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On the added mass of rippled discs

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Abstract. The problem of three-dimensional potential flow past a thin rigid screen is reduced to a hypersingular boundary integral equation. This equation is then projected onto a flat reference screen, which is taken to be a circular disc. Solutions are obtained for screens that are axisymmetric perturbations from the disc, so that the screen is rippled concentrically. The added mass is calculated for axisymmetric flow past such screens, correct to second order.

Keywords: Crack perturbations, hypersingular integral equations, potential flow, added mass.

1. Introduction

Lamb's *Hydrodynamics* gives the added mass of a flat circular disc as [1, Section 108]

$$M_0 = \tfrac{8}{3}\rho a^3, \tag{1}$$

where the disc has radius a and is moving perpendicular to its plane through an incompressible inviscid fluid of density ρ. Herein, we calculate corrections to M_0 when the disc is perturbed out of its plane into a wrinkled surface Ω. Specifically, we consider *rippled discs*, meaning that the disc perturbation is axisymmetric. Thus, Ω is given by

$$\Omega\colon z = \varepsilon f(r), \quad 0 \leqslant r \leqslant 1, \quad -\pi \leqslant \theta < \pi,$$

where (r, θ, z) are cylindrical polar coordinates, f is a given smooth function of one variable, and ε is a small dimensionless parameter. We suppose that the screen Ω translates with constant speed U along the z-axis, so that the resulting boundary-value problem for a velocity potential ϕ is axisymmetric. (Equivalently, we can hold Ω fixed in a uniform flow in the negative z-direction.)

It turns out that the added mass is given by

$$M = M_0 + \varepsilon^2 M_2 + \cdots,$$

for *any* wrinkled disc Ω (not merely rippled discs) when Ω translates along the z-axis. (For other translation directions, this result remains true for rippled discs, but, in general, the correction to M_0 is first order in ε.) Consequently, we have to work to second order if we want to obtain a non-trivial correction. We shall develop a method for carrying out this calculation, and present detailed results for quartic surfaces given by

$$z = \tfrac{1}{2}\varepsilon^2 r^2 (1 - \tfrac{1}{2}cr^2), \quad 0 \leqslant r \leqslant 1, \quad -\pi \leqslant \theta < \pi,$$

where c is a parameter. For this two-parameter family of surfaces, we find that

$$M = \tfrac{8}{3}\rho\{1 + \tfrac{1}{20}\varepsilon^2(3 - \tfrac{39}{7}c + \tfrac{97}{42}c^2)\}, \tag{2}$$

with an error of $O(\varepsilon^4)$. In particular, when $c = 0$, we recover a result in agreement with the exact solution due to Collins [2] for a spherical cap. The generalization of (2) for translations in other directions is given in Section 6.

There are other methods in the literature with a similar general aim. However, these are limited to first-order calculations (which are either trivial or can be performed explicitly for any f) or they are defective in some way. To put these remarks in context, let us begin by recalling that the classical problem of potential flow past a flat circular disc can be solved exactly by the method of separation of variables in oblate spheroidal coordinates [1] or by recasting the problem as a mixed boundary-value problem in a half-space $z > 0$ [3]. Attempts have been made to adapt the latter methodology to problems for which Ω is a non-planar perturbation of a circular disc D.

Jansson [4] imagined Ω to be a piece of an infinite interface separating two half-spaces, and then perturbed this transmission problem about the flat interface. (This is analogous to the theory of small-amplitude water waves [5, Chapter 2] and to the theory of scattering by slightly rough surfaces [6, Chapter 3].) However, the behaviour of the solution near the edge of Ω induces spurious singularities at the edge of D.

Beom and Earmme [7] began with assumed representations for ϕ, namely

$$\phi = \int_0^\infty A_\pm(\xi) J_0(\xi r)\, \mathrm{e}^{\mp\xi z}\, \mathrm{d}\xi \quad \text{for} \quad \pm z \geqslant \varepsilon f, \tag{3}$$

motivated by the use of such representations for flat discs [3, Chapter 3]. However, we can see that there will be points near Ω for which one of (3) will diverge.

In a previous paper [8], we began be reducing the exact boundary-value problem to a hypersingular integral equation for $[\phi]$, the discontinuity in the potential across Ω. We rewrote this equation by projecting onto the unperturbed (reference) surface, which is the disc D. This is an exact reformulation of the original boundary-value problem. Next, we introduced perturbation expansions, leading to a sequence of hypersingular boundary integral equations of the form $Hw_n = b_n$ where

$$[\phi] = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots$$

and H corresponds to potential flow past a rigid circular disc. We derived an explicit closed-form expression for the first-order correction w_1. We also derived explicit results for w_0, w_1 and w_2 for two particular geometries, namely, an inclined flat elliptical screen and a spherical cap. We calculated the added mass for these flows, and found agreement with known exact solutions.

The calculations in [8] are based on two-dimensional integral equations, and do not assume any symmetries in the geometry or the ambient flow. However, the second-order calculations are difficult. In this paper, we investigate axisymmetric problems with similar methods, in order to see whether this restricted class of problems allows second-order calculations to proceed more readily. The axial symmetry leads to one-dimensional hypersingular integral equations with kernels involving complete elliptic integrals. Their analysis is quite different from that described in [8]; in particular, we make essential use of certain integral representations of the complete elliptic integrals.

A second motivation for this study is as a model for other more complicated but more important physical applications. Thus, the basic methodology (namely, formulate an exact boundary integral equation, project exactly onto a reference surface and *then* introduce a regular perturbation expansion) has wide applicability, and it will succeed whenever one can solve the underlying boundary integral equation for the reference surface. For example, we can cite problems of Stokes flow [9, 10], where small obstacles are immersed in a viscous fluid (so that the Reynolds number is small); a lengthy analysis of such a flow past a perturbed sphere is given in [9, Section 5-9]. For another example, we can cite crack problems in elasticity theory; these are important because they arise in theories of crack stability and quasi-static propagation. Applications of the methodology described herein to problems involving perturbed penny-shaped cracks are currently being made. In-plane perturbations of circular discs and cracks are analysed in [11, 12].

2. Formulation

Let Ω be a thin rigid screen, defined by

$$\Omega\colon z = F(x, y), \qquad (x, y) \in D,$$

where (x, y, z) are Cartesian coordinates, D is the *unit disc* in the xy-plane, and F is a given smooth function; later, we shall restrict F to be a function of $r = \sqrt{x^2 + y^2}$. The problem is to solve Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

in the unbounded region exterior to Ω, subject to

$$\frac{\partial\phi}{\partial n} + \frac{\partial\phi_0}{\partial n} = 0 \quad \text{on } \Omega \tag{4}$$

and $\phi = O(R_3^{-1})$ as $R_3 \to \infty$, where $R_3^2 = r^2 + z^2$, ϕ_0 is the velocity potential of the given ambient flow, and $\partial/\partial n$ denotes normal differentiation. We also require that ϕ is bounded everywhere in the flow.

It is known that ϕ can be represented as a distribution of normal dipoles

$$\phi(P) = \frac{1}{4\pi}\int_\Omega [\phi(q)] \frac{\partial}{\partial n_q} G(P, q)\, \mathrm{d}S_q, \tag{5}$$

where $G(P, q) = |\boldsymbol{r} - \boldsymbol{q}|^{-1}$, $q \in \Omega$ has position vector $\boldsymbol{q}$ with respect to the origin O, and P has position vector $\boldsymbol{r}$. Furthermore, denote the two sides of Ω by Ω^+ and Ω^-, and define the unit normal vector on Ω, $\boldsymbol{n}$, to point from Ω^+ into the fluid. Then, we define the discontinuity in ϕ across Ω by

$$[\phi(q)] = \lim_{Q\to q^+} \phi(Q) - \lim_{Q\to q^-} \phi(Q),$$

where $q \in \Omega$, $q^{\pm} \in \Omega^{\pm}$ and Q is a point in the fluid.

Applying the boundary condition (4) to (5), we obtain

$$\frac{1}{4\pi} \int\!\!\!\!\!\!\times_{\Omega} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p,q)\,\mathrm{d}S_q = -\frac{\partial \phi_0}{\partial n_p}, \quad p \in \Omega, \tag{6}$$

which is the governing hypersingular integral equation for $[\phi]$. The integral in (6) must be interpreted in the finite-part sense. Also, (6) must be solved subject to $[\phi(q)] = 0$ for all $q \in \partial\Omega$, the edge of Ω. More information on (6) and its derivation, and on two-dimensional finite-part integrals can be found in [12] and [13].

Let us define a normal vector to Ω by

$$\boldsymbol{N} = \left(\frac{-\partial F}{\partial x}, \frac{-\partial F}{\partial y}, 1 \right),$$

whence $\boldsymbol{n} = \boldsymbol{N}/|\boldsymbol{N}|$ is a unit normal vector; this effectively specifies Ω^+. Suppose that $p \in \Omega$ and $q \in \Omega$ are at (x_0, y_0, z_0) and (x, y, z), respectively. Let

$$[\phi(q)] = w(x, y).$$

Then, we can project (6) onto D. Thus, using $z = F(x, y)$ and $z_0 = F(x_0, y_0)$, we rewrite (6), exactly, as an integral equation over D [8],

$$\frac{1}{4\pi} \int\!\!\!\!\!\!\times_{D} \mathcal{K}(x_0, y_0; x, y) w(x, y)\,\mathrm{d}A = b(x_0, y_0), \qquad (x_0, y_0) \in D, \tag{7}$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$,

$$\mathcal{K} = R_1^{-3} \{ \boldsymbol{N}(p) \cdot \boldsymbol{N}(q) \} - 3R_1^{-5} (\boldsymbol{N}(p) \cdot \boldsymbol{R}_1)(\boldsymbol{N}(q) \cdot \boldsymbol{R}_1), \tag{8}$$

$\boldsymbol{R}_1 = (x - x_0, y - y_0, F(x, y) - F(x_0, y_0))$, $R_1 = |\boldsymbol{R}_1|$, and

$$b(x, y) = -\boldsymbol{N} \cdot \operatorname{grad} \phi_0. \tag{9}$$

Equation (7) is to be solved subject to the edge condition $w(x, y) = 0$ for $r = 1$.

In the sequel, we take

$$\phi_0 = -Uz \quad \text{whence} \quad b = U.$$

We will then calculate an approximation to w by solving (7). The added mass itself is given exactly by [8]

$$M = -\frac{\rho}{U} \int_D w(x, y)\,\mathrm{d}A; \tag{10}$$

this formula comes by noting that, by definition, $T = \frac{1}{2} M U^2$, where T is the kinetic energy of the fluid motion [1, Section 44]. Exact solutions for M are known when Ω is a flat circular disc, a flat elliptical screen and a spherical cap; see [8] for references.

3. Axisymmetric problems: rippled discs

Assume that Ω is given by

$$\Omega\colon z = F(r), \quad 0 \leqslant r \leqslant 1, \quad -\pi \leqslant \theta < \pi.$$

Thus Ω is circularly symmetric - it is rippled. As $\phi_0 = -Uz$, the solution w is independent of θ

$$w(x, y) = w(r).$$

Then, the two-dimensional integral Equation (7) becomes

$$\frac{1}{4\pi} \mathop{\mathchoice{\times\kern-1.1em\int}{}{}{}}_0^1 L(r_0, r) w(r) r \, \mathrm{d}r = U, \quad 0 \leqslant r_0 < 1. \tag{11}$$

This is a one-dimensional hypersingular integral equation for $w(r)$; it is to be solved subject to $w(1) = 0$. The integral in (11) is a Hadamard finite-part integral. The kernel is given by

$$L(r_0, r) = \int_{-\pi}^{\pi} \mathcal{K}(x_0, y_0; x, y) \, \mathrm{d}\theta, \tag{12}$$

where $x = r \cos\theta$, $y = r \sin\theta$, $x_0 = r_0 \cos\theta_0$ and $y_0 = r_0 \sin\theta_0$.

It is well known that the standard boundary integral equations of axisymmetric potential theory can be reduced to one-dimensional integral equations in which the kernels involve complete elliptic integrals [14, 15]. The present situation is no exception, as we shall see.

In the Appendix, it is shown that

$$L = \tfrac{1}{2}\kappa^3 (rr_0)^{-3/2} (I_3^0 - F' F_0' I_3^1) - \tfrac{3}{8}\kappa^5 (rr_0)^{-5/2} (\mathcal{A} I_5^0 - \mathcal{B} I_5^1 + \mathcal{C} I_5^2), \tag{13}$$

where $F \equiv F(r)$, $F_0 \equiv F(r_0)$, $F' \equiv F'(r)$, $F_0' \equiv F'(r_0)$,

$$I_m^n \equiv I_m^n(\kappa) = \int_0^{\pi/2} \frac{\cos 2n\theta \, \mathrm{d}\theta}{(1 - \kappa^2 \sin^2 \theta)^{m/2}}, \tag{14}$$

$$\mathcal{A} = (F - F_0)^2 + (F - F_0)(F_0' r_0 - F' r) - \tfrac{3}{2} F' F_0' r r_0, \tag{15}$$

$$\mathcal{B} = (F - F_0)(F' r_0 - F_0' r) + F' F_0' (r^2 + r_0^2), \tag{16}$$

$$\mathcal{C} = -\tfrac{1}{2} F' F_0' r r_0 \tag{17}$$

and

$$\kappa^2 = \frac{4rr_0}{(r + r_0)^2 + (F - F_0)^2}; \tag{18}$$

note that

$$\kappa^2 \leqslant \frac{4rr_0}{(r + r_0)^2} = \frac{4rr_0}{(r - r_0)^2 + 4rr_0} \leqslant 1,$$

with $\kappa^2 = 1$ only when $r = r_0$.

The integrals I_m^n can be expressed in terms of complete elliptic integrals when m is an odd integer [16, Section 2.58]

$$I_3^0(k) = k'^{-2}E(k),$$

$$I_3^1(k) = k'^{-2}(1 - 2k^{-2})E(k) + 2k^{-2}K(k),$$

$$I_5^0(k) = \tfrac{2}{3}k'^{-4}(2 - k^2)E(k) - \tfrac{1}{3}k'^{-2}K(k),$$

$$I_5^1(k) = -\tfrac{2}{3}(kk'^2)^{-2}(1 - k^2 + k^4)E(k) + \tfrac{1}{3}(kk')^{-2}(2 - k^2)K(k),$$

$$I_5^2(k) = -\tfrac{2}{3}(kk')^{-4}(8 - 12k^2 + 2k^4 + k^6)E(k) + \tfrac{1}{3}(k^2k')^{-2}(16 - 16k^2 - k^4)K(k),$$

where $k'^2 = 1 - k^2$ and the complete elliptic integrals E and K are defined by

$$E(k) = \int_0^{\pi/2} (1 - k^2\,\sin^2\,\theta)^{1/2}\,\mathrm{d}\theta \quad \text{and} \quad K(k) = \int_0^{\pi/2} (1 - k^2\,\sin^2\,\theta)^{-1/2}\,\mathrm{d}\theta,$$

respectively.

3.1. The flat disc

If Ω is flat and lies parallel to the xy-plane, $F = F_0$. Hence, $F' = F_0' = 0$, $\mathcal{A} = \mathcal{B} = \mathcal{C} = 0$ and $\kappa = k$, where

$$k^2 = \frac{4rr_0}{(r + r_0)^2}. \tag{19}$$

Thus the kernel L simplifies to

$$L_0(r_0, r) \equiv \tfrac{1}{2}k^3(rr_0)^{-(3/2)}I_3^0(k) = \frac{4}{r + r_0}\frac{E(k)}{(r - r_0)^2} \tag{20}$$

and the integral equation (11) reduces to

$$\frac{1}{\pi} ⨍_0^1 \frac{rE(k)}{r + r_0}\frac{w(r)\,\mathrm{d}r}{(r - r_0)^2} = U, \quad 0 \leqslant r_0 < 1, \tag{21}$$

with $w(1) = 0$; here, we have used

$$k'^2 = 1 - k^2 = \frac{(r - r_0)^2}{(r + r_0)^2}.$$

The hypersingular integral equation (21) for axisymmetric potential flow past a rigid flat circular disc seems to be new, although it can be extracted from [15, Equation (6)]. (It also yields the crack-opening displacement w for a pressurized penny-shaped crack.)

As $r \to r_0$, $k \to 1$ and $E(1) = 1$, so that $rL_0(r_0, r)$ exhibits the basic hypersingularity in one dimension, namely $(r - r_0)^{-2}$.

The flat-disc integral equation (21) can be solved exactly, using several different methods. Thus, if we replace the constant U on the right-hand side by a given function $b(r_0)$, we have

$$w(r) = \frac{-4}{\pi}\int_r^1 \frac{1}{\sqrt{t^2-r^2}}\int_0^t \frac{b(s)s\,\mathrm{d}s}{\sqrt{t^2-s^2}}\,\mathrm{d}t, \tag{22}$$

for a derivation of this result, see, for example, [17] or [18]. In particular, when $b(r) = U$, we obtain $w(r) = -(4/\pi)U\sqrt{1-r^2}$.

3.2. The singularity of the kernel

We are interested in the singularity of the kernel $L(r_0, r)$ as $|r - r_0| \to 0$, for any $F(r)$. Let us define

$$R = r - r_0 \quad \text{and} \quad \Lambda = (F - F_0)/R,$$

so that Λ is bounded for all R. In particular, $\Lambda \to F_0'$ as $r \to r_0$. It follows that

$$\kappa'^2 \equiv 1 - \kappa^2 = \frac{R^2(1+\Lambda^2)}{(r+r_0)^2 + R^2\Lambda^2},$$

whence $\kappa' \to 0$ as $R \to 0$. In this limit, the complete elliptic integral K is singular: $K(\kappa) \sim \log(4/\kappa')$ as $\kappa' \to 0$.

A cursory glance at L suggests a very strong singularity, due to the terms containing κ'^{-4} in $I_5^n(\kappa)$. However, various cancellations occur. To see this, all quantities must be expanded for small R. Expanding about r_0, we have $\kappa'^2 \sim \frac{1}{4}\beta_0 r_0^{-2}R^2$ as $R \to 0$, where $\beta_0 = 1 + F_0'^2$. For the first term in (13), we have

$$\begin{aligned} \tfrac{1}{2}\kappa^3(rr_0)^{-(3/2)}(I_3^0 - F'F_0'I_3^1) &\sim \tfrac{1}{2}r_0^{-3}\beta_0\kappa'^{-2} \\ &\sim (2/r_0)R^{-2} \quad \text{as } R \to 0, \end{aligned} \tag{23}$$

thus, this term reduces to the flat-disc kernel for small R, as seen in (20).

For the second term in (13), we have

$$\mathcal{A} \sim -\tfrac{3}{2}\mathcal{D} + R^2 r_0 F_0'\tilde{\mathcal{A}},$$

$$\mathcal{B} \sim -2\mathcal{D} + R^2 r_0 F_0'\tilde{\mathcal{B}},$$

$$\mathcal{C} \sim -\tfrac{1}{2}\mathcal{D} + R^2 r_0 F_0'\tilde{\mathcal{C}}$$

as $R \to 0$, where

$$\mathcal{D} = r_0 F_0'\{r_0 F_0' + R(F_0' + r_0 F_0'')\},$$

$$\tilde{\mathcal{A}} = -\tfrac{5}{2}F_0'' - \tfrac{3}{4}r_0 F_0''',$$

$$\tilde{\mathcal{B}} = 3F_0'' + r_0 F_0''',$$

$$\tilde{\mathcal{C}} = -\tfrac{1}{2}F_0'' - \tfrac{1}{4}r_0 F_0''',$$

$F_0'' = F''(r_0)$ and $F_0''' = F'''(r_0)$; thus, $\mathcal{D}$, which is common to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, contains all the terms in R^0 and R^1. Then

$$\begin{aligned}\mathcal{A}I_5^0 - \mathcal{B}I_5^1 + \mathcal{C}I_5^2 &\sim \mathcal{D}Q_1 + R^2 r_0 F_0'(\tilde{\mathcal{A}}I_5^0 - \tilde{\mathcal{B}}I_5^1 + \tilde{\mathcal{C}}I_5^2)\\ &= \mathcal{D}Q_1 + R^2 r_0 F_0'(F_0''Q_2 + \tfrac{1}{2}r_0 F_0'''Q_1),\end{aligned}$$

where

$$\begin{aligned}Q_1 &= -\tfrac{3}{2}I_5^0 - 2I_5^1 - \tfrac{1}{2}I_5^2\\ &= \tfrac{8}{3}\kappa^{-4}(1+\kappa^2)E(\kappa) - \tfrac{4}{3}\kappa^{-4}(2+\kappa^2)K(\kappa),\\ Q_2 &= -\tfrac{5}{2}I_5^0 - 3I_5^1 - \tfrac{1}{2}I_5^2\\ &= \tfrac{2}{3}(\kappa^2\kappa')^{-2}(4+\kappa^2-6\kappa^4)E(\kappa) - \tfrac{2}{3}\kappa^{-4}(4+3\kappa^2)K(\kappa),\end{aligned}$$

whence $Q_1 \sim 4\log\kappa'$ and $Q_2 \sim -\frac{2}{3}\kappa'^{-2}$ as $\kappa' \to 0$. It follows that the second term in (13) has a logarithmic singularity, so that L has a dominant singularity given by (23), with additional (weaker) logarithmic terms.

4. Slightly rippled discs

The hypersingular integral equation (11) is exact. It is valid for axisymmetric flow past any rippled disc, and it could be solved numerically. Here, we suppose that the ripples are small, and write

$$F(r) = \varepsilon f(r),$$

where ε is a small dimensionless parameter and f is independent of ε. Then we look for approximate solutions of (11), valid for small ε.

It turns out that

$$L = L_0 + \varepsilon^2 L_2 + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0, \tag{24}$$

where L_0 is the flat-disc kernel given by (20) and L_2 is given by (25) below. To obtain (24), we start by setting

$$\Lambda = \varepsilon\lambda \quad \text{with} \quad \lambda = \frac{f(r) - f(r_0)}{R} \quad \text{and} \quad R = r - r_0.$$

Next, write $L = L^{(1)} + L^{(2)}$ where

$$L^{(1)} = \tfrac{1}{2}\kappa^3(rr_0)^{-3/2}I_3^0(\kappa) \quad \text{and} \quad L^{(2)} = L - L^{(1)}.$$

For small ε, we have

$$\kappa = k - \varepsilon^2\delta + O(\varepsilon^4) \quad \text{with} \quad \delta = \tfrac{1}{8}k^3(f - f_0)^2/(rr_0),$$

where $f \equiv f(r)$, $f_0 \equiv f(r_0)$, κ is defined by (18) and k is defined by (19). $L^{(2)}$ is quadratic in $F(r)$ and $F'(r)$, whence its contribution to L_2 will come by replacing F by f and κ by k. For $L^{(1)}$, we must take account of the difference between κ and k; this gives

$$L^{(1)}(r_0, r) = L_0(r_0, r) - \frac{\varepsilon^2 (f - f_0)^2 k^5}{16 k'^2 (r r_0)^{5/2}} \left\{ \frac{2}{k'^2}(2 - k^2) E(k) - K(k) \right\}.$$

Finally, simplification gives

$$L_2(r_0, r) = \frac{-2}{r + r_0} \left\{ \frac{S_1 E(k)}{(r - r_0)^2} + \frac{S_2 E(k)}{(r + r_0)^2} + \frac{S_3 K(k)}{(r + r_0)^2} \right\}, \tag{25}$$

where

$$S_1 = 6\lambda^2 - 4\lambda(f' + f_0') + \frac{r^2 + r_0^2}{r r_0} f' f_0',$$

$$S_2 = 6\lambda^2 - 4\lambda(r + r_0)\left(\frac{f' - f_0'}{r - r_0}\right),$$

$$S_3 = -3\lambda^2 + \lambda(r + r_0)\left(\frac{f' r_0 + f_0' r}{r r_0}\right) - \frac{(r + r_0)^2}{r r_0} f' f_0',$$

$f' \equiv f'(r)$ and $f_0' \equiv f'(r_0)$. Note that the apparent hypersingularity in (25) is removable because $S_1 = O(R^2)$ as $R \to 0$.

Having expanded the kernel for small ε, we next expand w as

$$w(r) = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots.$$

Then, (11) gives

$$\mathcal{L} w_0 = U, \qquad \mathcal{L} w_1 = 0, \quad \text{and} \quad \mathcal{L} w_2 = b_2,$$

where $\mathcal{L}$, defined by

$$(\mathcal{L} w)(r_0) = \frac{1}{4\pi} ⨍_0^1 L_0(r_0, r) w(r) r \, \mathrm{d}r = \frac{1}{\pi} ⨍_0^1 \frac{r E(k)}{r + r_0} \frac{w(r) \, \mathrm{d}r}{(r - r_0)^2},$$

is the basic hypersingular operator for axisymmetric potential flow past a rigid circular disc

$$b_2 = -\mathcal{L}_2 w_0 \quad \text{and} \quad (\mathcal{L}_2 w)(r_0) = \frac{1}{4\pi} \int_0^1 L_2(r_0, r) w(r) r \, \mathrm{d}r.$$

It follows immediately that

$$w_0(r) = -\left(\frac{4}{\pi}\right) U \sqrt{1 - r^2} \quad \text{and} \quad w_1 = 0.$$

For w_2, we can foresee that the most difficult part of the calculation will involve the evaluation of $b_2 = -\mathcal{L}_2 w_0$. In the next section, we describe this calculation for a certain quartic surface f.

Finally, we can calculate the second-order correction to the added mass from (10)

$$M_2 = -\frac{2\pi\rho}{U}\int_0^1 w_2(r) r \, \mathrm{d}r. \tag{26}$$

But the solution of $\mathcal{L}w_2 = b_2$ is given by (22) as

$$w_2(r) = \frac{-4}{\pi}\int_r^1 \frac{1}{\sqrt{t^2 - r^2}}\int_0^t \frac{b_2(s)s \, \mathrm{d}s}{\sqrt{t^2 - s^2}} \, \mathrm{d}t. \tag{27}$$

Substituting this expression in (26), and interchanging the order of integration twice, we obtain

$$M_2 = \frac{8\rho}{U}\int_0^1 s\sqrt{1 - s^2}\, b_2(s) \, \mathrm{d}s, \tag{28}$$

which avoids an explicit calculation of w_2.

5. A rippled quartic surface

Consider a quartic surface given by

$$z = \varepsilon f(r) \quad \text{with} \quad f(r) = \tfrac{1}{2}r^2(1 - \tfrac{1}{2}cr^2), \quad 0 \leqslant r \leqslant 1, \quad -\pi \leqslant \theta < \pi, \tag{29}$$

where c is a parameter. Thus $f' = r - cr^3$,

$$\lambda = \tfrac{1}{2}(r + r_0)\{1 - \tfrac{1}{2}c(r^2 + r_0^2)\},$$

$$S_1 = (r - r_0)^2\{\tfrac{1}{2} + \tfrac{1}{2}c(r^2 + 4rr_0 + r_0^2) - \tfrac{1}{8}c^2(r^2 + r_0^2)(5r^2 + 12rr_0 + 5r_0^2)\},$$

$$S_2 = (r + r_0)^2\{-\tfrac{1}{2} + \tfrac{1}{2}c(3r^2 + 4rr_0 + 3r_0^2) - \tfrac{1}{8}c^2(r^2 + r_0^2)(5r^2 + 8rr_0 + 5r_0^2)\},$$

$$S_3 = (r + r_0)^2\{-\tfrac{3}{4} + \tfrac{3}{4}c(r^2 + r_0^2) + \tfrac{1}{16}c^2(r^4 - 14r^2r_0^2 + r_0^4)\},$$

whence

$$\begin{aligned} L_2(r_0, r) \;=\; & -4c\{1 - \tfrac{5}{8}c(r^2 + r_0^2)\}(r + r_0)E(k) \\ & + \{\tfrac{3}{2} - \tfrac{3}{2}c(r^2 + r_0^2) - \tfrac{1}{8}c^2(r^4 - 14r^2r_0^2 + r_0^4)\}(r + r_0)^{-1}K(k). \end{aligned}$$

The next step is to evaluate b_2; we have

$$b_2(r_0) = -(\mathcal{L}_2 w_0)(r_0) = \frac{U}{\pi^2}\int_0^1 L_2(r_0, r)\sqrt{1 - r^2}\, r \, \mathrm{d}r.$$

The difficulty is that r occurs through k (defined by (19)) in the argument of the complete elliptic integrals. We proceed indirectly by using certain integral representations [19, p. 249]

$$\frac{K(k)}{r + r_0} = \tfrac{1}{2}\pi\int_0^\infty J_0(rt)J_0(r_0t) \, \mathrm{d}t,$$

$$(r + r_0)E(k) = (r^2 + r_0^2)\frac{K(k)}{r + r_0} - \pi r r_0 \int_0^\infty J_1(rt)J_1(r_0t)\,\mathrm{d}t,$$

where $J_n(x)$ is a Bessel function. These give

$$L_2 = \pi(\alpha_1 r + \alpha_2 r^3)\int_0^\infty J_1(rt)J_1(r_0t)\,\mathrm{d}t + \pi(\alpha_3 + \alpha_4 r^2 + \alpha_5 r^4) \times \int_0^\infty J_0(rt)J_0(r_0t)\,\mathrm{d}t,$$

where

$$\alpha_1 = 4cr_0(1 - \tfrac{5}{8}cr_0^2), \qquad \alpha_2 = -\tfrac{5}{2}c^2 r_0,$$

$$\alpha_3 = \tfrac{1}{16}(12 - 44cr_0^2 + 19c^2r_0^4), \qquad \alpha_4 = \tfrac{1}{8}(-22c + 27c^2r_0^2)$$

and $\alpha_5 = \frac{19}{16}c^2$. So, if we define

$$\mathcal{J}_n^m(t) = \int_0^1 J_n(rt)r^m\sqrt{1 - r^2}\,\mathrm{d}r,$$

we see that

$$b_2(r_0) = \frac{U}{\pi}\int_0^\infty J_1(r_0t)(\alpha_1\mathcal{J}_1^2 + \alpha_2\mathcal{J}_1^4)\,\mathrm{d}t + \frac{U}{\pi}\int_0^\infty J_0(r_0t)(\alpha_3\mathcal{J}_0^1 + \alpha_4\mathcal{J}_0^3 + \alpha_5\mathcal{J}_0^5)\,\mathrm{d}t.$$

The integrals $\mathcal{J}_n^m$ are standard [20, Equation (11.4.10)]

$$\mathcal{J}_0^1 = t^{-1}j_1(t), \qquad \mathcal{J}_0^3 = t^{-1}j_1(t) - 3t^{-2}j_2(t),$$

$$\mathcal{J}_0^5 = t^{-1}j_1(t) - 6t^{-2}j_2(t) + 15t^{-3}j_3(t),$$

$$\mathcal{J}_1^2 = t^{-1}j_2(t), \qquad \mathcal{J}_1^4 = t^{-1}j_2(t) - 3t^{-2}j_3(t),$$

where $j_n(x) = (\frac{1}{2}\pi/x)J_{n+1/2}(x)$ is a spherical Bessel function. Hence

$$b_2 = U(\gamma_1\mathcal{W}_{01}^1 + \gamma_2\mathcal{W}_{02}^2 + \gamma_3\mathcal{W}_{03}^3 + \gamma_4\mathcal{W}_{12}^1 + \gamma_5\mathcal{W}_{13}^2),$$

where $\gamma_1 = \alpha_3 + \alpha_4 + \alpha_5$, $\gamma_2 = -3(\alpha_4 + 2\alpha_5)$, $\gamma_3 = 15\alpha_5$, $\gamma_4 = \alpha_1 + \alpha_2$, $\gamma_5 = -3\alpha_2$ and

$$\mathcal{W}_{mn}^l(r_0) = \frac{1}{\pi}\int_0^\infty t^{-l}J_m(r_0t)j_n(t)\,\mathrm{d}t, \quad 0 \leqslant r_0 < 1.$$

$\mathcal{W}_{mn}^l$ is a Weber–Schafheitlin integral [20, Equation 11.4.34]; it can be expressed in terms of a hypergeometric function. In all the cases of interest to us, the hypergeometric function reduces to a polynomial

$$\mathcal{W}_{01}^1 = \tfrac{1}{8}(2 - r_0^2),$$

$$\mathcal{W}_{02}^2 = \tfrac{1}{128}(8 - 8r_0^2 + 3r_0^4),$$

$$\mathcal{W}_{03}^3 = \tfrac{1}{1536}(16 - 24r_0^2 + 18r_0^4 - 5r_0^6),$$

$$\mathcal{W}_{12}^1 = \tfrac{1}{32}r_0(4 - 3r_0^2),$$

$$\mathcal{W}_{13}^2 = \tfrac{1}{256}r_0(8 - 12r_0^2 + 5r_0^4).$$

Hence, we find that b_2 is a sextic polynomial given by

$$b_2(r_0) = U(p_0 + p_1 r_0^2 + p_2 r_0^4 + p_3 r_0^6),$$

where

$$p_0 = \tfrac{3}{16} - \tfrac{11}{64}c + \tfrac{19}{512}c^2,$$

$$p_1 = -\tfrac{3}{32} - \tfrac{23}{64}c + \tfrac{155}{1024}c^2,$$

$$p_2 = \tfrac{83}{512}c + \tfrac{491}{4096}c^2,$$

$$p_3 = -\tfrac{515}{8192}c^2.$$

We can now use (28) to calculate the second-order correction to the added mass. The result is

$$\begin{aligned} M_2 &= \tfrac{8}{3}\rho(p_0 + \tfrac{2}{5}p_1 + \tfrac{8}{35}p_2 + \tfrac{16}{105}p_3) \\ &= \rho(\tfrac{2}{5} - \tfrac{26}{35}c + \tfrac{97}{315}c^2), \end{aligned} \tag{30}$$

which gives (2); here, we have used

$$\int_0^1 s^{2m+1}\sqrt{1-s^2}\,\mathrm{d}s = \frac{m!\Gamma(3/2)}{2\Gamma(m+5/2)}.$$

When $c = 0$, the result (30) agrees with the known exact result for a spherical cap [2] when the cap is shallow; see [8] for more details. Another interesting case is $c = 2$, so that $f(0) = f(1) = 0$; then $M_2 = \frac{46}{315}\rho$. Also, when $c = \frac{117}{97}$, M_2 takes its minimum value of $-\frac{163}{3395} \simeq -0{\cdot}05$. Note also that M_2 vanishes for two positive values of c, approximately 0·8 and 1·6; at these values, the correction to the added mass is fourth order in ε

Finally, we can compute the second-order correction w_2. By substituting b_2 in (27), and evaluating the integrals, we find that

$$w_2(r) = -(U/\pi)\sqrt{1-r^2}(W_0 + W_1 r^2 + W_2 r^4 + W_3 r^6), \tag{31}$$

where

$$W_0 = \tfrac{2}{3} - \tfrac{211}{225}c + \tfrac{14011}{44100}c^2,$$

$$W_1 = -\tfrac{1}{6} - \tfrac{41}{75}c + \tfrac{9337}{29400}c^2,$$

$$W_2 = \tfrac{83}{450}c + \tfrac{1215}{11025}c^2,$$

$$W_3 = -\tfrac{103}{1960}c^2.$$

6. Discussion

In this paper, we have presented a perturbation method for calculating axisymmetric potential flow past a rippled disc. The method is general and takes proper account of the edge behaviour. At each perturbation order, one has to solve a one-dimensional hypersingular integral equation, $\mathcal{L}w_n = b_n$; the operator $\mathcal{L}$ corresponds to the unperturbed (flat) disc. The basic solution (w_0) is the solution for flow past a flat disc. The first-order correction (w_1) is identically zero. For the second-order correction (w_2), the main difficulty is in calculating b_2; this, in turn, is centred on the calculation of

$$\lambda = \frac{f(r) - f(r_0)}{r - r_0}.$$

This can be done for polynomial f; our explicit calculations are for quartic f. It seems that, although these calculations are tedious, they could be mechanised using a computer algebra package, and then one could obtain results for high-order polynomial approximations to quite general smooth rippled surfaces.

Finally, let us make a few remarks on *non-axisymmetric* flow past a rippled disc. Thus, suppose that

$$\phi_0(x, y, z) = U(x \sin\beta - z \cos\beta)$$

so that $\beta = 0$ gives the axisymmetric problem. Hence

$$b(x, y) = U \cos\beta + \varepsilon U \sin\beta f'(r) \cos\theta.$$

The first term gives an axisymmetric contribution to M. The second term gives a first-order correction to w, namely $\varepsilon w_1(r) \cos\theta$ where [8]

$$w_1(r) = \frac{-4}{\pi} Ur \sin\beta \int_r^1 \frac{\Psi(t)\,\mathrm{d}t}{t\sqrt{t^2 - r^2}}$$

and

$$\Psi(t) = \frac{1}{t}\int_0^t \frac{r^2 f'(r)\,\mathrm{d}r}{\sqrt{t^2 - r^2}}.$$

This does not give a first-order correction to M, but it does give a second-order correction [8]

$$\begin{aligned}\tilde{M}_2 &= -\left(\frac{\pi\rho}{U}\right) \sin\beta \int_0^1 w_1(r) f'(r) r\,\mathrm{d}r \\ &= 4\rho \sin^2\beta \int_0^1 \{\Psi(t)\}^2\,\mathrm{d}t,\end{aligned}$$

where we have substituted for w_1 and interchanged the order of integration. For the quartic f, given by (29), we have

$$\Psi(t) = \tfrac{2}{3}t^2(1 - \tfrac{4}{5}ct^2)$$

and

$$\tilde{M}_2 = \tfrac{16}{45}\rho \sin^2 \beta(1 - \tfrac{8}{7}c + \tfrac{16}{45}c^2).$$

Hence, correct to second order in ε, we find that

$$M = M_a \cos \beta + \varepsilon^2 \tilde{M}_2,$$

where M_a is the axisymmetric result given by (2).

Appendix. The kernel $L(r_0, r)$

The kernel L is defined by (12) in terms of $\mathcal{K}$ which is itself defined by (8). We have

$$\boldsymbol{N}(q) = (-F' \cos \theta, -F' \sin \theta, 1) \quad \text{and} \quad \boldsymbol{N}(p) = (-F_0' \cos \theta_0, -F_0' \sin \theta_0, 1),$$

in terms of Cartesian components, where $F' \equiv F'(r)$ and $F_0' \equiv F'(r_0)$. Hence

$$\boldsymbol{N}(p) \cdot \boldsymbol{N}(q) = 1 + F'F_0' \cos(\theta - \theta_0),$$

$$\boldsymbol{N}(q) \cdot \boldsymbol{R}_1 = F - F_0 - F'\{r - r_0 \cos(\theta - \theta_0)\},$$

$$\boldsymbol{N}(p) \cdot \boldsymbol{R}_1 = F - F_0 + F_0'\{r_0 - r \cos(\theta - \theta_0)\}$$

and

$$(\boldsymbol{N}(q) \cdot \boldsymbol{R}_1)(\boldsymbol{N}(p) \cdot \boldsymbol{R}_1) = \mathcal{A} + \mathcal{B} \cos(\theta - \theta_0) + \mathcal{C} \cos 2(\theta - \theta_0),$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined by (15), (16) and (17), respectively. Thus

$$L = \tilde{I}_3^0 + F'F_0'\tilde{I}_3^1 - 3\{\mathcal{A}\tilde{I}_5^0 + \mathcal{B}\tilde{I}_5^1 + \mathcal{C}\tilde{I}_5^2\},$$

where

$$\begin{aligned} \tilde{I}_m^n &= \int_{-\pi}^{\pi} R_1^{-m} \cos n(\theta - \theta_0)\,\mathrm{d}\theta \\ &= 2\int_0^{\pi} \frac{\cos n\varphi\,\mathrm{d}\varphi}{\{r^2 + r_0^2 + (F - F_0)^2 - 2rr_0 \cos \varphi\}^{m/2}}. \end{aligned}$$

In the denominator, replace $\cos \varphi$ by $2 \cos^2 \tfrac{1}{2}\varphi - 1$, and then change the integration variable using $\varphi = \pi - 2\theta$. The result is

$$\tilde{I}_m^n = 2^{2-m}\kappa^m(-1)^n(rr_0)^{-m/2}I_m^n(\kappa),$$

where κ is defined by (18) and $I_m^n(\kappa)$ is defined by (14), whence (13) follows.

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