beta_n),$$

$$u_{\mathrm{ev}}(r,\theta,\phi)=\sum_{m=1}^{M} B_m w(r,\theta,\phi;\mu_m,\nu_m), \tag{14}$$

$$u_{\mathrm{con}}(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathscr{C}(\mu,\nu)\ e^{ik(\mu x+\nu y+mz)}\frac{d\mu\, d\nu}{m(\kappa)}, \tag{15}$$

$$w(r,\theta,\phi;\mu,\nu)=\exp\{ikr\sin\theta\,[\mu\cos\phi+\nu\sin\phi]-kr\cos\theta\,\sqrt{\kappa^2-1}\},\tag{16}$$

and $\kappa=\sqrt{\mu^2+\nu^2}>1$ in Eq. (16). The first term in Eq. (13) is a sum of propagating waves; the coefficients A_n and the angles, α_n and β_n, are unknown in general. The second term in Eq. (13) is a sum of evanescent waves; B_m, μ_m, and ν_m are unknown in general. The third term in Eq. (13) is a continuous spectrum of plane waves; the unknown function $\mathscr{C}$ is continuous. See Sec. V for further comments.

Let us now return to the radiation condition. Having chosen an origin O, arbitrarily, we consider a large hemisphere H_r, with radius r and center O. We then require that all propagating plane-wave components $v(r,\theta,\phi;\alpha_n,\beta_n)$ in u propagate *outwards* through H_r, away from O. This is almost built into the decomposition (13): we have to be careful with grazing waves [$\alpha_n=\frac{1}{2}\pi$; see the discussion following Eq. (8)]. A simple way to impose our radiation condition is to split the half-space $z>0$ and the hemisphere H_r into four parts. Thus with

$$H_r^m=\left\{(r,\theta,\phi):0\leqslant\theta\leqslant\frac{1}{2}\pi,\frac{1}{2}(m-3)\pi\leqslant\phi<\frac{1}{2}(m-2)\pi\right\},\quad m=1,2,3,4,$$

being the surfaces of four octants of a sphere, we require the following conditions for the regions specified:

$$\begin{aligned}
&m=1: && \text{in } x<0,\ y\leqslant 0, && \text{use } -\pi\leqslant\beta_n<-\tfrac{1}{2}\pi,\\
&m=2: && \text{in } x\geqslant 0,\ y<0, && \text{use } -\tfrac{1}{2}\pi\leqslant\beta_n<0,\\
&m=3: && \text{in } x>0,\ y\geqslant 0, && \text{use } 0\leqslant\beta_n<\tfrac{1}{2}\pi,\\
&m=4: && \text{in } x\leqslant 0,\ y>0, && \text{use } \tfrac{1}{2}\pi\leqslant\beta_n<\pi.
\end{aligned}\tag{17}$$

This partitioning makes it easy to ensure that only plane waves propagating out through H_r^m are included. This is the form of radiation condition used to derive boundary integral equations.

C. Boundary integral equations

One way to determine the scattered field is to derive a boundary integral equation over the rough surface S. An appropriate fundamental solution is the free-space Green's function

$$G(P,Q)=G(\mathbf{y},\mathbf{x})=\frac{-1}{2\pi}\,\frac{\exp\{ik|\mathbf{x}-\mathbf{y}|\}}{|\mathbf{x}-\mathbf{y}|},$$

where $\mathbf{x}$ and $\mathbf{y}$ are the position vectors of Q and P, respectively, with respect to the origin O. Apply Green's theorem to u and G in the region D_r with boundary $\partial D_r=H_r\cup S_r\cup T_r$, where H_r is a large hemisphere H_r of radius r and center O,

$$S_r=\{(x,y,z):z=s(x,y),\ 0\leqslant x^2+y^2<r^2\}$$

is a truncated rough surface, and

$$T_r=\{(x,y,z):x^2+y^2=r^2,\ s(x,y)\leqslant z\leqslant 0\}\tag{18}$$

is the surface of a truncated circular cylinder joining H_r and S_r. The result is

$$2u(P)=\int_{\partial D_r}\left\{u(q)\,\frac{\partial G}{\partial n_q}(P,q)-\frac{\partial u}{\partial n}\,G(P,q)\right\}dS_q,$$

where $P\in D_r$, $q\in\partial D_r$, and $\partial/\partial n_q$ denotes normal differentiation at q. Use of the boundary condition (6) yields

$$2u(P)=\int_{S_r}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\mathrm{inc}}}{\partial n}G(P,q)\right\}dS_q+I(u;H_r)+I(u;T_r), \tag{19}$$

where

$$I(u;\mathscr{S})=\int_{\mathscr{S}}\left\{u(q)\frac{\partial G}{\partial n_q}(P,q)-\frac{\partial u}{\partial n}G(P,q)\right\}dS_q$$

and normal differentiation is taken in a direction away from the origin [so that $\partial/\partial n=\partial/\partial r$ on H_r, consistent with Eq. (2)].

The next step is to estimate $I(u;H_r)$ and $I(u;T_r)$ for large r. Before estimating $I(u;H_r)$, using Eq. (13), we consider a single propagating plane-wave component in Eq. (13). Thus we shall evaluate $I(v;H_r)$ as $r\to\infty$, where v is defined by Eq. (12). In fact, as $v(r,\theta,\phi;\alpha,\beta)=v(r,\theta,\phi-\beta;\alpha,0)$, we can assume without loss of generality that $\beta=0$; we write

$$v(r,\theta,\phi;\alpha,0)=v(r,\theta,\phi;\alpha)=v(\mathbf{x};\alpha),\quad 0\leqslant\alpha\leqslant\frac{1}{2}\pi.$$

Indeed, we shall evaluate the limit using two different methods; these are the method of stationary phase (Sec. III) and an expansion method (Sec. IV). The reasons for this twofold evaluation are (i) the results are surprising, and (ii) the expansion method is natural but it is complicated and it leads to some subtle nonuniform behavior. We shall discuss the evaluation of $I(u;H_r)$ itself for large r in Sec. V. We then consider the contribution from $I(u;T_r)$ to Eq. (19) in Sec. VI. Finally, we will derive boundary integral equations from Eq. (19) in Sec. VII.

III. THE METHOD OF STATIONARY PHASE

We use the method of stationary phase[7] to estimate $I(v;H_r)$. It turns out that there are three cases, depending on the angle α.

We are interested in large values of $r=|\mathbf{x}|$, for fixed $\mathbf{y}$ and k. We have

$$G(P,q)\simeq(B/r)\exp\{ik(r-\mathbf{y}\cdot\hat{\mathbf{x}})\}, \tag{20}$$

where $B=-\tfrac{1}{2}/\pi$. Hence for large r,

$$v\frac{\partial G}{\partial r}-G\frac{\partial v}{\partial r}\simeq ik\frac{B}{r}(1-\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})\exp\{ikr(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})\}\exp\{-ik\mathbf{y}\cdot\hat{\mathbf{x}}\}$$

and then

$$I(v;H_r)\simeq iB\,e^{ikr}L(kr),$$

where

$$L(\lambda)=\lambda\int_{\mathscr{D}}g(\theta,\phi)\,e^{i\lambda F(\theta,\phi)}\,d\theta\,d\phi, \tag{21}$$

$$g(\theta,\phi)=(1-\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})\exp\{-ik\mathbf{y}\cdot\hat{\mathbf{x}}\}\sin\theta, \tag{22}$$

$$F(\theta,\phi)=\hat{\mathbf{k}}\cdot\hat{\mathbf{x}}\quad\text{and}\quad\mathscr{D}=\{(\theta,\phi):0\leqslant\theta\leqslant\theta_0<\pi,-\pi\leqslant\phi\leqslant\pi\}$$

is the rectangular domain of integration; this corresponds to integrating over a spherical cap subtending an angle of $2\theta_0$ at the origin, for the hemisphere H_r, we set $\theta_0=\tfrac{1}{2}\pi$.

We now examine three cases in turn. These are $\alpha=0$, $0<\alpha<\tfrac{1}{2}\pi$, and $\alpha=\tfrac{1}{2}\pi$.

TABLE I. Stationary-phase points for $L(\lambda)$.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
θ	0	0	π	π	α	$\pi-\alpha$	$\pi-\alpha$
ϕ	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	$\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	0	π	$-\pi$

A. The method of stationary phase: $\alpha=0$

Consider the special case $\alpha=0$, corresponding to a plane wave v propagating along the z-axis. Then we have $F(\theta,\phi)=\cos\theta$, which means that the only dependence on ϕ is through $\mathbf{y}\cdot\hat{\mathbf{x}}$:

$$L(\lambda)=\lambda\int_0^{\theta_0}(1-\cos\theta)\; b(\theta;\mathbf{y})\; e^{i\lambda\cos\theta}\sin\theta\, d\theta, \tag{23}$$

where

$$b(\theta;\mathbf{y})=\int_{-\pi}^{\pi}\exp\{-ik\mathbf{y}\cdot\hat{\mathbf{x}}\}\, d\phi.$$

Introducing spherical polar coordinates for $\mathbf{y}$,

$$\mathbf{y}=\rho\hat{\mathbf{y}}=\rho(\sin\Theta\,\cos\Phi,\sin\Theta\,\sin\Phi,\cos\Theta), \tag{24}$$

we can integrate over ϕ to obtain

$$b(\theta;\mathbf{y})=2\pi\, e^{-ik\rho\cos\theta\,\cos\Theta}\, J_0(k\rho\sin\theta\,\sin\Theta).$$

Now, the integral (23) can be estimated for large λ using the (one-dimensional) method of stationary phase. The only stationary-phase point is at $\theta=0$; as the integrand vanishes at $\theta=0$, we deduce that $L(\lambda)=O(1)$ as $\lambda\to\infty$. In fact, an integration by parts shows that

$$L(\lambda)\sim i(1-\cos\theta_0)\; b(\theta_0;\mathbf{y})\; e^{i\lambda\cos\theta_0}\quad\text{as } \lambda\to\infty,$$

in particular, for the hemisphere ($\theta_0=\frac{1}{2}\pi$), we obtain

$$I(v;H_r)=e^{ikr}\, J_0(k\rho\sin\Theta)+O((kr)^{-1/2})\quad\text{as } kr\to\infty,\quad\text{for } \alpha=0. \tag{25}$$

B. The method of stationary phase: $0<\alpha<\frac{1}{2}\pi$

Return to the integral $L(\lambda)$, defined by Eq. (21). We estimate $L(\lambda)$ for large λ, using the method of stationary phase for two-dimensional integrals.[8] Thus we look for stationary-phase points $\mathbf{c}=(\theta,\phi)\in\mathscr{D}$ at which grad $F=\mathbf{0}$; such points may be in the interior of $\mathscr{D}$ or on the boundary, $\partial\mathscr{D}$. Each $\mathbf{c}$ contributes a term to $L(\lambda)$ proportional to

$$g(\mathbf{c})\; e^{i\lambda F(\mathbf{c})},$$

the next term being $O(\lambda^{-1})$.[9] In general, $\partial\mathscr{D}$ contributes terms of $O(\lambda^{-1/2})$, whereas corners of $\partial\mathscr{D}$ contribute terms of $O(\lambda^{-1})$; all these contributions are smaller than those from stationary-phase points.

We have

$$F(\theta,\phi)=\sin\theta\,\cos\phi\,\sin\alpha+\cos\theta\,\cos\alpha.$$

Elementary calculations show that grad $F=\mathbf{0}$ at seven points $\mathbf{c}_j$ $(j=1,2,\dots,7)$ in the range $0\leqslant\theta\leqslant\pi$, $|\phi|\leqslant\pi$ (which is larger than $\mathscr{D}$); see Table I. Substituting into Eq. (22), we see that

$$g(\mathbf{c}_1)=g(\mathbf{c}_2)=g(\mathbf{c}_3)=g(\mathbf{c}_4)=g(\mathbf{c}_5)=0$$

and

$$g(\mathbf{c}_6)=g(\mathbf{c}_7)=2\sin\alpha\ \exp\{i\mathbf{k}\cdot\mathbf{y}\}.$$

So, if we integrate over a region $\mathscr{D}$ that does *not* include $\mathbf{c}_6$ and $\mathbf{c}_7$, then $L(\lambda)=o(1)$ as $\lambda\to\infty$. This will be the case if

$$\alpha<\theta_0<\pi-\alpha,$$

in particular, for the hemisphere, we obtain

$$I(v;H_r)=O((kr)^{-1/2})\quad \text{as } kr\to\infty,\quad \text{for } 0<\alpha<\tfrac{1}{2}\pi.$$

C. The method of stationary phase: $\alpha=\frac{1}{2}\pi$

For the hemisphere, with $\alpha=\theta_0=\frac{1}{2}\pi$, we find that $\mathbf{c}_6$ and $\mathbf{c}_7$ are on $\partial\mathscr{D}$. Their contribution can be found;[10] the result is

$$I(v;H_r)=2\exp\{i\mathbf{k}\cdot\mathbf{y}\}+O((kr)^{-1/2})\quad \text{as } kr\to\infty,\quad \text{for } \alpha=\tfrac{1}{2}\pi.$$

D. Summary

The axis of symmetry of the hemisphere H_r is the z-axis. We have considered plane waves v propagating out of the hemisphere, at an angle α to the z-axis. We have seen that $I(v;H_r)\to 0$ as $r\to\infty$, for $0<\alpha<\frac{1}{2}\pi$. For $\alpha=\frac{1}{2}\pi$ (''grazing waves,'' with respect to the plane $z=0$), $I(v;H_r)\to 2\exp\{i\mathbf{k}\cdot\mathbf{y}\}$, a finite quantity, as $r\to\infty$. For $\alpha=0$ (''normal waves,'' with respect to $z=0$), $I(v;H_r)\sim e^{ikr}\,J_0(k\rho\sin\Theta)$, which means that $I(v;H_r)$ does not have a limit (in this case) as $r\to\infty$. This unpleasant result can be verified directly when $\rho=0$; in this special case, we have

$$\begin{aligned}L(\lambda)&=2\pi\lambda\int_0^{\pi/2}(1-\cos\theta)\ e^{i\lambda\cos\theta}\sin\theta\ d\theta\\&=2\pi i\{1-i\lambda^{-1}(1-e^{i\lambda})\}\end{aligned}$$

exactly, after an integration by parts. In fact, in this special case, we can evaluate I exactly, without using the approximation (20); the result is

$$I(v;H_r)=e^{ikr},\quad \rho=\alpha=0. \tag{26}$$

As in the two-dimensional case,[2] it is possible to derive a uniform approximation for α near $\frac{1}{2}\pi$.[11] The situation for small α is more complicated.[12] We shall not pursue these nonuniformities here.

IV. AN EXPANSION METHOD

We shall evaluate the integral over the hemisphere, $I(v;H_r)$, using suitable expansions of v and G in spherical polar coordinates. Thus

$$v(\mathbf{x};\alpha)=\exp\{i\mathbf{k}\cdot\mathbf{x}\}=\sum_{n=0}^{\infty}(2n+1)\ i^n\ j_n(kr)\ P_n(\cos\theta_1),$$

where $j_n(x)=(\frac{1}{2}\pi/x)^{1/2}\,J_{n+1/2}(x)$ is a spherical Bessel function,

$$\cos\theta_1=\hat{\mathbf{k}}\cdot\hat{\mathbf{x}}=\sin\theta\ \sin\alpha\ \cos\phi+\cos\theta\ \cos\alpha$$

and P_n is a Legendre polynomial. Similarly,

$$G(P,q)=G(\mathbf{y},\mathbf{x})=\frac{-ik}{2\pi}\sum_{n=0}^{\infty}(2n+1)\ j_n(k\rho)\ h_n(kr)\ P_n(\cos\theta_2)$$

for $r>\rho$, where

$$\cos\theta_2=\hat{\mathbf{y}}\cdot\hat{\mathbf{x}}=\sin\theta\ \sin\Theta\ \cos(\phi-\Phi)+\cos\theta\ \cos\Theta,$$

$h_n(x)\equiv h_n^{(1)}(x)=(\frac{1}{2}\pi/x)^{1/2}\ H_{n+1/2}^{(1)}(x)$ is a spherical Hankel function and $H_\nu^{(1)}$ is a Hankel function. Hence

$$I(v;H_r)=\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}(2l+1)(2s+1)\ i^s\ j_l(k\rho)\ W_{ls}(kr)\ A_{ls},$$

where

$$W_{ls}(w)=-iw^2\{j_s(w)h_l'(w)-j_s'(w)h_l(w)\} \tag{27}$$

and

$$A_{ls}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\pi/2}P_l(\cos\theta_2)\ P_s(\cos\theta_1)\ \sin\theta\ d\theta\ d\phi.$$

Note that W_{ll} is essentially a Wronskian, given by $W_{ll}=1$.

Let us evaluate A_{ls}. The addition theorem for Legendre polynomials gives

$$P_s(\cos\theta_1)=\sum_{m=0}^{s}\epsilon_m c_s^m P_s^m(\cos\theta)\ P_s^m(\cos\alpha)\ \cos m\phi,$$

$$P_l(\cos\theta_2)=\sum_{n=0}^{l}\epsilon_n c_l^n P_l^n(\cos\theta)\ P_l^n(\cos\Theta)\ \cos n(\phi-\Phi),$$

where P_n^m is an associated Legendre function, $\epsilon_0=1$, $\epsilon_m=2$ for $m\geqslant 1$, and

$$c_s^m=[(s-m)!]/[(s+m)!].$$

Hence, integrating over ϕ, we obtain

$$A_{ls}=\sum_{m=0}^{s}\epsilon_m c_l^m c_s^m P_l^m(\cos\Theta)\ P_s^m(\cos\alpha)\ B_{ls}^m\ \cos m\Phi, \tag{28}$$

where

$$B_{ls}^m=\int_0^{\pi/2}P_l^m(\cos\theta)\ P_s^m(\cos\theta)\ \sin\theta\ d\theta=\int_0^1 P_l^m(\mu)\ P_s^m(\mu)\ d\mu.$$

[Actually, the upper limit on the summation in Eq. (28) should be $\min\{l,s\}$, but this is of no consequence as $P_n^m\equiv 0$ for $m>n$.] B_{ls}^m has been evaluated by Hulme.[13] It turns out that

$$B_{2l+m,2s+m+1}^m=\frac{2^{2m}(-1)^{l+s+1}\Gamma(l+m+\frac{1}{2})\Gamma(s+m+\frac{3}{2})}{\pi(2l-2s-1)(l+s+m+1)\ l!\ s!}, \tag{29}$$

$$B_{2l+m+1,2s+m}^m=\frac{2^{2m}(-1)^{l+s+1}\Gamma(l+m+\frac{3}{2})\Gamma(s+m+\frac{1}{2})}{\pi(2s-2l-1)(l+s+m+1)\ l!\ s!}, \tag{30}$$

$B_{ll}^m=[(2l+1)\ c_l^m]^{-1}$ and $B_{ls}^m=0$ if $|l-s|$ is a positive even integer.

Now, returning to I, we see that it can be expressed in the form

$$I(v;H_r)=\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\sum_{m=0}^{s}\epsilon_m Q_{ls}^m\cos m\Phi,$$

where

$$Q_{ls}^m=(2l+1)(2s+1)\ i^s\ j_l(k\rho)\ W_{ls}(kr)\ c_l^m c_s^m\ B_{ls}^m\ P_l^m(\cos\Theta)\ P_s^m(\cos\alpha).$$

Interchanging the summations over s and m, we obtain

$$I(v;H_r)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=m}^{\infty}\epsilon_m Q_{ls}^m\cos m\Phi=\sum_{m=0}^{\infty}\epsilon_m\left\{\sum_{l=0}^{\infty}\sum_{s=m}^{\infty}Q_{ls}^m\right\}\cos m\Phi.$$

But $Q_{ls}^m\equiv 0$ for $m>l$, whence

$$I(v;H_r)=\sum_{m=0}^{\infty}\epsilon_m\Lambda_m\cos m\Phi,$$

where

$$\Lambda_m=\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}Q_{l+m,s+m}^m.$$

This shows that I can be expressed as a Fourier series in the azimuthal coordinate of the observation point $\mathbf{y}$, which is as expected.

A. A special case: $\rho=0$ ($\mathbf{y}=0$)

The expression for Λ_m is complicated. To gain some insight into its evaluation, we start with the special case of $\rho=0$. Then, only Q_{0s}^0 is not identically zero, whence

$$\begin{aligned}I(v;H_r)&=\sum_{s=0}^{\infty}Q_{0s}^0\\&=1+i\sum_{n=0}^{\infty}(4n+3)(-1)^n\ W_{0,2n+1}(kr)\ B_{0,2n+1}^0\ P_{2n+1}(\cos\alpha),\end{aligned}$$

where we have used $B_{00}^0=W_{00}=1$ and $B_{0,2n}^0=0$; also

$$B_{0,2n+1}^0=\int_0^1 P_{2n+1}(\mu)\ d\mu=\frac{(-1)^n\ \Gamma(n+\frac{1}{2})}{2\sqrt{\pi}\ (n+1)!}.$$

Given the definition (27), we define

$$S(\lambda;\alpha)=\sum_{n=0}^{\infty}(4n+3)(-1)^n\ B_{0,2n+1}^0\ j_{2n+1}(\lambda)\ P_{2n+1}(\cos\alpha)\tag{31}$$

so that

$$I(v;H_r)=1+\lambda\ e^{i\lambda}\{S+iS'+i\lambda^{-1}S\},\tag{32}$$

where we have simplified using $h_0(\lambda)=e^{i\lambda}/(i\lambda)$.

Proceeding formally, we substitute the known asymptotic approximation

$$j_m(\lambda) \sim \lambda^{-1} \sin\left(\lambda - \frac{1}{2} m\pi\right) \quad \text{as } \lambda \to \infty, \tag{33}$$

into Eq. (31), giving $S(\lambda;\alpha) \simeq -C(\alpha)\, \lambda^{-1} \cos\lambda$, where

$$C(\alpha) = \sum_{n=0}^{\infty} (4n+3)\, B^0_{0,2n+1}\, P_{2n+1}(\cos\alpha).$$

Hence, from Eq. (32),

$$I(v;H_r) \simeq 1 - C(\alpha) \quad \text{as } kr \to \infty.$$

But $C(\alpha)$ is a known Fourier–Legendre expansion:

$$C(\alpha) = \begin{cases} 1, & 0 < \alpha < \frac{1}{2}\pi, \\ -1, & \frac{1}{2}\pi < \alpha < \pi, \\ 0, & \alpha = 0, \ \frac{1}{2}\pi, \pi, \end{cases}$$

and is defined by periodicity for other values of α. Hence we deduce that

$$I(v;H_r) = o(1) \quad \text{as } kr \to \infty, \quad \text{for } \rho = 0 \text{ and } \quad 0 < \alpha < \tfrac{1}{2}\pi,$$

which is correct, but

$$I(v;H_r) = 1 + o(1) \quad \text{as } kr \to \infty, \quad \text{for } \rho = 0 \text{ and } \alpha = 0, \tfrac{1}{2}\pi,$$

which is incorrect. This last result follows from $C(0) = 0$, giving $S(\lambda;0) = o(\lambda^{-1})$ as $\lambda \to \infty$. In fact, we have

$$S(\lambda;0) = \sum_{n=0}^{\infty} \frac{(4n+3)\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}\,(n+1)!}\, j_{2n+1}(\lambda) = \frac{1-\cos\lambda}{\lambda}, \tag{34}$$

exactly. This can be shown either by using a formula due to Gegenbauer,[14] namely

$$\left(\frac{1}{2}\lambda\right)^{\gamma} = \sum_{n=0}^{\infty} \frac{(2n+\gamma)\,\Gamma(n+\gamma)}{n!}\, J_{2n+\gamma}(\lambda)$$

with $\gamma = -\frac{1}{2}$, or by an application of the Mellin transform with respect to λ (the resulting series can be summed using the known formula for $F(a,b;c;1)$ where F is the Gauss hypergeometric function; the sum can be inverted using the Mellin convolution theorem).

The formal calculation above shows that we cannot expect to obtain the correct result if we simply replace the spherical Bessel functions by their large-argument asymptotic approximations, at least in the special cases of grazing ($\alpha = \frac{1}{2}\pi$) and normal ($\alpha = 0$) plane waves. We examine the latter case next.

B. Another special case: $\alpha = 0$

When $\alpha = 0$, only those terms with $m = 0$ contribute, giving

$$I(v;H_r) = \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} Q^0_{ls} = \sum_{l=0}^{\infty} (2l+1)\, i^l\, b_l(kr)\, j_l(k\rho)\, P_l(\cos\Theta),$$

where

$$b_l(\lambda)=\sum_{s=0}^{\infty}(2s+1)\; i^{s-l}\; W_{ls}(\lambda)\; B_{ls}^0$$
$$=1+\lambda^2\{h_l'(\lambda)\; S_l(\lambda)-h_l(\lambda)\; S_l'(\lambda)\},\tag{35}$$

$$S_l(\lambda)=\sum_{\substack{s=0\\ s\neq l}}^{\infty}(2s+1)(-i)^{l-s+1}\; B_{ls}^0\; j_s(\lambda),$$

and we have used Eq. (27). It remains to evaluate $S_l(\lambda)$, and thence $b_l(\lambda)$ for large λ. Before doing this, let us consider the expected final answer, namely Eq. (25). From another formula due to Gegenbauer,[15] we have

$$\int_0^{\pi} J_0(w\sin\theta)\; P_n(\cos\theta)\; \sin\theta\; d\theta=\begin{cases}2\,(-1)^{n/2}\; P_n(0)\; j_n(w), & n \text{ even},\\ 0, & n \text{ odd},\end{cases}$$

whence

$$J_0(w\sin\theta)=\sum_{n=0}^{\infty}(4n+1)(-1)^n\; P_{2n}(0)\; j_{2n}(w)\; P_{2n}(\cos\theta).$$

Thus we are expecting to find that

$$b_{2l}(\lambda)=e^{i\lambda}\; P_{2l}(0)=e^{i\lambda}\;\frac{(-1)^l\;\Gamma(l+\frac{1}{2})}{\sqrt{\pi}\; l!}\quad\text{and}\quad b_{2l+1}=0.\tag{36}$$

Now, consider S_{2l+1}. From Eq. (30) with $m=0$, we have

$$S_{2l+1}(\lambda)=\frac{\Gamma(l+\frac{3}{2})}{2\pi\; l!}\sum_{n=0}^{\infty}\frac{(4n+1)\;\Gamma(n+\frac{1}{2})}{(n+l+1)(n-l-\frac{1}{2})n!}j_{2n}(\lambda).\tag{37}$$

Using

$$[\Gamma(n+a)]/[\Gamma(n+b)]\sim n^{a-b}\quad\text{as } n\to\infty,\tag{38}$$

we see that, for large n, the terms in the series (37) behave like $n^{-3/2}\,j_{2n}(\lambda)$. Given that $|J_\nu(\lambda)|\leqslant 1$ for all real λ and for all positive ν,[16] we deduce that the series (37) is absolutely and uniformly convergent. Thus we can replace $j_{2n}(\lambda)$ by its large-argument asymptotic approximation (33) to obtain

$$S_{2l+1}(\lambda)\sim\mathscr{A}_l\lambda^{-1}\sin\lambda\quad\text{as }\lambda\to\infty,$$

where

$$\mathscr{A}_l=\frac{\Gamma(l+\frac{3}{2})}{\pi\; l!}\sum_{n=0}^{\infty}\frac{(2n+\frac{1}{2})\;\Gamma(n+\frac{1}{2})}{(n+l+1)(n-l-\frac{1}{2})}\;\frac{(-1)^n}{n!}$$

is a real constant. Substituting into Eq. (35), using $h_n(\lambda)\sim\lambda^{-1}(-i)^{n+1}\,e^{i\lambda}$ as $\lambda\to\infty$, we find that

$$b_{2l+1}(\lambda)=1+(-1)^l\mathscr{A}_l+o(1)\quad\text{as }\lambda\to\infty.\tag{39}$$

Finally, we evaluate $\mathscr{A}_l$ using a contour-integral method. Splitting into partial fractions, the series can be written as

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n+l+1} \frac{(-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n-l-\frac{1}{2}} \frac{(-1)^n}{n!}. \tag{40}$$

Then, consider integrating the function $\Gamma(\frac{1}{2}-z)\ \Gamma(z)/(l-z+1)$ around a large circular contour in the complex z-plane. $\Gamma(z)$ has simple poles at $z=-n$, $n=0,1,2,\ldots,$ with residue $(-1)^n/n!$. Thus the residues at the poles of $\Gamma(z)$ and $\Gamma(\frac{1}{2}-z)$ give rise to the first and second sums, respectively, in Eq. (40). The residue from the pole at $z=l+1$ is

$$-\Gamma(-\tfrac{1}{2}-l)\ \Gamma(l+1)=(l+\tfrac{1}{2})^{-1} l!\ \Gamma(\tfrac{1}{2}-l).$$

Combining Eq. (38) with the formula

$$\Gamma(\tfrac{1}{2}+z)\ \Gamma(\tfrac{1}{2}-z)=\pi \sec \pi z \tag{41}$$

shows that the contribution from integrating around the large circular contour vanishes as the circle expands to infinity. Hence the calculus of residues gives

$$\frac{\pi l!}{\Gamma(l+\frac{3}{2})} \mathscr{A}_l + \frac{l!\ \Gamma(\frac{1}{2}-l)}{l+\frac{1}{2}}=0$$

whence $\mathscr{A}_l=(-1)^{l+1}$, using Eq. (41). Hence Eq. (39) gives $b_{2l+1}(\lambda)\to 0$ as $\lambda\to\infty$, as predicted by Eq. (36).

Next, consider S_{2l}. From Eq. (29) with $m=0$, we have

$$S_{2l}(\lambda)=\frac{\Gamma(l+\frac{1}{2})}{2\pi\ l!} \sum_{n=0}^{\infty} \frac{(4n+3)\ \Gamma(n+\frac{3}{2})}{(n+l+1)(n-l+\frac{1}{2})\ n!} j_{2n+1}(\lambda). \tag{42}$$

The terms in this series decay like $n^{-1/2}\ j_{2n+1}(\lambda)$ as $n\to\infty$, which is not fast enough to guarantee uniform convergence. However, we have

$$S_0(\lambda)=S(\lambda;0)=\lambda^{-1}(1-\cos\lambda)$$

from Eq. (34), so that

$$S_{2l}(\lambda)=\frac{\Gamma(l+\frac{1}{2})}{\sqrt{\pi}\ l!} S_0(\lambda)+T_{2l}(\lambda),$$

where

$$T_{2l}(\lambda)=\frac{\Gamma(l+\frac{1}{2})}{2\pi\ l!} \sum_{n=0}^{\infty} \frac{(4n+3)\ \Gamma(n+\frac{3}{2})}{n!} \left\{ \frac{1}{(n+l+1)(n-l+\frac{1}{2})} - \frac{1}{(n+1)(n+\frac{1}{2})} \right\} j_{2n+1}(\lambda),$$

this series is absolutely and uniformly convergent, whence $T_{2l}(\lambda)\sim \mathscr{B}_l \lambda^{-1}\cos\lambda$ as $\lambda\to\infty$, where

$$\mathscr{B}_l=\frac{\Gamma(l+\frac{1}{2})}{\pi\ l!} \sum_{n=0}^{\infty} \Gamma(n+\tfrac{3}{2}) \left\{ \frac{1}{n+1} + \frac{1}{n+\frac{1}{2}} - \frac{1}{n+l+1} - \frac{1}{n-l+\frac{1}{2}} \right\} \frac{(-1)^n}{n!}$$

is a real constant. Substituting back into Eq. (35), we find that

$$b_{2l}(\lambda)=\frac{(-1)^l\ \Gamma(l+\frac{1}{2})}{\sqrt{\pi}\ l!} e^{i\lambda}+1-\frac{(-1)^l\ \Gamma(l+\frac{1}{2})}{\sqrt{\pi}\ l!}+(-1)^l \mathscr{B}_l+o(1) \tag{43}$$

as $\lambda\to\infty$. The first term on the right-hand side is in accord with the prediction (36). The next three terms sum to zero, as expected. This can be shown by evaluating $\mathscr{B}_l$ using a contour-integral method (as for $\mathscr{A}_l$): use

$$\Gamma\left(\frac{3}{2}-z\right)\ \Gamma(z)\{(1-z)^{-1}-(l-z+1)^{-1}\}.$$

This completes the proof for $\alpha=0$ using an expansion method. It demonstrates that the expansion method is much more complicated to use (and more subtle, due to nonuniform convergence of one of the component series) than an approach based on the method of stationary phase.

V. ASYMPTOTIC BEHAVIOR OF $I(u;H_r)$

When a plane wave is reflected by a rough surface S, we can use the angular-spectrum representation (13) for the reflected field in $z>0$. Thus we have

$$I(u;H_r)=I(u_{\text{pr}};H_r)+I(u_{\text{ev}};H_r)+I(u_{\text{con}};H_r).$$

For $I(u_{\text{ev}};H_r)$, with u_{ev} defined by Eq. (14), we have

$$I(w;H_r)\simeq iB\lambda\ e^{i\lambda}\int_{\mathscr{D}} g(\theta,\phi)\ e^{i\lambda F(\theta,\phi)}\ d\theta\ d\phi,$$

where w is defined by Eq. (16), $\lambda=kr$, $B=-\frac{1}{2}/\pi$,

$$F(\theta,\phi)=[\mu\cos\phi+\nu\sin\phi]\sin\theta+i\cos\theta\ \sqrt{\kappa^2-1},$$

$$g(\theta,\phi)=[1-F(\theta,\phi)]\ \exp\{-ik\mathbf{y}\cdot\hat{\mathbf{x}}\}\ \sin\theta,$$

and $\mathscr{D}$ is defined below Eq. (22) with $\theta_0=\frac{1}{2}\pi$. As $\kappa>1$, grad $F\neq\mathbf{0}$ in $\mathscr{D}$, whence the vector

$$\mathbf{f}(\theta,\phi)=g(\text{grad } F)/|\text{grad } F|^2$$

is well defined. Then, use of the identity

$$ge^{i\lambda F}=(i\lambda)^{-1}\{\text{div } (e^{i\lambda F}\mathbf{f})-e^{i\lambda F}\text{div } \mathbf{f}\},$$

together with the divergence theorem in $\mathscr{D}$, shows that $I(w;H_r)=o(1)$ as $kr\to\infty$.[17] Hence from Eq. (14),

$$I(u_{\text{ev}};H_r)\to 0 \quad \text{as } r\to\infty.$$

As $\mathscr{C}$ in Eq. (15) is continuous, we know that u_{con} satisfies the Sommerfeld radiation condition.[18] It follows that

$$I(u_{\text{con}};H_r)\to 0 \quad \text{as } r\to\infty.$$

Finally, consider $I(u_{\text{pr}};H_r)$. If $0\leq\alpha_n<\frac{1}{2}\pi$, the results of the previous sections are immediately applicable. For a typical grazing-wave component v, propagating at some angle β, our radiation condition implies that we consider $I(v;H_r^m)$, with m chosen according to Eq. (17). The stationary-phase analysis of Sec. III shows that $I(v;H_r^m)\to 0$ as $r\to\infty$. For the other three values of m, v will either be incoming through the corresponding H_r^m, and so its contribution must be discarded; or the contribution is negligible, using the method of stationary phase again.

Thus, in summary, let us extract the normal waves $(\alpha=0)$ from u_{pr}, and write

$$u_{\text{pr}}(r,\theta,\phi)=A_0\ e^{ikr\cos\theta}+\text{other propagating waves with } 0<\alpha<\tfrac{1}{2}\pi, \tag{44}$$

where the coefficient A_0 is unknown. Then

$$I(u;H_r) \sim A_0\, e^{ikr}\, \mathscr{U}_0(P) \quad \text{as } r\to\infty, \tag{45}$$

where

$$\mathscr{U}_0(P) \equiv \mathscr{U}_0(\rho,\Theta,\Phi) = J_0(k\rho\sin\Theta). \tag{46}$$

VI. ASYMPTOTIC BEHAVIOR OF $I(u;T_r)$

The truncated cylindrical surface T_r is defined by Eq. (18). A point $q\in T_r$, with position vector $\mathbf{x}$, has cylindrical polar coordinates (r,ϕ,z). Then, for large r,

$$|\mathbf{x}-\mathbf{y}| \simeq r - \rho\sin\Theta\,\cos(\phi-\Phi),$$

where the fixed point P has position vector $\mathbf{y}$ and spherical polar coordinates (ρ,Θ,Φ) defined by Eq. (24). Hence as $\partial/\partial n = \partial/\partial r$ on T_r,

$$u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \simeq \frac{B}{r}\, e^{ikr}\left(iku - \frac{\partial u}{\partial r}\right)\exp\{-ik\rho\sin\Theta\,\cos(\phi-\Phi)\},$$

where $B = -\tfrac{1}{2}/\pi$. Let

$$\mathscr{E}(r,\phi) = \int_s^0 \left(\frac{\partial u}{\partial r} - iku\right) dz,$$

where the lower limit is $s(r\cos\phi, r\sin\phi)$. Then

$$I(u;T_r) \sim \frac{1}{2\pi}\, e^{ikr}\int_{-\pi}^{\pi} \mathscr{E}(r,\phi)\, \exp\{-ik\rho\sin\Theta\,\cos(\phi-\Phi)\}\, d\phi \quad \text{as } r\to\infty.$$

When is it true that $I(u;T_r)\to 0$ as $r\to\infty$? One sufficient condition is that

$$s\to 0 \;\text{ as }\; r\to\infty, \quad \text{for all } \phi,$$

this means that the rough surface approaches the flat plane $z=0$ at large distances, in all directions.

Another sufficient condition is that

$$\frac{\partial u}{\partial r} - iku = o(1) \;\text{ as } r\to\infty, \quad \text{for all } \phi \;\text{ and }\; s\leqslant z\leqslant 0,$$

so that u satisfies a form of radiation condition in directions parallel to $z=0$. In particular, this will be the case if u comprises outgoing cylindrical or spherical waves.

Next, consider a typical plane wave v, defined by Eq. (12); thus let

$$v = e^{ikr\sin\alpha\,\cos\phi}\, e^{ikz\cos\alpha} \;\text{ with }\; 0<\alpha\leqslant \tfrac{1}{2}\pi.$$

Then, a Fourier expansion in ϕ leads to Fourier components involving $J_n(kr\sin\alpha)$ and $J_n'(kr\sin\alpha)$. As $J_n(w) = O(w^{-1/2})$ as $w\to\infty$, we deduce that $I(v;T_r)\to 0$ as $r\to\infty$. This result includes grazing waves, but not normal waves for which $\alpha=0$. When $\alpha=0$, $v=e^{ikz}$ whence $\partial v/\partial r = 0$ and the integration over z in $\mathscr{E}$ is trivial. The result is

$$I(v;T_r) \sim \frac{1}{2\pi}\, e^{ikr}\int_{-\pi}^{\pi} (e^{iks}-1)\, \exp\{-ik\rho\sin\Theta\,\cos(\phi-\Phi)\}\, d\phi$$

$$= \frac{1}{2\pi}\, e^{ikr}\int_{-\pi}^{\pi} e^{iks}\, \exp\{-ik\rho\sin\Theta\,\cos(\phi-\Phi)\}\, d\phi - e^{ikr}\, J_0(k\rho\sin\Theta) \;\text{ as } r\to\infty.$$

Thus, in summary, suppose that for $s\leqslant z\leqslant 0$ and large r,

$$u=A_0\ e^{ikz}+\tilde{u},$$

where the coefficient A_0 is the same as in Eq. (44) and $\tilde{u}$ is such that $I(\tilde{u};T_r)\rightarrow 0$ as $r\rightarrow\infty$. Then

$$I(u;T_r)\sim A_0\ e^{ikr}\{\mathscr{U}_s(r;P)-\mathscr{U}_0(P)\}\quad \text{as } r\rightarrow\infty, \tag{47}$$

where $\mathscr{U}_0$ is defined by Eq. (46) and

$$\mathscr{U}_s(r;P)\equiv\mathscr{U}_s(r;\rho,\Theta,\Phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{iks}\ \exp\{-ik\rho\sin\Theta\ \cos(\phi-\Phi)\}\ d\phi; \tag{48}$$

the dependence on r comes from the exponent $s=s(r\cos\phi,r\sin\phi)$.

VII. BOUNDARY INTEGRAL EQUATIONS

In Sec. II, we used Green's theorem to obtain the integral representation

$$2u(P)=\int_{S_r}\left\{u(q)\ \frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\text{inc}}}{\partial n}\ G(P,q)\right\}\ dS_q+I(u;H_r)+I(u;T_r), \tag{49}$$

when $P\in D_r$, the region bounded by the hemisphere H_r, the truncated rough surface S_r, and the truncated circular cylinder T_r. Note that the left-hand side of this equation does not depend on r, so that the right-hand side of the equation must have a limit as $r\rightarrow\infty$. Before taking this limit, it is instructive to consider a very simple example.

A. An example

Consider a plane wave *normally* incident upon a *flat* surface at $z=-h$. Thus

$$u_{\text{inc}}=e^{-ikz}\quad\text{and } u=A_0\ e^{ikz}\quad\text{with } A_0=e^{2ikh}.$$

This is the exact solution. Let us see how this solution is reconstructed by the representation (49). For simplicity, we take P at the origin; this will permit all the integrals to be evaluated exactly (without any asymptotic approximations). Thus, from Eq. (26), we have

$$I(u;H_r)=A_0\ e^{ikr}. \tag{50}$$

On S_r, we have $u=A_0\ e^{-ikh}$, $\partial u/\partial n=-iku$,

$$G=BR_\sigma^{-1}e^{ikR_\sigma},\quad \partial G/\partial n=hR_\sigma^{-2}(ik-R_\sigma^{-1})G$$

and $R_\sigma=\sqrt{\sigma^2+h^2}$, whence the integral over S_r is

$$\begin{aligned}
I(u;S_r)&=2\pi A_0B\ e^{-ikh}\int_0^r e^{ikR_\sigma}\{ik+hR_\sigma^{-1}(ik-R_\sigma^{-1})\}\frac{\sigma d\sigma}{R_\sigma}\\
&=-A_0\ e^{-ikh}\int_h^R e^{ikt}\{ik+ht^{-1}(ik-t^{-1})\}dt\\
&=-A_0\ e^{-ikh}\ [e^{ikt}+ht^{-1}\ e^{ikt}]_{t=h}^R\\
&=2A_0-A_0\left(1+\frac{h}{R}\right)e^{ik(R-h)},
\end{aligned} \tag{51}$$

where $R=\sqrt{r^2+h^2}$.

On T_r, we have $u=A_0\ e^{ikz}$ and $\partial u/\partial n=\partial u/\partial r=0$, whence

$$I(u;T_r)=2\pi A_0 Br\int_{-h}^{0} e^{ikz}\frac{\partial}{\partial r}\left(\frac{e^{ikR_z}}{R_z}\right)dz$$
$$=-A_0\,[(1-zR_z^{-1})\;e^{ik(R_z+z)}]_{z=-h}^{0}$$
$$=-A_0\;e^{ikr}+A_0\left(1+\frac{h}{R}\right)e^{ik(R-h)},\tag{52}$$

where $R_z=\sqrt{r^2+z^2}$.

Adding Eqs. (50)–(52), we see that their sum is exactly $2A_0$, which is $2u(P)$ evaluated at the origin. Note that, as $r\to\infty$,

$$I(u;S_r)\sim 2A_0-A_0\;e^{ik(r-h)},$$
$$I(u;T_r)\sim -A_0\;e^{ikr}+A_0\;e^{ik(r-h)},$$

and $I(u;H_r)$ does not simplify further. Thus the boundary integral over the truncated rough surface does not have a limit as $r\to\infty$. Moreover, the integral over the truncated cylinder T_r does not have a limit as $r\to\infty$, and it is not negligible. This is a genuine three-dimensional effect, which is not seen in the two-dimensional case.[2]

B. Taking the limit: A new finite-part integral

Letting $r\to\infty$ in Eq. (49), we obtain

$$2u(P)=\oint_S\left\{u(q)\;\frac{\partial G}{\partial n_q}(P,q)+\frac{\partial u_{\text{inc}}}{\partial n}\;G(P,q)\right\}\;dS_q\,,\quad P\in D_\infty\,,\tag{53}$$

where D_∞ is the unbounded region $z>s$,

$$\oint_S\cdots dS=\lim_{r\to\infty}\left\{\int_{S_r}\cdots dS+A_0\;e^{ikr}\;\mathscr{U}_s\right\}\tag{54}$$

and $\mathscr{U}_s$ is defined by Eq. (48). Note that the terms involving $\mathscr{U}_0$ in Eqs. (45) and (47) cancel.

The definition (54) is a nonstandard form of finite-part integral. It reduces to the standard definition of a principal-value integral at infinity if the coefficient A_0 vanishes; in other words, the standard definition is only appropriate if the scattered field does *not* include a discrete plane wave propagating up the z-axis. For a general rough surface, we do not know *a priori* whether $A_0=0$ or not.

We can express A_0 in terms of the values of $u(r,\theta,\phi)$ on $z=0$ $(\theta=\frac{1}{2}\pi)$. Thus write

$$u\left(r,\frac{1}{2}\pi,\phi\right)=\sum_{n=-\infty}^{\infty}u_n(r)\;e^{in\phi}.$$

Hence, as a plane wave propagating along the z-axis is constant on planes of constant z, we have

$$A_0=\lim_{r\to\infty}u_0(r)=\frac{1}{2\pi}\lim_{r\to\infty}\int_{-\pi}^{\pi}u\left(r,\frac{1}{2}\pi,\phi\right)\;d\phi.$$

However, this formula is not useful, as we do not know u on $z=0$. What we would like is a similar formula, involving the boundary values of u on S.

Let us combine the identity

$$A_0=\frac{2}{r^2}\int_0^r u_0(\sigma)\;\sigma\;d\sigma+\frac{2}{r^2}\int_0^r[A_0-u_0(\sigma)]\sigma\;d\sigma$$

with an assumption that $u_0(r)=A_0+O(r^{-\delta})$ as $r\to\infty$, where $\delta>0$. Then, the second integral is $o(1)$ as $r\to\infty$, whence

$$A_0=\lim_{r\to\infty}\frac{2}{r^2}\int_0^r u_0(\sigma)\ \sigma\ d\sigma$$

$$=\frac{1}{\pi}\lim_{r\to\infty}\frac{1}{r^2}\int_{\mathscr{D}_r} u\left(\sigma,\frac{1}{2}\pi,\phi\right)\ \sigma\ d\sigma\ d\phi, \tag{55}$$

where $\mathscr{D}_r$ is the circular disk on $z=0$ with $x^2+y^2<r^2$. This is another formula for A_0 in terms of u on $z=0$. However, this formula involves u everywhere on $\mathscr{D}_r$, so we can relate it to an integral over S_r, using Green's theorem.

Consider the truncated circular cylinder V_r with axis along the z-axis and radius r, between $z=0$ and $z=s$. Thus one end of V_r is S_r and the other is the circular disk $\mathscr{D}_r$. Apply Green's theorem in V_r to u and $\sin kz$, giving

$$k\int_{\mathscr{D}_r} u\ dS+\int_{S_r}\left\{u\ \frac{\partial}{\partial n}(\sin kz)-\frac{\partial u}{\partial n}\sin kz\right\}\ dS-\int_{T_r}\frac{\partial u}{\partial n}\sin kz\ dS=0,$$

where T_r, defined by Eq. (18), is the curved part of V_r on which $x^2+y^2=r^2$. The third integral is $O(r)$ as $r\to\infty$, as it is bounded by $2\pi rM\max(s)$, where M is a bound on $|\partial u/\partial r|$. Hence, comparing with Eq. (55), and making use of Eq. (6), we see that

$$A_0=\lim_{r\to\infty}\mathscr{A}(r)\quad\text{where}\quad \mathscr{A}(r)=\frac{-1}{\pi kr^2}\int_{S_r}\left\{u\ \frac{\partial}{\partial n}(\sin kz)+\frac{\partial u_{\text{inc}}}{\partial n}\sin kz\right\}\ dS, \tag{56}$$

which is a formula for A_0 in terms of the boundary values of u.

For the example in Sec. VII A, one can check that Eq. (56) does produce the correct value for A_0.

C. A boundary integral equation for the scattered field

Letting $P\to p\in S$ in Eq. (53) gives

$$u(p)=\oint_S\left\{u(q)\ \frac{\partial G}{\partial n_q}(p,q)+\frac{\partial u_{\text{inc}}}{\partial n}\ G(p,q)\right\}\ dS_q,\quad p\in S. \tag{57}$$

This would be a boundary integral equation for u on S if we knew A_0. A_0 is defined by the formula (56), which requires $u(q)$ for $q\in S$; but we can only find $u(q)$ by solving Eq. (57).

Two possible ways to proceed are as follows. First, one could *assume* that $A_0=0$. In that case, the integral in Eq. (57) becomes an ordinary principal-value integral at infinity. However, the integral will diverge if it turns out that $A_0\neq 0$; see the example in Sec. VII A.

Second, from Eq. (56), we could replace the constant A_0 by $\mathscr{A}(r)$, where r is the radius of the truncated rough surface. This gives

$$u(p)=\int_{S_r}\left\{u(q)\ \frac{\partial\tilde{G}}{\partial n_q}(r;p,q)+\frac{\partial u_{\text{inc}}}{\partial n}\ \tilde{G}(r;p,q)\right\}dS_q,\quad p\in S_r, \tag{58}$$

where

$$\tilde{G}(r;P,Q)=G(P,Q)-(\pi kr^2)^{-1}\ e^{ikr}\ \mathscr{U}_s(r;P)\ \sin kz, \tag{59}$$

$Q=(x,y,z)$ and $\mathscr{U}_s$ is defined by Eq. (48). Thus we have a plausible boundary integral equation for u on S_r. Note that, in any numerical treatment, the rough surface would have to be truncated to S_r, so our ''plausible'' integral equation seems to be optimal in some sense. It remains to carry out numerical experiments.

VIII. CONCLUSION

In this paper we have considered the three-dimensional problem of the reflection of a plane wave by an infinite two-dimensional rough surface, defined by $z=s(x,y)$ with $-h\leqslant s\leqslant 0$. We have shown that the derivation of a boundary integral equation for this problem, akin to the Helmholtz integral equation for scattering by a bounded obstacle, is by no means straightforward. In particular, if the scattered field includes a plane wave propagating along the z-axis away from the rough surface (''normal waves''), then the usual Helmholtz integral equation is not valid: the boundary integral diverges. We have offered a modified integral equation which reduces to the standard Helmholtz integral equation when normal waves are absent.

The situation just described is unsatisfactory, even though the mathematical difficulty may be overcome. Indeed, this difficulty is due entirely to the unphysical problem posed at the outset: *plane-wave* reflection by an *infinite* rough surface. Clearly, we can realize neither a plane wave nor an infinite rough surface. Moreover, the mathematical difficulty disappears if we consider either point-source insonification or a finite patch of roughness on an otherwise flat surface.[2]

Several papers have been written in which a Helmholtz integral equation was used to provide ''exact'' or ''benchmark'' numerical solutions for plane-wave reflection by an infinite one-dimensional rough surface, the purpose being to validate various approximate theories.[19] We have shown here that the same integral equation cannot be used for two-dimensional surfaces, in general, unless one changes the problem, as suggested in the previous paragraph.

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[1] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory* (Wiley, New York, 1983).
[2] J. A. DeSanto and P. A. Martin, ''On the derivation of boundary integral equations for scattering by an infinite one-dimensional rough surface,'' J. Acoust. Soc. Am. **102**, 67–77 (1997).
[3] Sec. I A of Ref. 2.
[4] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1944), 2nd ed.
[5] P. C. Clemmow, *The Plane Wave Spectrum Representation of Electromagnetic Fields* (Pergamon, Oxford, 1966).
[6] J. A. DeSanto and P. A. Martin, ''On angular-spectrum representations for scattering by infinite rough surfaces,'' Wave Motion **24**, 421–433 (1996).
[7] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).
[8] Sec. 8.4 of Ref. 7.
[9] Eqs. 42, 44, and 46 in Sec. 8.4 of Ref. 7.
[10] Eq. (8.4.46) of Ref. 7.
[11] Sec. 9.6 of Ref. 7.
[12] See Exercise 9.15(b) in Ref. 7.
[13] A. Hulme, ''The wave forces acting on a floating hemisphere undergoing forced periodic oscillations,'' J. Fluid Mech. **121**, 443–463 (1982).
[14] Sec. 5.2 of Ref. 4.
[15] Eq. 2 on p. 379 of Ref. 4, with $\nu=\frac{1}{2}$ and $\psi=\frac{1}{2}\pi$.
[16] See p. 406 of Ref. 4.
[17] See Sec. 8.2 of Ref. 7 for details.
[18] G. C. Sherman, J. J. Stamnes, and É. Lalor, ''Asymptotic approximations to angular-spectrum representations,'' J. Math. Phys. **17**, 760–776 (1976).
[19] See Ref. 2 for a discussion and references.