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On potential flow past wrinkled discs

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The problem of three-dimensional potential flow past a thin rigid screen is reduced to a hypersingular boundary integral equation. This equation is then projected onto a flat reference screen, which is taken to be a circular disc. Solutions are obtained for screens that are perturbations from the disc. Explicit results are obtained for inclined elliptical screens and for spherical caps, correct to second order.

Keywords: perturbation theory; hypersingular integral equations; cracks

1. Introduction

Potential flow past a rigid sphere of radius a is a textbook problem. It can be solved exactly using the method of separation of variables. However, this method is not immediately applicable when the sphere is perturbed so that the new boundary is given by

$$S : r = a(1 + \varepsilon f(\theta, \varphi)), \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi < 2\pi,$$

where (r, θ, φ) are spherical polar coordinates, f is a given function and ε is a small parameter. Nevertheless, such problems can be solved approximately, exploiting the size of ε. Conventionally, this is done by the 'boundary perturbation technique', in which the boundary condition on S is Taylor-expanded about the unperturbed boundary $r = a$. Then, an expansion of the potential as

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots$$

leads to a sequence of boundary-value problems for ϕ_n in the unperturbed domain $(r > a)$; ϕ_0 is the unperturbed solution and subsequent ϕ_n are forced by ϕ_m with $m < n$.

The boundary perturbation technique was used by Erma (1963) for the above potential problem. However, the idea itself is much older. For example, it is used in the theory of small-amplitude water waves, wherein the nonlinear boundary conditions on the (unknown) free surface $z = F(x, y)$ are 'linearized' about the mean free surface $z = 0$ (Stoker 1957, ch. 2). Another example occurs in the theory of scattering by rough surfaces, wherein an incident acoustic wave is reflected by a corrugated surface $z = F(x, y)$ (Ogilvy 1991, ch. 3). The work of G. H. Darwin is also of interest. He wrote five papers in 1879 on the motions of an incompressible viscous fluid inside a perturbed sphere (Darwin 1908). He neglected the inertia terms in the Navier–Stokes equations (Stokes's approximation) giving a linear interior boundary-value problem, which he solved for 'small deviations from sphericity', to first order in ε. He argued that one could take account of these deviations by imposing certain tractions on $r = a$, and then solved the corresponding problem inside the sphere. His method is applicable to any f although he was mainly interested in spheroidal surfaces.†

Two remarks concerning the boundary perturbation technique are worth noting. First, both the perturbed surface S and the given data on S are assumed to be smooth. Second, the technique implicitly assumes that the unknown potential can be continued analytically through S, as necessary, as far as the unperturbed boundary. For further information, see the paper by Lebovitz (1982).

In this paper, we develop an alternative method. First, we reduce the boundary-value problem to a boundary integral equation over S. We rewrite this equation by projecting onto the unperturbed (reference) surface. At this stage, we have an exact reformulation of the original boundary-value problem. Next, we introduce perturbation expansions, leading to a sequence of boundary integral equations, $L\phi_n = b_n$, $n = 0, 1, 2, \ldots$. Each integral equation involves the same operator L but different forcing functions b_n; L corresponds to the unperturbed boundary-value problem. Any convenient method can be used to invert L.

Our development is concerned with problems where the obstacle is *thin*. Thus, we replace the closed surface S by an open surface Ω. We suppose that Ω is a non-planar perturbation of a circular disc D. The derivation of a sequence of boundary-value problems for such geometries is difficult, due to the presence of the edge of Ω. This approach has been attempted by Jansson (1996). He imagined Ω to be a piece of an infinite interface separating two half-spaces, and then perturbed this transmission problem about the flat interface. However, the edge behaviour of the solution induces singularities at the edge of D, whose strength increases with the perturbation order. (This is apparent in the terms involving $\partial^2 u_0^n/\partial z^2$ in his equation (8c).)

Another approach was taken by Beom & Earmme (1992). They considered axisymmetric perturbations, and assumed that ϕ could be written as

$$\phi(x, y, z) = \int_0^\infty A_\pm(\xi) J_0(\xi r) \mathrm{e}^{\mp\xi z}\,\mathrm{d}\xi \quad \text{for } \pm z \geqslant F, \tag{1.1}$$

where $r = \sqrt{(x^2+y^2)}$, the functions $A_\pm$ are to be determined, and J_0 is a Bessel function. Such representations are commonly used for mixed boundary-value problems involving *flat* circular discs (Sneddon 1966, ch. 3). However, if we suppose without loss of generality that part of Ω is such that $F < 0$, then (1.1), with $z \geqslant F$, will diverge for those z with $0 > z \geqslant F$.

We proceed by reducing the exact boundary-value problem to a hypersingular integral equation for $[\phi]$, the discontinuity in the potential across Ω. After projection, we obtain a sequence of hypersingular integral equations of the form $Hw_n = b_n$ where Ω is given by

$$\Omega : z = \varepsilon\, f(x, y), \quad (x, y) \in D,$$
$$[\phi] = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots,$$

and H corresponds to potential flow past a rigid circular disc. We derive an explicit closed-form expression for the first-order correction w_1.

To verify the method, we derive explicit results for w_0, w_1 and w_2 for two problems, namely, an inclined flat elliptical screen and a spherical cap. In particular, we calculate the added mass for these flows, and find agreement with known exact solutions.

† I am indebted to Professor G. F. Roach for drawing my attention to Darwin's work.

Moreover, our result for uniform flow past a shallow spherical cap in any direction interpolates between the known results for axial flow and flow perpendicular to the axis of the cap.

The use of hypersingular integral equations leads to a simpler formulation than would follow from the use of regularized integral equations. This advantage should make the treatment of crack problems in elasticity feasible. Xu *et al*. (1994) have used a perturbation theory based on a regularized integral equation for a dislocation density (analogous to the tangential gradient of $[\phi]$), but they were only able to find the first-order correction (w_1) for a semi-infinite crack. We expect that our method will extend to non-planar perturbations of a penny-shaped crack; this is the subject of ongoing work. We have considered in-plane perturbations of a penny-shaped crack elsewhere (Martin 1995, 1996).

2. Formulation

We consider potential flow past a thin rigid screen Ω; we model the screen as a smooth simply connected bounded surface with a smooth edge $\partial\Omega$. Thus, the problem is to solve Laplace's equation in three dimensions,

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

subject to

$$\frac{\partial\phi}{\partial n} + \frac{\partial\phi_0}{\partial n} = 0, \quad \text{on } \Omega, \tag{2.1}$$

and $\phi = O(r^{-1})$ as $r \to \infty$, where $r^2 = x^2 + y^2 + z^2$, ϕ_0 is the velocity potential of the given ambient flow, and $\partial/\partial n$ denotes normal differentiation.

Exact solutions for ϕ are known when Ω is a flat circular disc, a flat elliptical screen and a spherical cap. These solutions will be mentioned later.

We shall be interested mainly in situations where the ambient flow is uniform, so that

$$\phi_0(x, y, z) = U(x \sin\beta - z\cos\beta), \tag{2.2}$$

where U and β are given constants.

(*a*) *The added mass*

When a rigid body is in irrotational motion through an incompressible fluid of density ρ, the kinetic energy of the fluid motion is given by (Lamb 1932, § 44)

$$T = \tfrac{1}{2}\rho \int_S \phi \frac{\partial\phi_0}{\partial n}\, \mathrm{d}S. \tag{2.3}$$

Here, S is the surface of the body and $\partial/\partial n$ denotes normal differentiation on S, in the direction from S into the fluid.

Now, suppose that S degenerates into a thin body with zero volume, namely Ω. Denote the two sides of Ω by Ω^+ and Ω^-, and define the unit normal vector on Ω, $\boldsymbol{n}$, to point from Ω^+ into the fluid. Finally, define the discontinuity in ϕ across Ω by

$$[\phi(q)] = \lim_{Q\to q^+} \phi(Q) - \lim_{Q\to q^-} \phi(Q),$$

where $q \in \Omega$, $q^{\pm} \in \Omega^{\pm}$ and Q is a point in the fluid. Then, (2.3) becomes

$$T = \frac{1}{2}\rho \int_{\Omega} [\phi] \frac{\partial \phi_0}{\partial n} \, \mathrm{d}S. \tag{2.4}$$

For a body translating in a straight line with speed U, we can write

$$T = \tfrac{1}{2} M' U^2, \tag{2.5}$$

where M' is the *added mass* of the body: if its actual mass is M, the total kinetic energy of the system (body plus fluid) is $\frac{1}{2}(M + M')U^2$.

The added mass is known for a few three-dimensional bodies. For example, if S is an ellipsoid, one can obtain M' by applying the method of separation of variables in ellipsoidal coordinates to Laplace's equation. In particular, for a solid sphere of radius a (Lamb 1932, § 92)

$$M' = \tfrac{2}{3}\pi\rho a^3, \tag{2.6}$$

whereas for a circular disc of radius c, translating perpendicular to its plane (Lamb 1932, § 108)

$$M' = \tfrac{8}{3}\rho c^3. \tag{2.7}$$

Further results are known for a thin rigid spherical cap. In spherical polar coordinates (r, θ, φ), the cap is defined by

$$r = a, \quad 0 \leqslant \theta \leqslant \alpha, \quad 0 \leqslant \varphi < 2\pi.$$

Then, for the axisymmetric problem of a cap translating along its axis, Collins (1959) has shown that

$$M' = \rho a^3 (2\alpha - \sin 2\alpha + \tfrac{4}{3}\sin^3 \alpha). \tag{2.8}$$

One can check that (2.8) reduces to (2.7) when $\alpha \to 0$ and $a \to \infty$ in such a way that $a\alpha \to c$ (see (6.5)). When $\alpha \to \pi$, the cap closes up and (2.8) gives

$$M' = 2\pi\rho a^3 = \tfrac{4}{3}\pi\rho a^3 + \tfrac{2}{3}\pi\rho a^3;$$

the first term is the mass of the enclosed fluid whereas the second term gives the added mass for a solid sphere, in agreement with (2.6).

Collins (1961) has also obtained M' for a cap moving *perpendicular* to its axis, in the direction $\varphi = 0$. For this non-axisymmetric problem, he finds that

$$M' = \rho a^3 \left\{ \frac{(2\alpha - \sin 2\alpha)(2\alpha + 4\sin\alpha + \sin 2\alpha)}{2(\alpha + \sin\alpha)} - \tfrac{8}{3}\sin^3\alpha \right\}. \tag{2.9}$$

3. Governing integral equation

(a) Integral representation

For any harmonic function ϕ, satisfying $\phi = O(r^{-1})$ as $r \to \infty$, we have the integral representation

$$\phi(P) = \frac{1}{4\pi} \int_S \left\{ \phi(q) \frac{\partial}{\partial n_q} G(P, q) - G(P, q) \frac{\partial \phi}{\partial n} \right\} \mathrm{d}S_q \tag{3.1}$$

where $G(P,q) = |\boldsymbol{r} - \boldsymbol{q}|^{-1}$, $q \in S$ has position vector $\boldsymbol{q}$ with respect to an origin O, and P has position vector $\boldsymbol{r}$ and spherical polar coordinates (r, θ, φ). Then

$$\phi = \frac{1}{4\pi r}\int_S \frac{\partial \phi_0}{\partial n}\,\mathrm{d}S + \frac{1}{4\pi r^3}\boldsymbol{r}\cdot\int_S\left(\phi\boldsymbol{n} + \frac{\partial\phi_0}{\partial n}\boldsymbol{q}\right)\mathrm{d}S_q + O(r^{-3}),$$

as $r \to \infty$. The first integral on the right-hand side vanishes as $\nabla^2\phi_0 = 0$ inside S. Hence

$$\phi(r,\theta,\varphi) \sim r^{-2}(g_0\cos\theta + g_1\sin\theta\,\cos\varphi + g_2\sin\theta\,\sin\varphi) \quad \text{as } r\to\infty, \tag{3.2}$$

where g_0, g_1 and g_2 do not depend on θ or φ. In particular,

$$g_0 = \frac{1}{4\pi}\int_S\left\{\phi(\boldsymbol{n}\cdot\hat{\boldsymbol{z}}) + \frac{\partial\phi_0}{\partial n}(\boldsymbol{q}\cdot\hat{\boldsymbol{z}})\right\}\mathrm{d}S,$$

where $\hat{\boldsymbol{z}}$ is a unit vector along the positive z-axis.

Next, apply Green's theorem to ϕ and ϕ_0 in the region between S and S_∞, a large sphere of radius r and centre O. In the limit as $r \to \infty$, this gives

$$M' + M_\mathrm{d} = -4\pi\rho g_0/U, \tag{3.3}$$

where M_d is the mass of fluid displaced by the body. This result is general, being valid for uniform flow past an arbitrary body S. It is a special case of results given by Lamb (1932, § 121a); see also Newman (1977, § 4.14). In particular, when S degenerates into a thin body with zero volume, $M_\mathrm{d} = 0$. Thus, we see that the added mass is simply related to the far-field coefficient g_0.

Now, for a thin screen Ω, (3.1) reduces to

$$\phi(P) = \frac{1}{4\pi}\int_\Omega [\phi(q)]\frac{\partial}{\partial n_q}G(P,q)\,\mathrm{d}S_q. \tag{3.4}$$

To be more explicit, we suppose that the surface Ω is given by

$$\Omega : z = F(x,y), \quad (x,y)\in D,$$

where D is the *unit disc* in the xy-plane. We define a normal vector to Ω by

$$\boldsymbol{N} = (-\partial F/\partial x, -\partial F/\partial y, 1),$$

and then $\boldsymbol{n} = \boldsymbol{N}/|\boldsymbol{N}|$ is a unit normal vector. Suppose that P and $q \in \Omega$ are at (x_0, y_0, z_0) and (x, y, z), respectively. Let

$$[\phi(q)] = w(x,y).$$

Then, we find that (3.4) becomes

$$\phi(x_0,y_0,z_0) = \frac{1}{4\pi}\int_D w(x,y)\boldsymbol{N}(q)\cdot\boldsymbol{R}_2\frac{\mathrm{d}A}{R_2^3},$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, $\boldsymbol{R}_2 = (x_0 - x, y_0 - y, z_0 - F(x,y))$ and $R_2 = |\boldsymbol{R}_2|$.

(*b*) *Integral equation*

Application of the boundary condition (2.1) to (3.4) gives

$$\frac{1}{4\pi}\,⨎_\Omega [\phi(q)]\frac{\partial^2}{\partial n_p\partial n_q}G(p,q)\,\mathrm{d}S_q = -\frac{\partial\phi_0}{\partial n_p}, \quad p\in\Omega, \tag{3.5}$$

where the integral must be interpreted in the finite-part sense. Equation (3.5) is the governing hypersingular integral equation for $[\phi]$; it is to be solved subject to the edge condition

$$[\phi(q)] = 0 \quad \text{for all } q \in \partial\Omega.$$

Projecting onto D, (3.5) becomes

$$\frac{1}{4\pi} ⨍_D K(x_0, y_0; x, y) w(x, y)\, \mathrm{d}A = b(x_0, y_0), \quad (x_0, y_0) \in D, \tag{3.6}$$

where

$$\begin{aligned} K &= R_1^{-3}\{\boldsymbol{N}(p) \cdot \boldsymbol{N}(q)\} - 3R_1^{-5}(\boldsymbol{N}(p) \cdot \boldsymbol{R}_1)(\boldsymbol{N}(q) \cdot \boldsymbol{R}_1), \\ \boldsymbol{R}_1 &= (x - x_0, y - y_0, F(x, y) - F(x_0, y_0)), \quad R_1 = |\boldsymbol{R}_1| \\ b(x, y) &= -\partial\phi_0/\partial N = U(\cos\beta + (\partial F/\partial x)\sin\beta), \end{aligned} \tag{3.7}$$

when ϕ_0 is given by (2.2). Equation (3.6) is to be solved subject to the edge condition

$$w(x, y) = 0 \quad \text{for } r = \sqrt{(x^2 + y^2)} = 1.$$

Let

$$F_1 = \partial F/\partial x \quad \text{and} \quad F_2 = \partial F/\partial y \quad \text{evaluated at } (x, y), \tag{3.8}$$

with F_1^0 and F_2^0 being the corresponding quantities at (x_0, y_0). Then $\boldsymbol{N}(q) = (-F_1, -F_2, 1)$ and $\boldsymbol{N}(p) = (-F_1^0, -F_2^0, 1)$. Let $R = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}$ and $\Lambda = \{F(x, y) - F(x_0, y_0)\}/R$. Also, define the angle Θ by

$$x - x_0 = R\cos\Theta \quad \text{and} \quad y - y_0 = R\sin\Theta,$$

whence $\boldsymbol{R}_1 = R(\cos\Theta, \sin\Theta, \Lambda)$. Hence

$$\begin{aligned} K = \frac{1}{R^3}\bigg\{ &\frac{1 + F_1F_1^0 + F_2F_2^0}{(1 + \Lambda^2)^{3/2}} \\ &- 3\frac{(F_1\cos\Theta + F_2\sin\Theta - \Lambda)(F_1^0\cos\Theta + F_2^0\sin\Theta - \Lambda)}{(1 + \Lambda^2)^{5/2}}\bigg\}. \end{aligned} \tag{3.9}$$

This formula is exact. If we expand K for small R about p, we find that

$$K \sim R^{-3}\sigma(p; \Theta),$$

where

$$\sigma(p; \Theta) = \frac{1 + (F_1^0)^2 + (F_2^0)^2}{1 + (F_1^0\cos\Theta + F_2^0\sin\Theta)^2}.$$

In particular, $\sigma \equiv 1$ when F is constant. Thus, for non-constant F, the singularity in the kernel of the integral equation (3.6) is essentially different from that occurring in the integral equation for constant F. A similar phenomenon was noted previously (Martin 1995, 1996) when the integral equation for a *flat* but *non-circular* Ω was mapped onto the unit disc D. In that case, the difficulty was resolved by using a *conformal mapping*. Here, we are projecting onto D, so that the mapping from Ω onto D is prescribed. However, we can make progress by supposing that Ω is *almost* flat.

It is worth remarking that the situation described above does not arise with two-dimensional problems leading to one-dimensional hypersingular integral equations over smooth curves. When the curve is parametrized, one obtains an equation identical to the equation for a straight line-segment, apart from an additional weakly singular kernel.

4. Wrinkled discs

Suppose that

$$F(x,y) = \varepsilon f(x,y),$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda, \quad \text{with } \lambda = \{f(x,y) - f(x_0,y_0)\}/R, \tag{4.1}$$

we find that

$$K = R^{-3}\{1 + \varepsilon^2 K_2 + O(\varepsilon^4)\} \quad \text{as } \varepsilon \to 0,$$

where

$$K_2 = f_1 f_1^0 + f_2 f_2^0 - \tfrac{3}{2}\lambda^2 - 3(f_1\cos\Theta + f_2\sin\Theta - \lambda)(f_1^0\cos\Theta + f_2^0\sin\Theta - \lambda) \tag{4.2}$$

and f_j, f_j^0 are defined similarly to F_j, F_j^0 (see (3.8)).

We expand b similarly. Assuming that ϕ_0 does not depend on ε, we have

$$b(x,y) = b_0(x,y) + \varepsilon b_1(x,y)$$

exactly; indeed, for uniform ambient flow, (3.7) gives

$$b_0 = U\cos\beta \quad \text{and} \quad b_1 = U f_1 \sin\beta.$$

Then, if we expand w as

$$w(x,y) = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots,$$

we find from (3.6) that

$$Hw_0 = b_0, \quad Hw_1 = b_1, \quad \text{and} \quad Hw_2 = -\mathcal{K}_2 w_0,$$

where

$$(Hw)(x_0,y_0) = \frac{1}{4\pi} \oint\!\!\!\!\!\!\times_D w(x,y)\frac{\mathrm{d}A}{R^3}$$

is the basic hypersingular operator for potential flow past a rigid circular disc and

$$(\mathcal{K}_2 w)(x_0,y_0) = \frac{1}{4\pi} \oint\!\!\!\!\!\!\times_D K_2(x,y;x_0,y_0)w(x,y)\frac{\mathrm{d}A}{R^3}.$$

For uniform ambient flow, $b_0 = U\cos\beta$ is a constant, and so we can determine w_0 immediately by solving $Hw_0 = b_0$:

$$w_0(x,y) = -(4/\pi)b_0\surd(1-r^2).$$

Next, we calculate w_1. For w_2, we can foresee that the most difficult part of the calculation will involve the evaluation of $\mathcal{K}_2 w_0$. The simplest results are obtained when f is a polynomial. We shall illustrate this subsequently using two examples, namely linear and quadratic surfaces.

(a) The first-order correction

General methods for solving $Hw = b$ are available (see Martin 1996 and references therein), and these can be used to solve for w_1 for any ambient flow and any disc perturbation f. Thus, introduce plane polar coordinates on D, so that $x = r\cos\theta$ and $y = r\sin\theta$, and then expand b as

$$b(r,\theta) = B_0(r) + \sum_{n=1}^{\infty}\{B_n(r)\cos n\theta + \tilde{B}_n(r)\sin n\theta\}.$$

Then the solution of $Hw = b$ is given by

$$w(r,\theta) = W_0(r) + \sum_{n=1}^{\infty}\{W_n(r)\cos n\theta + \tilde{W}_n(r)\sin n\theta\},$$

where

$$W_n(r) = -\frac{4}{\pi}r^n \int_r^1 \frac{1}{t^{2n}\sqrt{(t^2-r^2)}} \int_0^t \frac{B_n(s)s^{n+1}\,\mathrm{d}s}{\sqrt{(t^2-s^2)}}\,\mathrm{d}t, \tag{4.3}$$

with a similar relation between $\tilde{W}_n$ and $\tilde{B}_n$. These formulae are convenient, although the summation over n can be evaluated to give

$$w(r,\theta) = -\frac{2}{\pi^2}\int_r^1 \frac{1}{\sqrt{(t^2-r^2)}} \int_0^t \frac{s(t^4-r^2s^2)}{\sqrt{(t^2-s^2)}} \int_0^{2\pi} \frac{b(s,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}s\,\mathrm{d}t}{t^4+r^2s^2-2t^2rs\cos(\theta-\varphi)}.$$

All these formulae have been derived by Guidera & Lardner (1975). In particular, replacing b by b_1 gives the first-order correction w_1. For example, if the disc is *rippled* so that $f(x,y) = f(r)$, and the ambient flow is uniform, then

$$w_1(r,\theta) = -\frac{4}{\pi}Ux\sin\beta \int_r^1 \frac{1}{t^2\sqrt{(t^2-r^2)}} \int_0^t \frac{s^2 f'(s)\,\mathrm{d}s}{\sqrt{(t^2-s^2)}}\,\mathrm{d}t. \tag{4.4}$$

(b) The added mass

From (2.4) and (3.7), we obtain

$$T = -\tfrac{1}{2}\rho U \int_D w(x,y)\{\cos\beta + F_1\sin\beta\}\,\mathrm{d}A.$$

So, writing $T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \dots$ gives

$$T_0 = -\tfrac{1}{2}\rho U\cos\beta\int_D w_0\,\mathrm{d}A = \tfrac{1}{2}43\rho U^2\cos^2\beta,$$

$$T_1 = -\tfrac{1}{2}\rho U\int_D\{w_1\cos\beta + f_1w_0\sin\beta\}\,\mathrm{d}A, \tag{4.5}$$

$$T_2 = -\tfrac{1}{2}\rho U\int_D\{w_2\cos\beta + f_1w_1\sin\beta\}\,\mathrm{d}A. \tag{4.6}$$

T_0 is the kinetic energy of the flow around a flat circular disc of unit radius. Note that if the disc is rippled ($f = f(r)$), then $T_1 \equiv 0$.

5. Example 1: inclined ellipse

Suppose that Ω is an ellipse on the plane $z = x \tan\gamma$. Let X and Y be Cartesian coordinates on this plane, so that

$$X = x\cos\gamma + z\sin\gamma, \quad Y = y \quad \text{and} \quad Z = z\cos\gamma - x\sin\gamma,$$

where Z is a coordinate perpendicular to the plane. Then, the ellipse Ω with $\partial\Omega$ given by

$$X^2\cos^2\gamma + Y^2 = 1$$

can be specified by

$$z = F(x, y) = x\tan\gamma, \quad (x, y) \in D.$$

From (3.9), we have

$$K = \frac{1}{R^3}\frac{1 + \tan^2\gamma}{(1 + \tan^2\gamma\cos^2\Theta)^{3/2}}$$

exactly.

For small inclinations of the ellipse to the plane $z = 0$, set $\varepsilon = \tan\gamma$ and $f(x, y) = x$, whence

$$K_2 = \tfrac{1}{4}(1 - 3\cos 2\Theta),$$

$b_1 = U\sin\beta$, $w_1 = -(4/\pi)b_1\surd(1 - r^2)$ and

$$T_1 = \tfrac{4}{3}\rho U^2\sin 2\beta.$$

Next, define operators $\mathcal{H}_0$, $\mathcal{H}_\mathrm{c}$ and $\mathcal{H}_\mathrm{s}$ by

$$\mathcal{H}_0 u = \frac{1}{4\pi}⨎_D \surd(1 - r^2)\frac{u}{R^3}\,\mathrm{d}A,$$
$$\mathcal{H}_\mathrm{c} u = \frac{1}{4\pi}⨎_D \surd(1 - r^2)\frac{u\,\cos 2\Theta}{R^3}\,\mathrm{d}A,$$
$$\mathcal{H}_\mathrm{s} u = \frac{1}{4\pi}⨎_D \surd(1 - r^2)\frac{u\,\sin 2\Theta}{R^3}\,\mathrm{d}A.$$

Then

$$-\mathcal{K}_2 w_0 = (b_0/\pi)\{\mathcal{H}_0 1 - 3\mathcal{H}_\mathrm{c} 1\}.$$

We have

$$\mathcal{H}_0 1 = -\tfrac{1}{4}\pi \quad \text{and} \quad \mathcal{H}_\mathrm{c} 1 = 0$$

from Martin (1996) and Martin (1995), respectively. (The operators $\mathcal{H}_\mathrm{c}$ and $\mathcal{H}_\mathrm{s}$ occur in the coupled integral equations for the shear loading of a penny-shaped crack.) Hence $Hw_2 = -\frac{1}{4}b_0$ and so $w_2 = -\frac{1}{4}w_0$. Substituting into (4.6) then gives

$$T_2 = \rho U^2(\tfrac{1}{2} - \tfrac{5}{6}\cos 2\beta).$$

Thus, combining the above results with (2.5), we find that the added mass is given by

$$M' = \tfrac{8}{3}\rho\{\cos^2\beta + \varepsilon\sin 2\beta + \varepsilon^2(\tfrac{3}{4}\cos^2\beta - \cos 2\beta)\} + O(\varepsilon^3). \tag{5.1}$$

This agrees with the known exact solution (see the appendix).

6. Example 2: spherical cap

Consider a spherical cap given by

$$z = F(x, y) = a - \surd(a^2 - x^2 - y^2), \quad (x, y) \in D,$$

where a is the radius of the sphere. The cap subtends a solid angle of $2\pi(1 - \cos\alpha)$ at the centre of the sphere, where $\sin\alpha = a^{-1}$. For the uniform ambient flow (2.2), the added mass is given by (2.8) and (2.9) for $\beta = 0$ and $\beta = \frac{1}{2}\pi$, respectively.

We shall consider a shallow spherical cap, given approximately by $z = \varepsilon f(x, y)$ with

$$f(x, y) = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}r^2 \quad \text{and} \quad \varepsilon = a^{-1} = \sin\alpha.$$

This is an example of a rippled surface. We have $f_1 = x$, $f_2 = y$, $b_1 = Ux\sin\beta$ and

$$w_1(x, y) = -\tfrac{8}{3}\pi^{-1}Ux\sin\beta\,\surd(1 - r^2).$$

Thus, from (4.5),

$$T_1 = \tfrac{5}{3}\rho\frac{U^2}{\pi}\sin 2\beta\int_D x\surd(1 - r^2)\,\mathrm{d}A = 0,$$

as expected. So, $T = T_0 + O(\varepsilon^2)$, for all β.

The second-order correction is given by (4.6); write it as

$$T_2 = T_2^{(1)} + T_2^{(2)},$$

where

$$\begin{aligned} T_2^{(1)} &= -\tfrac{1}{2}\rho U\sin\beta\int_D f_1 w_1\,\mathrm{d}A \\ &= \frac{4}{3\pi}\rho U^2\sin^2\beta\int_D x^2\surd(1 - r^2)\,\mathrm{d}A \\ &= \tfrac{4}{3}\rho U^2\sin^2\beta\int_0^1 r^3\surd(1 - r^2)\,\mathrm{d}r = \tfrac{8}{45}\rho U^2\sin^2\beta \end{aligned}$$

and

$$T_2^{(2)} = -\tfrac{1}{2}\rho U\cos\beta\int_D w_2\,\mathrm{d}A. \tag{6.1}$$

Next, we solve $Hw_2 = -\mathcal{K}_2 w_0$ for w_2. From (4.1), we obtain

$$\lambda = \tfrac{1}{2}\{(x + x_0)\cos\Theta + (y + y_0)\sin\Theta\},$$

whence

$$f_1\cos\Theta + f_2\sin\Theta - \lambda = \tfrac{1}{2}R.$$

Then, substituting from (4.2), we obtain

$$K_2 = P_0 + P_\mathrm{c}\cos 2\Theta + P_\mathrm{s}\sin 2\Theta,$$

where P_0, P_c and P_s are quadratic polynomials:

$$\begin{aligned} P_0 &= \tfrac{9}{16}(x^2 + y^2 + x_0^2 + y_0^2) - \tfrac{7}{8}(xx_0 + yy_0), \\ P_\mathrm{c} &= -\tfrac{3}{16}(x^2 - y^2 + 2xx_0 - 2yy_0 + x_0^2 - y_0^2), \\ P_\mathrm{s} &= -\tfrac{3}{8}(xy + xy_0 + yx_0 + x_0y_0). \end{aligned}$$

Hence

$$-\mathcal{K}_2 w_0 = (4b_0/\pi)\{\mathcal{H}_0 P_0 + \mathcal{H}_c P_c + \mathcal{H}_s P_s\}.$$

Next, we use known results for penny-shaped cracks (Martin 1995, 1996) to evaluate the following integrals:

$$\begin{aligned}
\mathcal{H}_0 1 &= -\tfrac{1}{4}\pi, \quad \mathcal{H}_c 1 = \mathcal{H}_s 1 = 0,\\
\mathcal{H}_0 x &= -\tfrac{3}{8}\pi x_0, \quad \mathcal{H}_0 y = -\tfrac{3}{8}\pi y_0,\\
\mathcal{H}_c x &= -\tfrac{1}{16}\pi x_0, \quad \mathcal{H}_c y = +\tfrac{1}{16}\pi y_0,\\
\mathcal{H}_s x &= -\tfrac{1}{16}\pi y_0, \quad \mathcal{H}_s y = -\tfrac{1}{16}\pi x_0,\\
\mathcal{H}_0(x^2+y^2) &= \tfrac{1}{16}\pi(2-9r_0^2),\\
\mathcal{H}_0(x^2-y^2) &= -\tfrac{15}{32}\pi(x_0^2-y_0^2), \quad \mathcal{H}_0(xy) = -\tfrac{15}{32}\pi x_0 y_0,\\
\mathcal{H}_c(x^2-y^2) &= \tfrac{1}{16}\pi(1-\tfrac{5}{2}r_0^2), \quad \mathcal{H}_c(xy) = 0,\\
\mathcal{H}_s(xy) &= \tfrac{1}{32}\pi(1-\tfrac{5}{2}r_0^2), \quad \mathcal{H}_s(x^2-y^2) = 0.
\end{aligned}$$

Here, $r_0^2 = x_0^2 + y_0^2$. After some simplification, we find that

$$Hw_2 = -\mathcal{K}_2 w_0 = \tfrac{3}{32} b_0 (2 - r_0^2). \tag{6.2}$$

Solving this equation then gives

$$w_2 = -\tfrac{1}{6}(b_0/\pi)(4-r^2)\sqrt{(1-r^2)}$$

for the second-order solution. Integrating over the unit disc D, (6.1) gives

$$T_2^{(2)} = \tfrac{1}{5}\rho U^2 \cos^2\beta.$$

Finally, we find that

$$M' = \rho\{\tfrac{8}{3}\cos^2\beta + \varepsilon^2(\tfrac{16}{45}\sin^2\beta + \tfrac{2}{5}\cos^2\beta)\}, \tag{6.3}$$

correct to second order in ε.

For the axisymmetric problem ($\beta = 0$), we find that

$$M' \simeq \rho(\tfrac{8}{3} + \tfrac{2}{5}\varepsilon^2). \tag{6.4}$$

The exact solution is given by (2.8); for small α, this formula gives

$$M' \simeq \rho(a\alpha)^3(\tfrac{8}{3} - \tfrac{14}{15}\alpha^2). \tag{6.5}$$

But $1 = a\sin\alpha \sim a\alpha(1-\frac{1}{6}\alpha^2)$, whence $a\alpha \sim 1 + \frac{1}{6}\alpha^2$ and $(a\alpha)^3 \sim 1 + \frac{1}{2}\alpha^2$. Also, $\varepsilon = \sin\alpha$ whence $\alpha \sim \varepsilon + \frac{1}{6}\varepsilon^3$. Hence

$$M' \simeq \rho(1+\tfrac{1}{2}\alpha^2)(\tfrac{8}{3} - \tfrac{14}{15}\alpha^2),$$

and this agrees with (6.4).

For flow perpendicular to the axis of the cap ($\beta = \frac{1}{2}\pi$), we find that

$$M' \simeq \tfrac{16}{45}\rho\varepsilon^2. \tag{6.6}$$

The exact solution is given by (2.9); for small α, this formula gives

$$M' \simeq \tfrac{16}{45}\rho(a\alpha)^3\alpha^2,$$

and this agrees with (6.6).

7. Discussion

In this paper, we have presented a perturbation method for calculating potential flow past a wrinkled disc. The method is general and takes proper account of the edge behaviour. At each perturbation order, one has to solve a hypersingular integral equation, $Hw_n = b_n$, over the unperturbed (flat) disc. The basic solution (w_0) is the solution for flow past a flat disc. The first-order correction (w_1) is easily obtained for any smooth disc perturbation f, as the forcing function b_1 is simple. For the second-order correction (w_2), the main difficulty is in calculating b_2; this, in turn, is centred on the calculation of

$$\lambda = \frac{f(x,y) - f(x_0,y_0)}{\sqrt{((x-x_0)^2 + (y-y_0)^2)}}.$$

It seems that λ can be calculated for polynomial f, although we have not attempted to delimit this class. (Our explicit calculations are for $f(x,y) = x$ and $f(x,y) = \frac{1}{2}(x^2 + y^2)$.) We have devised special techniques for concentric (axisymmetric) perturbations (Martin 1998). It should be worthwhile to extend these methods to non-planar perturbations of a penny-shaped crack under mixed-mode loadings.

Appendix A. Elliptical disc

Consider the elliptical disc $(X/A)^2 + (Y/B)^2 \leqslant 1$ on the plane $Z = 0$. The incident flow is given by

$$\phi_0(X,Y,Z) = U(X \sin\beta_0 - Z\cos\beta_0).$$

Then, it is known (see, for example, Martin 1986) that the potential jump across the ellipse is

$$[\phi] = -(2/E)BU\cos\beta_0\{1 - (X/A)^2 - (Y/B)^2\}^{1/2},$$

where $E(k)$ is the complete elliptic integral of the second kind and $k = \{1 - (B/A)^2\}^{1/2}$. Hence, the added mass is given exactly by (2.4) and (2.5) as

$$M' = \tfrac{4}{3}\pi\rho\cos^2\beta_0\, AB^2/E(k). \tag{A 1}$$

This result includes (2.7) as a special case ($A = B = c$, $\beta_0 = 0$).

To compare with Example 1, we take

$$A = \sec\gamma \sim 1 + \tfrac{1}{2}\varepsilon^2, \quad B = 1 \quad \text{and} \quad \beta_0 = \beta - \gamma,$$

whence $k \sim \varepsilon$ and $\cos^2\beta_0 \sim \cos^2\beta + \varepsilon\sin 2\beta - \varepsilon^2\cos 2\beta$ using $\gamma = \varepsilon + O(\varepsilon^3)$ as $\varepsilon \to 0$. Also, $E(k) \sim \frac{1}{2}\pi(1 - \frac{1}{4}k^2)$ as $k \to 0$. Substituting these approximations into (A 1), we obtain (5.1).

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