ELECTROMAGNETIC SCATTERING BY A HOMOGENEOUS CHIRAL OBSTACLE: BOUNDARY INTEGRAL EQUATIONS AND LOW-CHIRALITY APPROXIMATIONS*

C. ATHANASIADIS†, P. A. MARTIN‡, AND I. G. STRATIS†

Abstract. Time-harmonic electromagnetic waves are scattered by a homogeneous chiral obstacle. The corresponding transmission problem is reduced to a pair of coupled integral equations over S, where S is the interface between the obstacle and the surrounding medium. This is done using a generalization of the Stratton–Chu representation that is valid for chiral media. The integral equations obtained are a generalization of those obtained by Müller for a homogeneous dielectric obstacle. Finally, we develop approximations for low-chirality obstacles. These approximations can be computed using simple modifications to existing codes for solving Müller's equations.

Key words. chiral media, boundary integral equations, transmission problem

AMS subject classifications. 35Q60, 45F05, 78A45

PII. S003613999833633X

1. Introduction. Chiral media are everywhere: "We know today that almost every substance found in living organisms is a carbon compound possessing a basic asymmetry, or 'chirality' as chemists and physicists prefer to call it, a term coined by Lord Kelvin" [11, p. 113]. Nonorganic materials can also be chiral; good examples are certain man-made composites and coatings. Chiral media support two kinds of electromagnetic waves. Thus, when "a linearly polarized wave is incident normally upon a slab of chiral medium, two waves are generated in the medium. One is a left-circularly polarized wave and the other is a right-circularly polarized wave of a different phase velocity" [6, p. 2]. This phenomenon, known as *optical activity*, can be modeled using appropriate constitutive relations; we use the Drude–Born–Fedorov relations.

In this paper, we consider the scattering of time-harmonic electromagnetic waves by a bounded three-dimensional chiral obstacle surrounded by free space. This problem was first solved by Bohren [7] for a spherical obstacle. Subsequent work, using integral equations and T-matrix methods, is described in [16] and [15]; see also [19]. It is possible to allow the exterior to be chiral, too; a uniqueness theorem covering this case is proved in appendix A.

We assume that the obstacle is homogeneous. This allows us to reduce the problem to a pair of coupled integral equations over the interface between the obstacle and its surroundings. We derive these equations using a generalization of the Stratton–Chu representation to chiral media; such generalizations, involving scalar Green's functions, have been given before [17], [25], [1], although we give two different proofs. Alternative representations, involving dyadic Green's functions, are also available [16], [15], [24].

Many different pairs of coupled boundary integral equations can be derived and studied; reviews are available for the acoustic transmission problem [14] and for electromagnetic scattering by a dielectric obstacle [12], [20]. (These papers also describe methods for solving transmission problems using a single integral equation.) One such pair, obtained by using an *ansatz* in each region, has been analyzed by Ola [23]. Here, we derive a generalization of Müller's equations. This pair is contrived so that all hypersingular operators are combined in such a way that their strong singularities cancel. Thus, as for the pure electromagnetic transmission problem, we obtain a pair of equations that has good theoretical properties and that is attractive for computations.

We give a dimensionless formulation of the transmission problem. It is seen to depend on four dimensionless parameters, which we can take as $k_e a$, ka, ρ, and $k\beta$. Here, k_e is the exterior wavenumber; k is the analogous quantity for the chiral obstacle, which itself has diameter $2a$; ρ^2 is the ratio of the two magnetic permeabilities; and β is the chirality measure. We allow ka, ρ, and $k\beta$ to be complex. In practice, $|k\beta| \ll 1$, so we develop approximations exploiting this fact. Indeed, we show that the first-order correction in $k\beta$ can be computed by solving Müller's equations (for a dielectric obstacle) with different right-hand functions. This allows existing codes to be extended easily so as to treat low-chirality obstacles.

2. Chiral media. A homogeneous isotropic chiral medium is characterized by three (complex) parameters. These are the electric permittivity ε, the magnetic permeability μ, and the chirality measure β. Thus, we use the Drude–Born–Fedorov constitutive relations

$$\boldsymbol{D} = \varepsilon(\widetilde{\boldsymbol{E}} + \beta\,\mathrm{curl}\,\widetilde{\boldsymbol{E}}) \quad \text{and} \quad \boldsymbol{B} = \mu(\widetilde{\boldsymbol{H}} + \beta\,\mathrm{curl}\,\widetilde{\boldsymbol{H}}),$$

where $\widetilde{\boldsymbol{E}}$ is the electric field, $\widetilde{\boldsymbol{H}}$ is the magnetic field, $\boldsymbol{B}$ is the magnetic flux density, and $\boldsymbol{D}$ is the electric flux density. For free space (where the medium is achiral), we have $\beta = 0$.

In a source-free region, we also have

$$\mathrm{curl}\,\widetilde{\boldsymbol{E}} - \mathrm{i}\omega\,\boldsymbol{B} = \boldsymbol{0} \quad \text{and} \quad \mathrm{curl}\,\widetilde{\boldsymbol{H}} + \mathrm{i}\omega\,\boldsymbol{D} = \boldsymbol{0},$$

where we have suppressed a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ throughout. Hence, eliminating $\boldsymbol{B}$ and $\boldsymbol{D}$, we obtain

$$\mathrm{curl}\,\widetilde{\boldsymbol{E}} - \mathrm{i}\omega\mu(\widetilde{\boldsymbol{H}} + \beta\,\mathrm{curl}\,\widetilde{\boldsymbol{H}}) = \boldsymbol{0} \tag{2.1}$$

and

$$\mathrm{curl}\,\widetilde{\boldsymbol{H}} + \mathrm{i}\omega\varepsilon(\widetilde{\boldsymbol{E}} + \beta\,\mathrm{curl}\,\widetilde{\boldsymbol{E}}) = \boldsymbol{0}. \tag{2.2}$$

Put $\widetilde{\boldsymbol{E}} = \sqrt{\mu}\,\boldsymbol{E}$ and $\widetilde{\boldsymbol{H}} = \sqrt{\varepsilon}\,\boldsymbol{H}$ whence (2.1) and (2.2) become

$$\mathrm{curl}\,\boldsymbol{E} - \mathrm{i}k(\boldsymbol{H} + \beta\,\mathrm{curl}\,\boldsymbol{H}) = \boldsymbol{0} \tag{2.3}$$

and

$$\mathrm{curl}\,\boldsymbol{H} + \mathrm{i}k(\boldsymbol{E} + \beta\,\mathrm{curl}\,\boldsymbol{E}) = \boldsymbol{0}, \tag{2.4}$$

where k is defined by

$$k = \omega\sqrt{\mu\varepsilon}.$$

This shows that the key dimensionless parameter for chiral media is $k\beta$.

It is convenient to introduce fields $\boldsymbol{U}$ and $\boldsymbol{U}'$ (the dual of $\boldsymbol{U}$) as follows:

$$\text{if } \boldsymbol{U} = \boldsymbol{E}, \text{ then } \boldsymbol{U}' = \mathrm{i}\,\boldsymbol{H};$$

$$\text{if } \boldsymbol{U} = \boldsymbol{H}, \text{ then } \boldsymbol{U}' = -\mathrm{i}\,\boldsymbol{E}.$$

Here, $\{\boldsymbol{E}, \boldsymbol{H}\}$ are regarded as solutions of (2.3) and (2.4). With this notation, we can write (2.3) and (2.4) as

$$(2.5) \qquad\qquad \operatorname{curl}\boldsymbol{U} = k\boldsymbol{U}' + k\beta\operatorname{curl}\boldsymbol{U}'.$$

As $\left(\boldsymbol{U}'\right)' = \boldsymbol{U}$, we also have

$$(2.6) \qquad\qquad \operatorname{curl}\boldsymbol{U}' = k\boldsymbol{U} + k\beta\operatorname{curl}\boldsymbol{U}.$$

Eliminating $\operatorname{curl}\boldsymbol{U}'$, we obtain

$$(2.7) \qquad\qquad \operatorname{curl}\boldsymbol{U} = \gamma^2\beta\boldsymbol{U} + (\gamma^2/k)\,\boldsymbol{U}',$$

where

$$\gamma^2 = k^2(1 - k^2\beta^2)^{-1}.$$

We always assume that $|k\beta| < 1$.

Finally, taking the curl of (2.7), using (2.6), we obtain

$$\operatorname{curl}\operatorname{curl}\boldsymbol{U} - 2\gamma^2\beta\operatorname{curl}\boldsymbol{U} - \gamma^2\boldsymbol{U} = \boldsymbol{0},$$

which is the governing differential equation for both $\boldsymbol{E}$ and $\boldsymbol{H}$.

In chiral (or "handed") media, left-handed and right-handed waves both can propagate independently and with different phase speeds. To see this, let

$$\boldsymbol{Q}_L = \boldsymbol{E} + \mathrm{i}\,\boldsymbol{H} \quad\text{and}\quad \boldsymbol{Q}_R = \boldsymbol{E} - \mathrm{i}\,\boldsymbol{H},$$

whence

$$(2.8) \qquad\qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{Q}_R + \boldsymbol{Q}_L) \quad\text{and}\quad \boldsymbol{H} = \tfrac{1}{2}\mathrm{i}(\boldsymbol{Q}_R - \boldsymbol{Q}_L).$$

Then, forming (2.3) $\pm$ (2.4), we obtain

$$(2.9) \qquad\qquad \operatorname{curl}\boldsymbol{Q}_L = \gamma_L\,\boldsymbol{Q}_L \quad\text{and}\quad \operatorname{curl}\boldsymbol{Q}_R = -\gamma_R\,\boldsymbol{Q}_R,$$

where

$$(2.10) \qquad \gamma_L = \frac{k}{1 - k\beta}, \qquad \gamma_R = \frac{k}{1 + k\beta}, \qquad\text{and}\qquad \gamma^2 = \gamma_L\gamma_R.$$

Thus, $\boldsymbol{Q}_L$ is a left-handed field, with wavenumber γ_L, whereas $\boldsymbol{Q}_R$ is a right-handed field, with wavenumber γ_R. Note that k itself is not a wavenumber. Note also that our splitting into left-handed and right-handed fields was done with a dimensionless version of the well-known Bohren transformation.

We can relate $\boldsymbol{Q}_L$ and $\boldsymbol{Q}_R$ to solutions of Maxwell's equations. Thus, let us say that $[\boldsymbol{E}, \boldsymbol{H}, k]$ solves Maxwell's equations if $\boldsymbol{E}$ and $\boldsymbol{H}$ solve (2.3) and (2.4) with $\beta = 0$ and the specified value of k. Then it is easily verified that both

$$(2.11) \qquad [\boldsymbol{Q}_L, -\mathrm{i}\boldsymbol{Q}_L, \gamma_L] \quad\text{and}\quad [\boldsymbol{Q}_R, \mathrm{i}\boldsymbol{Q}_R, \gamma_R] \quad\text{solve Maxwell's equations.}$$

These results allow representations for chiral fields to be found using known achiral representations.

3. Integral representations. There are two standard representations for electromagnetic fields in achiral media in terms of surface integrals. These are the Stratton–Chu formula (which involves acoustic single-layer potentials) and another formula based on a dyadic Green's function. Here, we concentrate on the former, although connections with dyadic Green's functions are discussed in appendix B.

3.1. Achiral media. Consider a bounded three-dimensional domain B_i with a smooth closed boundary, S. Suppose that B_i is filled with achiral material, with electromagnetic parameters ε and μ.

We shall use the following notation: p, q denote points of S; P, Q denote points not on S. We choose the origin O at some point in B_i; $\boldsymbol{r}_P$ is the position vector of P with respect to O, $r_P = |\boldsymbol{r}_P|$, and $\hat{\boldsymbol{r}}_P = \boldsymbol{r}_P/r_P$.

Introduce the fundamental solution G, defined by

$$G(P, Q; k) = \exp(\mathrm{i}kR)/(2\pi R),$$

where $R = |\boldsymbol{r}_P - \boldsymbol{r}_Q|$ is the distance between P and Q. Next, define a single-layer potential by

$$(3.1) \qquad (S\nu)(P) = \int_S \nu(q)\, G(P, q; k)\, ds_q,$$

where $q \in S$ and $\nu(q)$ is a continuous density function. Let $\boldsymbol{a}(q)$ be a tangential vector density, so that $\boldsymbol{a}(q) \cdot \boldsymbol{n}(q) = 0$ for all $q \in S$, where $\boldsymbol{n}(q)$ denotes the unit normal at q pointing out of B_i. Define

$$(C\boldsymbol{a})(P) = \mathrm{curl}\,\{S\boldsymbol{a}\} \quad \text{and} \quad (F\boldsymbol{a})(P) = \mathrm{curl}\,\{C\boldsymbol{a}\}.$$

Then, the Stratton–Chu formula can be written as [10, Theorem 4.1]

$$(3.2) \qquad -2\boldsymbol{U}(P) = C\{\boldsymbol{n} \times \boldsymbol{U}\} + k^{-1}F\{\boldsymbol{n} \times \boldsymbol{U}'\}$$

for $P \in B_i$. This expresses the fields $\{\boldsymbol{E}, \boldsymbol{H}\}$ in B_i in terms of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ on S.

An alternative representation, using a dyadic Green's function, is given in appendix B.

3.2. Chiral media. Suppose now that B_i is filled with a chiral medium, with parameters ε, μ, and β. From (2.11) and (3.2), we have the representations

$$(3.3) \qquad -2\boldsymbol{Q}_L(P) = \left(C_L + \gamma_L^{-1}F_L\right)\{\boldsymbol{n} \times \boldsymbol{Q}_L\}$$

and

$$(3.4) \qquad -2\boldsymbol{Q}_R(P) = \left(C_R - \gamma_R^{-1}F_R\right)\{\boldsymbol{n} \times \boldsymbol{Q}_R\}$$

for $P \in B_i$, where, for $\alpha = L, R$,

$$(3.5) \qquad (C_\alpha \boldsymbol{a})(P) = \mathrm{curl}\,\{S_\alpha \boldsymbol{a}\}, \qquad (F_\alpha \boldsymbol{a})(P) = \mathrm{curl}\,\{C_\alpha \boldsymbol{a}\},$$

and

$$(S_\alpha \boldsymbol{a})(P) = \int_S \boldsymbol{a}(q)\, G(P, q; \gamma_\alpha)\, ds_q.$$

It is worth noting that $C_L\boldsymbol{a}$ and $F_L\boldsymbol{a}$ are *not* left-handed fields but the combination $C_L\boldsymbol{a} + \gamma_L^{-1}F_L\boldsymbol{a}$ *is* such a field. Similarly, $C_R\boldsymbol{a} - \gamma_R^{-1}F_R\boldsymbol{a}$ is a right-handed field.

Next, from (2.8), we obtain

$$-2\boldsymbol{E}(P) = \tfrac{1}{2}\left(C_L + \gamma_L^{-1}F_L\right)\{\boldsymbol{n}\times\boldsymbol{Q}_L\} + \tfrac{1}{2}\left(C_R - \gamma_R^{-1}F_R\right)\{\boldsymbol{n}\times\boldsymbol{Q}_R\}$$

and

$$-2\boldsymbol{H}(P) = -\tfrac{1}{2}\mathrm{i}\left(C_L + \gamma_L^{-1}F_L\right)\{\boldsymbol{n}\times\boldsymbol{Q}_L\} + \tfrac{1}{2}\mathrm{i}\left(C_R - \gamma_R^{-1}F_R\right)\{\boldsymbol{n}\times\boldsymbol{Q}_R\}$$

for $P \in B_i$. Finally, using (2.8) again, we obtain

$$
\begin{aligned}
(3.6) \qquad -2\boldsymbol{U}(P) = {}& \tfrac{1}{2}C_L\{\boldsymbol{n}\times\boldsymbol{U}\} + \tfrac{1}{2}C_R\{\boldsymbol{n}\times\boldsymbol{U}\} \\
&+ \tfrac{1}{2}\gamma_L^{-1}F_L\{\boldsymbol{n}\times\boldsymbol{U}\} - \tfrac{1}{2}\gamma_R^{-1}F_R\{\boldsymbol{n}\times\boldsymbol{U}\} \\
&+ \tfrac{1}{2}C_L\{\boldsymbol{n}\times\boldsymbol{U}'\} - \tfrac{1}{2}C_R\{\boldsymbol{n}\times\boldsymbol{U}'\} \\
&+ \tfrac{1}{2}\gamma_L^{-1}F_L\{\boldsymbol{n}\times\boldsymbol{U}'\} + \tfrac{1}{2}\gamma_R^{-1}F_R\{\boldsymbol{n}\times\boldsymbol{U}'\}
\end{aligned}
$$

for $P \in B_i$. This is the generalization of the Stratton–Chu formula to chiral media. It reduces to (3.2) when $\beta = 0$.

The vector fields $C_\alpha\boldsymbol{a}$ and $F_\alpha\boldsymbol{a}$ have well-known properties. In particular, the tangential components of these fields evaluated on S when $\boldsymbol{a}$ is itself a tangential vector density are calculated readily. Thus, for continuous tangential densities, we have

$$\boldsymbol{n}\times C_\alpha\boldsymbol{a} = \pm\boldsymbol{a} + M_\alpha\boldsymbol{a},$$

where the upper (lower) sign corresponds to $P \to p \in S$ from B_e (B_i), B_e is the (unbounded) exterior of B_i, and M_α is a boundary integral operator defined by

$$(3.7) \qquad (M_\alpha\boldsymbol{a})(p) = \boldsymbol{n}(p)\times\operatorname{curl}\{S_\alpha\boldsymbol{a}\}, \quad p \in S.$$

For sufficiently smooth tangential densities $\boldsymbol{a}$, we also have

$$\boldsymbol{n}\times F_\alpha\boldsymbol{a} = P_\alpha\boldsymbol{a}$$

on S, where

$$(3.8) \qquad (P_\alpha\boldsymbol{a})(p) = \boldsymbol{n}(p)\times\operatorname{curl}\operatorname{curl}\{S_\alpha\boldsymbol{a}\}, \quad p \in S.$$

Note that M_α and P_α are related to the operators $\mathbf{M}_\alpha$ and $\mathbf{N}_\alpha$ in [10, section 2.7] by $\mathbf{M}_\alpha\boldsymbol{a} = 2M_\alpha\boldsymbol{a}$ and $\mathbf{N}_\alpha\boldsymbol{a} = 2P_\alpha\{\boldsymbol{n}\times\boldsymbol{a}\}$.

4. Statement of the problem. Let B_i denote a bounded three-dimensional domain with a smooth closed boundary, S, and connected exterior, B_e. B_e is filled with an achiral medium, with constant electromagnetic parameters ε_e and μ_e. B_i is filled with a chiral medium, with parameters ε, μ, and β. A given electromagnetic field is incident upon the obstacle; it is partly scattered and partly transmitted into the obstacle. This leads to the following (dimensional) transmission problem.

Transmission problem. Find electric fields $\widetilde{\boldsymbol{E}}_e$ and $\widetilde{\boldsymbol{E}}_i$ and magnetic fields $\widetilde{\boldsymbol{H}}_e$ and $\widetilde{\boldsymbol{H}}_i$ that satisfy Maxwell's equations in B_e,

$$\operatorname{curl}\widetilde{\boldsymbol{E}}_e - \mathrm{i}\mu_e\omega\widetilde{\boldsymbol{H}}_e = \boldsymbol{0} \quad\text{and}\quad \operatorname{curl}\widetilde{\boldsymbol{H}}_e + \mathrm{i}\varepsilon_e\omega\widetilde{\boldsymbol{E}}_e = \boldsymbol{0}, \quad P \in B_e,$$

a modified form of these equations in B_i,

$$\operatorname{curl}\widetilde{\boldsymbol{E}}_i - \mathrm{i}\mu\omega(\gamma/k)^2\widetilde{\boldsymbol{H}}_i - \beta\gamma^2\widetilde{\boldsymbol{E}}_i = \boldsymbol{0}$$

and

$$\operatorname{curl}\widetilde{\boldsymbol{H}}_i + \mathrm{i}\varepsilon\omega(\gamma/k)^2\widetilde{\boldsymbol{E}}_i - \beta\gamma^2\widetilde{\boldsymbol{H}}_i = \boldsymbol{0}, \quad P \in B_i,$$

and two transmission conditions on the interface,

$$(4.1) \qquad \boldsymbol{n} \times \widetilde{\boldsymbol{E}}_t = \boldsymbol{n} \times \widetilde{\boldsymbol{E}}_i \quad \text{and} \quad \boldsymbol{n} \times \widetilde{\boldsymbol{H}}_t = \boldsymbol{n} \times \widetilde{\boldsymbol{H}}_i, \quad p \in S,$$

where the total fields in B_e are given by

$$(4.2) \qquad \widetilde{\boldsymbol{E}}_t(P) = \widetilde{\boldsymbol{E}}_e + \widetilde{\boldsymbol{E}}_{\mathrm{inc}}, \quad \widetilde{\boldsymbol{H}}_t(P) = \widetilde{\boldsymbol{H}}_e + \widetilde{\boldsymbol{H}}_{\mathrm{inc}}, \quad P \in B_e,$$

and $\{\widetilde{\boldsymbol{E}}_{\mathrm{inc}}, \widetilde{\boldsymbol{H}}_{\mathrm{inc}}\}$ is the given incident field. In addition, the scattered fields $\{\widetilde{\boldsymbol{E}}_e, \widetilde{\boldsymbol{H}}_e\}$ must satisfy a Silver–Müller radiation condition [10, section 4.2],

$$(4.3) \qquad \sqrt{\mu_e}\hat{\boldsymbol{r}}_P \times \widetilde{\boldsymbol{H}}_e + \sqrt{\varepsilon_e}\widetilde{\boldsymbol{E}}_e = o(r_P^{-1}) \quad \text{as } r_P \to \infty,$$

uniformly for all directions $\hat{\boldsymbol{r}}_P$.

We assume that the constants ε_e and μ_e are positive, whereas the constants ε, μ, and β can be complex. Thus, with the chiral wavenumbers γ_L and γ_R defined by (2.10), with $k^2 = \omega^2\mu\varepsilon$ and $\gamma^2 = \gamma_L\gamma_R$, we assume that [4]

$$\operatorname{Re}\gamma_L > 0, \quad \operatorname{Re}\gamma_R > 0, \quad \operatorname{Im}\gamma_L \geq 0, \quad \operatorname{Im}\gamma_R \geq 0, \quad \text{and} \quad \eta > 0,$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the (real) intrinsic impedance of the chiral medium. Athanasiadis and Stratis [4] have proved that, with these assumptions, the transmission problem has precisely one solution for all frequencies. Ola [23] has obtained similar results, assuming that the chiral parameters are all real. Ammari and Nédélec [1] have proved unique solvability for inhomogeneous obstacles, assuming that μ, ε, and β are twice-continuously differentiable real functions of position everywhere in space, taking constant values outside a bounded region. A uniqueness theorem for a homogeneous chiral obstacle embedded in a different homogeneous chiral medium is proved in appendix A. This theorem also permits more general transmission conditions than (4.1).

It is convenient to consider a dimensionless version of the transmission problem. Thus, scale all lengths using a, a typical length-scale for the chiral obstacle, and then put

$$\widetilde{\boldsymbol{E}}_e = \sqrt{\mu_e}\boldsymbol{E}_e, \quad \widetilde{\boldsymbol{H}}_e = \sqrt{\varepsilon_e}\boldsymbol{H}_e, \quad \widetilde{\boldsymbol{E}}_i = \sqrt{\mu}\boldsymbol{E}, \quad \widetilde{\boldsymbol{H}}_i = \sqrt{\varepsilon}\boldsymbol{H},$$

with similar scalings for $\widetilde{\boldsymbol{E}}_t$, $\widetilde{\boldsymbol{H}}_t$, $\widetilde{\boldsymbol{E}}_{\mathrm{inc}}$, and $\widetilde{\boldsymbol{H}}_{\mathrm{inc}}$. These scalings reduce the transmission problem to the following problem.

Dimensionless transmission problem. Find electric fields $\boldsymbol{E}_e$ and $\boldsymbol{E}$ and magnetic fields $\boldsymbol{H}_e$ and $\boldsymbol{H}$ that satisfy Maxwell's equations in B_e,

$$\operatorname{curl}\boldsymbol{E}_e - \mathrm{i}(k_e a)\boldsymbol{H}_e = \boldsymbol{0} \quad \text{and} \quad \operatorname{curl}\boldsymbol{H}_e + \mathrm{i}(k_e a)\boldsymbol{E}_e = \boldsymbol{0}, \quad P \in B_e,$$

a modified form of these equations in B_i,

$$\operatorname{curl}\boldsymbol{E} - \mathrm{i}(ka)(\gamma/k)^2\boldsymbol{H} - \beta a\gamma^2\boldsymbol{E} = \boldsymbol{0}$$

and

$$\operatorname{curl} \boldsymbol{H} + \mathrm{i}(ka)(\gamma/k)^2 \boldsymbol{E} - \beta a \gamma^2 \boldsymbol{H} = \boldsymbol{0}, \quad P \in B_i,$$

and two transmission conditions on the interface,

(4.4) $$\rho \boldsymbol{n} \times \boldsymbol{E}_t = \boldsymbol{n} \times \boldsymbol{E} \quad \text{and} \quad \delta \boldsymbol{n} \times \boldsymbol{H}_t = \boldsymbol{n} \times \boldsymbol{H}, \quad p \in S,$$

where the total fields in B_e are given by $\boldsymbol{E}_t = \boldsymbol{E}_e + \boldsymbol{E}_{\mathrm{inc}}$ and $\boldsymbol{H}_t = \boldsymbol{H}_e + \boldsymbol{H}_{\mathrm{inc}}$,

$$k_e = \omega\sqrt{\mu_e \varepsilon_e}, \quad \rho = \sqrt{\mu_e/\mu}, \quad \text{and} \quad \delta = \sqrt{\varepsilon_e/\varepsilon}.$$

In addition, the scattered fields $\{\boldsymbol{E}_e, \boldsymbol{H}_e\}$ must satisfy a Silver–Müller radiation condition

(4.5) $$\hat{\boldsymbol{r}}_P \times \boldsymbol{H}_e + \boldsymbol{E}_e = o(r_P^{-1}) \quad \text{as } r_P \to \infty.$$

As $\delta = k_e a/(\rho k a)$ and $(\gamma/k)^2 = (1 - k^2\beta^2)^{-1}$, we see that there are four dimensionless parameters, which we can take as

$$k_e a, \quad ka, \quad k\beta, \quad \text{and} \quad \rho.$$

For an achiral obstacle, $\beta = 0$ so that there are then only three dimensionless parameters. Moreover, for chiral obstacles, one typically has

$$|k\beta| \ll 1,$$

so this suggests that "low-chirality" approximations will be useful.

Alternative approximations are based on the assumption that the frequency is low, so that $k_e a \ll 1$; see [3] and [5].

In practice, it is common to consider an impedance-matched obstacle, so that

$$\eta = \eta_e, \quad \text{where} \quad \eta = \sqrt{\mu/\varepsilon} \quad \text{and} \quad \eta_e = \sqrt{\mu_e/\varepsilon_e}.$$

This implies that $\rho = \delta$ and so $k_e a = \rho^2 ka$, which again reduces the problem to one involving three dimensionless parameters.

Henceforth, we assume that all lengths have been scaled using a, and so we can set $a = 1$.

5. Boundary integral equations. We are going to reduce the transmission problem to a pair of coupled boundary integral equations over S. This can be done in many ways. Our preferred choice is a direct method (meaning that the unknowns are physically relevant) leading to a generalization of Müller's equations for scattering by an achiral obstacle.

Thus, in the exterior achiral region, we use the Stratton–Chu representation. When this is applied in B_e to $\{\boldsymbol{E}_e, \boldsymbol{H}_e\}$, it gives

(5.1) $$2\boldsymbol{U}_e(P) = C_e\{\boldsymbol{n} \times \boldsymbol{U}_e\} + k_e^{-1} F_e\{\boldsymbol{n} \times \boldsymbol{U}_e'\}, \quad P \in B_e,$$

where C_e and F_e are defined by (3.5) with γ_α replaced by k_e. This representation satisfies the Silver–Müller radiation conditions. It can be used for $\boldsymbol{U}_e = \boldsymbol{E}_e$ (in which case $\boldsymbol{U}_e' = \mathrm{i}\boldsymbol{H}_e$) or for $\boldsymbol{U}_e = \boldsymbol{H}_e$ (in which case $\boldsymbol{U}_e' = -\mathrm{i}\boldsymbol{E}_e$).

An application in B_i to $\{\boldsymbol{E}_{\text{inc}}, \boldsymbol{H}_{\text{inc}}\}$ (with the exterior material) yields a similar formula which, when added to (5.1), gives

$$(5.2) \qquad 2\boldsymbol{U}_e(P) = C_e\{\boldsymbol{n} \times \boldsymbol{U}_t\} + k_e^{-1}F_e\{\boldsymbol{n} \times \boldsymbol{U}_t'\}, \quad P \in B_e.$$

Computing the tangential components of (5.2) on S, we obtain

$$(5.3) \qquad (I - M_e)\{\boldsymbol{n} \times \boldsymbol{U}_t\} - k_e^{-1}P_e\{\boldsymbol{n} \times \boldsymbol{U}_t'\} = 2\boldsymbol{n} \times \boldsymbol{U}_{\text{inc}},$$

where $\boldsymbol{U}_t$, $\boldsymbol{U}_t'$, $\boldsymbol{U}_{\text{inc}}$, M_e, and P_e are defined in an obvious manner.

In the interior chiral region, we use the generalization of the Stratton–Chu formula (3.6). Computing the tangential components on S, we obtain

$$(5.4) \qquad (I + M_+ + P_-)\{\boldsymbol{n} \times \boldsymbol{U}\} + (M_- + P_+)\{\boldsymbol{n} \times \boldsymbol{U}'\} = \boldsymbol{0},$$

where

$$M_\pm = \tfrac{1}{2}(M_L \pm M_R) \quad \text{and} \quad P_\pm = \tfrac{1}{2}(\gamma_L^{-1}P_L \pm \gamma_R^{-1}P_R).$$

Next, we use the transmission conditions (4.4) in (5.3) and (5.4). Setting $\boldsymbol{U}_t = \boldsymbol{E}_t$ and $\boldsymbol{U}_t = \boldsymbol{H}_t$ in (5.3), we obtain

$$(5.5) \qquad (I - M_e)\boldsymbol{J} - \mathrm{i}k_e^{-1}P_e\boldsymbol{M} = 2\boldsymbol{J}_{\text{inc}}$$

and

$$(5.6) \qquad (I - M_e)\boldsymbol{M} + \mathrm{i}k_e^{-1}P_e\boldsymbol{J} = 2\boldsymbol{M}_{\text{inc}},$$

where, as is customary, we have defined $\boldsymbol{J}$, $\boldsymbol{M}$, $\boldsymbol{J}_{\text{inc}}$, and $\boldsymbol{M}_{\text{inc}}$ by

$$\boldsymbol{J}(p) = \boldsymbol{n} \times \boldsymbol{H}_t, \quad \boldsymbol{M}(p) = -\boldsymbol{n} \times \boldsymbol{E}_t, \quad \boldsymbol{J}_{\text{inc}} = \boldsymbol{n} \times \boldsymbol{H}_{\text{inc}}, \quad \text{and} \quad \boldsymbol{M}_{\text{inc}} = -\boldsymbol{n} \times \boldsymbol{E}_{\text{inc}}$$

for $p \in S$. Similarly, (5.4) gives

$$(5.7) \qquad (I + M_+ + P_-)\boldsymbol{J} + \mathrm{i}(\rho/\delta)(M_- + P_+)\boldsymbol{M} = \boldsymbol{0}$$

and

$$(5.8) \qquad (I + M_+ + P_-)\boldsymbol{M} - \mathrm{i}(\delta/\rho)(M_- + P_+)\boldsymbol{J} = \boldsymbol{0}.$$

Equations (5.5)–(5.8) are four boundary integral equations for the two unknowns $\boldsymbol{J}(q)$ and $\boldsymbol{M}(q)$. We shall choose two linear combinations of these equations, namely,

$$(5.9) \qquad \alpha_1(5.5) + \alpha_2(5.6) + \alpha_3(5.7) + \alpha_4(5.8)$$

and

$$(5.10) \qquad \alpha_1'(5.5) + \alpha_2'(5.6) + \alpha_3'(5.7) + \alpha_4'(5.8),$$

where α_j and α_j', $j = 1, 2, 3, 4$, are constants to be specified.

When $\beta = 0$, several choices have been investigated, both theoretically and numerically; see [12] and [20] for reviews. For all these choices, we always have *existence*: $\boldsymbol{J}$ and $\boldsymbol{M}$ are just the tangential components of $\boldsymbol{H}_t$ and $\boldsymbol{E}_t$, respectively, and we already know that the transmission problem always has precisely one solution. However, the question of *uniqueness* is less obvious.

One good choice is by Müller [21]; see also [13, section 6.29]. Müller took

(5.11) $\alpha_1 = \mu_e, \quad \alpha_3 = \mu, \quad \alpha_2' = \varepsilon_e, \quad \alpha_4' = \varepsilon, \quad \text{and} \quad \alpha_2 = \alpha_4 = \alpha_1' = \alpha_3' = 0.$

This choice leads to a Fredholm system of the second kind: by construction, the hypersingular operators P_α occur only in the combination $(P_e - P_i)$, and this difference is known to be compact in the space of Hölder-continuous tangential vector densities [20]. Numerical results obtained from Müller's system of boundary integral equations are given in [8].

The chiral case is more complicated. For choice (5.9), we obtain

$$\{(\alpha_1 + \alpha_3)I + \mathcal{M}_1 + \mathcal{P}_1\}\,\boldsymbol{J} + \{(\alpha_2 + \alpha_4)I + \mathcal{M}_2 + \mathcal{P}_2\}\,\boldsymbol{M} = 2\alpha_1\boldsymbol{J}_{\text{inc}} + 2\alpha_2\boldsymbol{M}_{\text{inc}},$$

where

(5.12) $$\mathcal{M}_1 = -\alpha_1 M_e + \alpha_3 M_+ - \mathrm{i}(\delta/\rho)\alpha_4 M_-,$$

(5.13) $$\mathcal{M}_2 = -\alpha_2 M_e + \mathrm{i}(\rho/\delta)\alpha_3 M_- + \alpha_4 M_+,$$

(5.14) $$\mathcal{P}_1 = \mathrm{i}(\alpha_2/k_e)P_e + \alpha_3 P_- - \mathrm{i}(\delta/\rho)\alpha_4 P_+,$$

(5.15) $$\mathcal{P}_2 = -\mathrm{i}(\alpha_1/k_e)P_e + \mathrm{i}(\rho/\delta)\alpha_3 P_+ + \alpha_4 P_-.$$

We want to eliminate the hypersingularities in $\mathcal{P}_1$ and $\mathcal{P}_2$, so we require (from (5.14)) that

(5.16) $$\mathrm{i}(\alpha_2/k_e) + \tfrac{1}{2}\alpha_3(\gamma_L^{-1} - \gamma_R^{-1}) - \tfrac{1}{2}\mathrm{i}(\delta/\rho)\alpha_4(\gamma_L^{-1} + \gamma_R^{-1}) = 0$$

and (from (5.15))

(5.17) $$-\mathrm{i}(\alpha_1/k_e) + \tfrac{1}{2}\mathrm{i}(\rho/\delta)\alpha_3(\gamma_L^{-1} + \gamma_R^{-1}) + \tfrac{1}{2}\alpha_4(\gamma_L^{-1} - \gamma_R^{-1}) = 0.$$

We impose similar constraints on α_j'. Moreover, to obtain a second-kind system, we also require that

(5.18) $$\begin{vmatrix} \alpha_1 + \alpha_3 & \alpha_2 + \alpha_4 \\ \alpha_1' + \alpha_3' & \alpha_2' + \alpha_4' \end{vmatrix} \neq 0.$$

The constraints (5.16) and (5.17) simplify to

(5.19) $$\alpha_2 - \delta^2\alpha_4 + \mathrm{i}k_e\beta\alpha_3 = 0 \quad \text{and} \quad \alpha_1 - \rho^2\alpha_3 - \mathrm{i}k_e\beta\alpha_4 = 0,$$

with similar constraints on α_j'.

We want a pair of equations that reduces to Müller's pair when $\beta = 0$. Therefore, we choose

$$\alpha_1 = (k\rho/\gamma)^2, \quad \alpha_2 = 0, \quad \alpha_3 = 1, \quad \alpha_4 = \mathrm{i}(\rho/\delta)k\beta$$

and

$$\alpha_1' = 0, \quad \alpha_2' = (k\delta/\gamma)^2, \quad \alpha_3' = -\mathrm{i}(\delta/\rho)k\beta, \quad \alpha_4' = 1.$$

These choices satisfy (5.19). They reduce to Müller's choice (5.11) when $\beta = 0$ (apart from a constant factor). The condition (5.18) will also be satisfied if

$$(1 + \rho^2)(1 + \delta^2) \neq k_e^2\beta^2.$$

Substituting for α_j and α'_j into (5.9) and (5.10), respectively, and simplifying, we obtain

(5.20) $\{I + (k\rho/\gamma)^2(I - M_e) + \mathcal{A}\}\boldsymbol{J} + \mathrm{i}(\rho/\delta)\mathcal{B}\boldsymbol{M} = 2(k\rho/\gamma)^2\boldsymbol{J}_{\mathrm{inc}},$

(5.21) $-\mathrm{i}(\delta/\rho)\mathcal{B}\boldsymbol{J} + \{I + (k\delta/\gamma)^2(I - M_e) + \mathcal{A}\}\boldsymbol{M} = 2(k\delta/\gamma)^2\boldsymbol{M}_{\mathrm{inc}},$

where

$$\mathcal{A} = \tfrac{1}{2}(k/\gamma^2)(\gamma_L M_L + \gamma_R M_R + P_L - P_R),$$
$$\mathcal{B} = \tfrac{1}{2}(k/\gamma^2)(\gamma_L M_L - \gamma_R M_R + P_L + P_R - 2P_e).$$

Equations (5.20) and (5.21) are our generalized Müller equations for solving the problem of electromagnetic scattering by a chiral obstacle. When the obstacle is achiral ($\beta = 0$), the equations reduce to Müller's equations:

(5.22) $\{I + \rho^2(I - M_e) + \mathcal{A}_0\}\boldsymbol{J} + \mathrm{i}(\rho/\delta)\mathcal{B}_0\boldsymbol{M} = 2\rho^2\boldsymbol{J}_{\mathrm{inc}},$

(5.23) $-\mathrm{i}(\delta/\rho)\mathcal{B}_0\boldsymbol{J} + \{I + \delta^2(I - M_e) + \mathcal{A}_0\}\boldsymbol{M} = 2\delta^2\boldsymbol{M}_{\mathrm{inc}};$

here, $\mathcal{A}_0 = M$, $\mathcal{B}_0 = k^{-1}(P - P_e)$, and M and P are defined by (3.7) and (3.8), respectively, with γ_α replaced by k.

5.1. Uniqueness. The pair (5.20) and (5.21) is uniquely solvable. Existence follows as when $\beta = 0$; see the paragraph following (5.10). To prove uniqueness, we adapt fairly standard arguments. Thus, let $\boldsymbol{J}_0$ and $\boldsymbol{M}_0$ solve the homogeneous forms of (5.20) and (5.21), and then construct the following eight fields:

$$-C_e\boldsymbol{M}_0 + \mathrm{i}k_e^{-1}F_e\boldsymbol{J}_0 = \begin{cases} 2\boldsymbol{E}_e^0(P), & P \in B_e, \\ -2\widehat{\boldsymbol{E}}(P), & P \in B_i, \end{cases}$$

$$C_e\boldsymbol{J}_0 + \mathrm{i}k_e^{-1}F_e\boldsymbol{M}_0 = \begin{cases} 2\boldsymbol{H}_e^0(P), & P \in B_e, \\ -2\widehat{\boldsymbol{H}}(P), & P \in B_i, \end{cases}$$

$$-\tfrac{1}{2}\rho(C_L + C_R + \gamma_L^{-1}F_L - \gamma_R^{-1}F_R)\boldsymbol{M}_0$$
$$+ \tfrac{1}{2}\mathrm{i}\delta(C_L - C_R + \gamma_L^{-1}F_L + \gamma_R^{-1}F_R)\boldsymbol{J}_0 = \begin{cases} 2\widehat{\boldsymbol{E}}_e(P), & P \in B_e, \\ -2\boldsymbol{E}^0(P), & P \in B_i, \end{cases}$$

$$\tfrac{1}{2}\delta(C_L + C_R + \gamma_L^{-1}F_L - \gamma_R^{-1}F_R)\boldsymbol{J}_0$$
$$+ \tfrac{1}{2}\mathrm{i}\rho(C_L - C_R + \gamma_L^{-1}F_L + \gamma_R^{-1}F_R)\boldsymbol{M}_0 = \begin{cases} 2\widehat{\boldsymbol{H}}_e(P), & P \in B_e, \\ -2\boldsymbol{H}^0(P), & P \in B_i. \end{cases}$$

Evaluating the tangential components of these fields on S, we find first that

(5.24) $\boldsymbol{J}_0 = \boldsymbol{n} \times \boldsymbol{H}_e^0 + \boldsymbol{n} \times \widehat{\boldsymbol{H}} = \delta^{-1}(\boldsymbol{n} \times \widehat{\boldsymbol{H}}_e + \boldsymbol{n} \times \boldsymbol{H}^0),$

(5.25) $-\boldsymbol{M}_0 = \boldsymbol{n} \times \boldsymbol{E}_e^0 + \boldsymbol{n} \times \widehat{\boldsymbol{E}} = \rho^{-1}(\boldsymbol{n} \times \widehat{\boldsymbol{E}}_e + \boldsymbol{n} \times \boldsymbol{E}^0).$

Next, from the governing integral equations for $\boldsymbol{J}_0$ and $\boldsymbol{M}_0$ (in the form of (5.9) and (5.10)), we calculate that

$$\alpha_1\,\boldsymbol{n} \times \widehat{\boldsymbol{H}} - \alpha_2\,\boldsymbol{n} \times \widehat{\boldsymbol{E}} + \delta^{-1}\alpha_3\,\boldsymbol{n} \times \widehat{\boldsymbol{H}}_e - \rho^{-1}\alpha_4\,\boldsymbol{n} \times \widehat{\boldsymbol{E}}_e = \boldsymbol{0} \quad \text{on } S,$$

with another equation obtained by changing α_j to α'_j. As $\alpha_2 = \alpha'_1 = 0$, these are seen to be generalized transmission conditions in the form of (A.1) and (A.2), in which

$$c_{11} = \frac{-\alpha'_4}{\rho\alpha'_2}, \quad c_{12} = \frac{\alpha'_3}{\delta\alpha'_2}, \quad c_{21} = \frac{\alpha_4}{\rho\alpha_1}, \quad \text{and} \quad c_{22} = \frac{-\alpha_3}{\delta\alpha_1}.$$

It follows that $\{\widehat{\boldsymbol{E}}_e, \widehat{\boldsymbol{H}}_e\}$ and $\{\widehat{\boldsymbol{E}}, \widehat{\boldsymbol{H}}\}$ solve a homogeneous generalized transmission problem in which the materials have been interchanged (achiral obstacle embedded in a chiral medium), and so these fields must vanish identically by the uniqueness theorem proved in appendix A.

Now, consider the fields $\{\boldsymbol{E}_e^0, \boldsymbol{H}_e^0\}$ and $\{\boldsymbol{E}^0, \boldsymbol{H}^0\}$. Direct calculation shows that

$$\rho\,\boldsymbol{n} \times \boldsymbol{E}_e^0 - \boldsymbol{n} \times \boldsymbol{E}^0 = \boldsymbol{n} \times \widehat{\boldsymbol{E}}_e - \rho\,\boldsymbol{n} \times \widehat{\boldsymbol{E}} = \boldsymbol{0} \quad \text{on } S,$$

as $\widehat{\boldsymbol{E}}_e \equiv \boldsymbol{0}$ and $\widehat{\boldsymbol{E}} \equiv \boldsymbol{0}$. Similarly, $\delta\,\boldsymbol{n} \times \boldsymbol{H}_e^0 = \boldsymbol{n} \times \boldsymbol{H}^0$ on S. Hence, as $\{\boldsymbol{E}_e^0, \boldsymbol{H}_e^0\}$ and $\{\boldsymbol{E}^0, \boldsymbol{H}^0\}$ solve the homogeneous transmission problem, they also must vanish identically. Finally, (5.24) and (5.25) imply that $\boldsymbol{J}_0 \equiv \boldsymbol{0}$ and $\boldsymbol{M}_0 \equiv \boldsymbol{0}$, as required.

6. Low-chirality approximations. In practice, the chirality parameter $k\beta$ is usually small. This motivates the development of low-chirality approximations. Thus, corrections to first-order in $k\beta$ for sources above a slightly chiral half-space have been found [18], [26]. Here, we consider how to compute such corrections for scattering by a slightly chiral obstacle.

It is known that the solution of the transmission problem depends continuously on β as $\beta \to 0$ through positive real values [2]. In fact, the solution is actually an analytic function of β. To see this, consider the Bohren field $\boldsymbol{Q}_L$. From (2.9), we see that $\boldsymbol{Q}_L$ satisfies

$$(\nabla^2 + \gamma_L^2)\,\boldsymbol{Q}_L = \boldsymbol{0}.$$

Thus, $\boldsymbol{Q}_L$ is an analytic function of γ_L; see [28], for example. But, from (2.10), γ_L is a rational function of β, whence $\boldsymbol{Q}_L$ is an analytic function of β. A similar argument applies to $\boldsymbol{Q}_R$. Since k is fixed and $|k\beta|$ is small, it follows from the Bohren decomposition (2.8) that the solution is an analytic function of $k\beta$ for $|k\beta| < 1$.

Rather than seek power-series solutions of the transmission problem directly, we expand the solutions of the governing integral equations. Thus, we write

$$\boldsymbol{J} \simeq \boldsymbol{J}_0 + \mathrm{i}k\beta\boldsymbol{J}_1 \quad \text{and} \quad \boldsymbol{M} \simeq \boldsymbol{M}_0 + \mathrm{i}k\beta\boldsymbol{M}_1,$$

where the error is $O((k\beta)^2)$ as $k\beta \to 0$. For small $k\beta$, we have

$$\gamma_L \sim k(1 + k\beta), \quad \mathrm{e}^{\mathrm{i}\gamma_L R} \sim \mathrm{e}^{\mathrm{i}kR}(1 + \mathrm{i}k^2\beta R),$$

and

$$S_L\boldsymbol{a} \simeq S\boldsymbol{a} + \mathrm{i}k\beta\,T\boldsymbol{a},$$

where

$$(T\boldsymbol{a})(P) = \frac{k}{2\pi} \int_S \boldsymbol{a}(q)\,\mathrm{e}^{\mathrm{i}kR}\,ds_q.$$

Similarly, $S_R\boldsymbol{a} \simeq S\boldsymbol{a} - \mathrm{i}k\beta\,T\boldsymbol{a}$. Thus, we obtain

$$M_L \simeq M + \mathrm{i}k\beta\,L, \qquad P_L \simeq P + \mathrm{i}k\beta\,Q,$$
$$M_R \simeq M - \mathrm{i}k\beta\,L, \text{ and } P_R \simeq P - \mathrm{i}k\beta\,Q$$

with an error of $O((k\beta)^2)$, where

$$L\boldsymbol{a} = \boldsymbol{n} \times \operatorname{curl}\{T\boldsymbol{a}\} \quad \text{and} \quad Q\boldsymbol{a} = \boldsymbol{n} \times \operatorname{curl}\operatorname{curl}\{T\boldsymbol{a}\}.$$

Note that L is an integral operator with a continuous kernel, whereas Q has a weak singularity.

Substituting these approximations into the exact boundary integral equations, (5.20) and (5.21), we find that $\{J_0, M_0\}$ solves Müller's equations, (5.22) and (5.23). The first-order correction $\{J_1, M_1\}$ also solves Müller's equations but with different functions on the right-hand sides: replace

$$2\rho^2 J_{\text{inc}} \quad \text{by} \quad -k^{-1}Q J_0 - (\rho/\delta)(M + \mathrm{i}L)M_0$$

and

$$2\delta^2 M_{\text{inc}} \quad \text{by} \quad -k^{-1}Q M_0 + (\delta/\rho)(M + \mathrm{i}L)J_0.$$

In principle, one can calculate higher-order approximations, but the new right-hand sides will be much more complicated. However, we can see that computing the first-order correction is fairly straightforward.

The virtue of this strategy is clear. Sophisticated codes have been developed for solving Müller's equations [8]. By a simple modification, these can be used to compute the correction due to chirality of the obstacle. Note that the approximations obtained for J and M can be inserted into (3.2) and (3.6) so as to generate exact solutions of the governing equations in B_e and B_i.

Appendix A. A chiral/chiral uniqueness theorem. Consider a chiral obstacle surrounded by a different unbounded chiral medium. The parameters of the external medium are distinguished by the subscript e. We are going to prove that, under certain conditions, the following transmission problem has only the trivial solution. Note that we consider more general interface conditions than previously; they are needed in our analysis of the system of integral equations in section 5.1.

Homogeneous transmission problem. Find electric fields E_e and E and magnetic fields H_e and H that satisfy

$$\operatorname{curl} E_e - \mathrm{i}(\gamma_e^2/k_e)H_e - \beta_e\gamma_e^2 E_e = 0$$

and

$$\operatorname{curl} H_e + \mathrm{i}(\gamma_e^2/k_e)E_e - \beta_e\gamma_e^2 H_e = 0$$

in B_e,

$$\operatorname{curl} E - \mathrm{i}(\gamma^2/k)H - \beta\gamma^2 E = 0$$

and

$$\operatorname{curl} H + \mathrm{i}(\gamma^2/k)E - \beta\gamma^2 H = 0$$

in B_i, and two generalized transmission conditions on the interface S,

(A.1) $$n \times E = c_{11}\, n \times E_e + c_{12}\, n \times H_e$$

and

(A.2) $$n \times H = c_{21}\, n \times E_e + c_{22}\, n \times H_e,$$

where c_{11}, c_{12}, c_{21}, and c_{22} are constants. In addition, $\{\boldsymbol{E}_e, \boldsymbol{H}_e\}$ must satisfy a Silver–Müller radiation condition (4.5).

We proceed in a standard way [4], [23], although some deviations are discussed in detail. Thus, let S_R denote a large sphere of radius R enclosing B_i, and let $B_{e,R}$ denote the region bounded internally by S and externally by S_R.

An application of the divergence theorem in B_i to div $(\boldsymbol{E} \times \overline{\boldsymbol{H}})$ gives

$$\int_S \boldsymbol{n} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}})\, ds = I(\boldsymbol{E}, \boldsymbol{H}, \beta\gamma^2, \gamma^2/k; B_i) \equiv I_i,$$

say, where the overbar denotes complex conjugation, and

$$(A.3) \qquad I(\boldsymbol{E}, \boldsymbol{H}, \kappa_1, \kappa_2; B) = \mathrm{i} \int_B \left\{ 2\,\mathrm{Im}\,(\kappa_1)\, \boldsymbol{E} \cdot \overline{\boldsymbol{H}} + \kappa_2 |\boldsymbol{H}|^2 - \bar{\kappa}_2 |\boldsymbol{E}|^2 \right\} dV.$$

Similarly, an application of the divergence theorem in $B_{e,R}$ to div $(\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e})$ gives

$$\int_{S_R} \hat{\boldsymbol{r}} \cdot (\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e})\, ds - \int_S \boldsymbol{n} \cdot (\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e})\, ds = I(\boldsymbol{E}_e, \boldsymbol{H}_e, \beta_e \gamma_e^2, \gamma_e^2/k_e; B_{e,R}) \equiv I_e,$$

say. Hence

$$(A.4) \qquad \mathrm{Re} \int_{S_R} \boldsymbol{E}_e \cdot (\hat{\boldsymbol{r}} \times \overline{\boldsymbol{H}_e})\, ds + \mathrm{Re}\,\{I_e\} + \mathrm{Re} \int_S \boldsymbol{n} \cdot (\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e})\, ds = 0$$

and

$$(A.5) \qquad \mathrm{Re}\,\{I_i\} = \mathrm{Re} \int_S \boldsymbol{n} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}})\, ds.$$

Now, we eliminate the integrals over S using the transmission conditions. These give

$$\boldsymbol{n} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}}) = c_{11} \bar{c}_{22}\, \boldsymbol{n} \cdot (\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e}) - c_{12} \bar{c}_{21}\, \boldsymbol{n} \cdot (\overline{\boldsymbol{E}_e} \times \boldsymbol{H}_e).$$

Assume that

$$c_{11} \bar{c}_{22} \qquad \text{and} \qquad c_{12} \bar{c}_{21} \qquad \text{are real}$$

and that

$$(A.6) \qquad \Delta \equiv c_{11} \bar{c}_{22} - c_{12} \bar{c}_{21} > 0.$$

Then $\mathrm{Re}\,\{\boldsymbol{n} \cdot (\boldsymbol{E} \times \overline{\boldsymbol{H}})\} = \Delta\,\mathrm{Re}\,\{\boldsymbol{n} \cdot (\boldsymbol{E}_e \times \overline{\boldsymbol{H}_e})\}$ whence (A.4) and (A.5) give

$$(A.7) \qquad \mathrm{Re} \int_{S_R} \boldsymbol{E}_e \cdot (\hat{\boldsymbol{r}} \times \overline{\boldsymbol{H}_e})\, ds + \mathrm{Re}\,\{I_e\} + \Delta^{-1}\,\mathrm{Re}\,\{I_i\} = 0.$$

Next, we use the radiation condition. This gives

$$\int_{S_R} |\hat{\boldsymbol{r}} \times \overline{\boldsymbol{H}_e} + \boldsymbol{E}_e|^2\, ds = \int_{S_R} \left(|\hat{\boldsymbol{r}} \times \overline{\boldsymbol{H}_e}|^2 + |\boldsymbol{E}_e|^2 \right) ds + 2\,\mathrm{Re} \int_{S_R} \boldsymbol{E}_e \cdot (\hat{\boldsymbol{r}} \times \overline{\boldsymbol{H}_e})\, ds = o(1)$$

as $R \to \infty$. We can eliminate the last integral using (A.7). From (A.3), we have

$$\mathrm{Re}\,\{I\} = -\mathrm{Im}\,(\kappa_2) \int_B \left(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 \right) dV - 2\,\mathrm{Im}\,(\kappa_1) \int_B \mathrm{Im}\,(\boldsymbol{E} \cdot \overline{\boldsymbol{H}}) dV.$$

But, using the Bohren transformation (2.8), we obtain

$$2|\boldsymbol{E}|^2 + 2|\boldsymbol{H}|^2 = |\boldsymbol{Q}_L|^2 + |\boldsymbol{Q}_R|^2 \quad \text{and} \quad 4\,\mathrm{Im}\,(\boldsymbol{E}\cdot\overline{\boldsymbol{H}}) = |\boldsymbol{Q}_L|^2 - |\boldsymbol{Q}_R|^2,$$

whence

$$2\,\mathrm{Re}\,\{I\} = -\mathrm{Im}\,(\kappa_2 + \kappa_1)\int_B |\boldsymbol{Q}_L|^2\,dV - \mathrm{Im}\,(\kappa_2 - \kappa_1)\int_B |\boldsymbol{Q}_R|^2\,dV.$$

It follows that

$$\int_{S_R}\left(|\hat{\boldsymbol{r}}\times\overline{\boldsymbol{H}_e}|^2 + |\boldsymbol{E}_e|^2\right)ds + \mathrm{Im}\,(\gamma_{eL})\int_{B_{e,R}}|\boldsymbol{Q}_{eL}|^2\,dV + \mathrm{Im}\,(\gamma_{eR})\int_{B_{e,R}}|\boldsymbol{Q}_{eR}|^2\,dV$$

$$+\Delta^{-1}\left\{\mathrm{Im}\,(\gamma_L)\int_{B_i}|\boldsymbol{Q}_L|^2\,dV + \mathrm{Im}\,(\gamma_R)\int_{B_i}|\boldsymbol{Q}_R|^2\,dV\right\} = o(1)$$

as $R \to \infty$.

Thus, assuming that all the coefficients multiplying the above integrals are nonnegative, we deduce that

(A.8)
$$\int_{S_R}|\boldsymbol{E}_e|^2\,ds = o(1) \quad \text{as } R \to \infty.$$

Now, we have used the Silver–Müller condition (4.5). However, we also know that $\{\boldsymbol{E}_e, \boldsymbol{H}_e\}$ satisfies the other Silver–Müller condition, namely,

(A.9)
$$\hat{\boldsymbol{r}}_P \times \boldsymbol{E}_e - \boldsymbol{H}_e = o(r_P^{-1}) \quad \text{as } r_P \to \infty.$$

(This can be shown as follows. First, note that the generalized Stratton–Chu formula (3.6) holds in B_e, if we replace the left-hand side by $+2\boldsymbol{U}(P)$. From the representations (3.3) and (3.4), we can show by direct calculation that

$$\hat{\boldsymbol{r}}_P \times \boldsymbol{Q}_{eL} + \mathrm{i}\boldsymbol{Q}_{eL} \quad \text{and} \quad \hat{\boldsymbol{r}}_P \times \boldsymbol{Q}_{eR} - \mathrm{i}\boldsymbol{Q}_{eR}$$

are both $o(r_P^{-1})$ as $r_P \to \infty$. From these, we can derive (4.5) and (A.9), using (2.8).) Then, repeating the calculation, we deduce that

$$\int_{S_R}|\boldsymbol{H}_e|^2\,ds = o(1) \quad \text{as } R \to \infty.$$

When this is combined with (A.8), we obtain

$$\int_{S_R}|\boldsymbol{Q}_{eL}|^2\,ds = o(1) \quad \text{and} \quad \int_{S_R}|\boldsymbol{Q}_{eR}|^2\,ds = o(1)$$

as $R \to \infty$. But, from the curl of (2.9), we have

$$(\nabla^2 + \gamma_{eL}^2)\boldsymbol{Q}_{eL} = \boldsymbol{0} \quad \text{and} \quad (\nabla^2 + \gamma_{eR}^2)\boldsymbol{Q}_{eR} = \boldsymbol{0}$$

in B_e. Hence, Rellich's lemma implies that $\boldsymbol{Q}_{eL} \equiv \boldsymbol{0}$ and $\boldsymbol{Q}_{eR} \equiv \boldsymbol{0}$ in B_e. The rest of the argument is standard, and so we obtain uniqueness, assuming that

$$\mathrm{Im}\,(\gamma_L) \geq 0, \quad \mathrm{Im}\,(\gamma_R) \geq 0,$$
$$\mathrm{Re}\,(\gamma_{eL}) > 0, \quad \mathrm{Re}\,(\gamma_{eR}) > 0,$$
$$\mathrm{Im}\,(\gamma_{eL}) \geq 0, \quad \mathrm{Im}\,(\gamma_{eR}) \geq 0$$

and that (A.6) holds.

For the standard transmission conditions (4.4), we have $c_{11} = \rho$, $c_{12} = c_{21} = 0$, and $c_{22} = \delta$ whence $\Delta = c_{11}\bar{c}_{22} = \rho\bar{\delta}$ and $c_{12}\bar{c}_{21} = 0$. Then (A.6) reduces to the condition that $\rho\bar{\delta}$ be real. In terms of the intrinsic impedances η and η_e, (A.6) is equivalent to

$$\eta_e/\eta \text{ is real and positive.}$$

Appendix B. Dyadic Green's functions. For Maxwell's equations, we have the dyadic Green's function $\mathbf{G}_0$, defined by

$$(\text{B.1}) \qquad \mathbf{G}_0(P, Q; k) = (\mathbf{I} + k^{-2}\nabla\nabla) \, G(P, Q; k),$$

where $\mathbf{I}$ is the unit tensor; see, for example, [22] or [9, Chapter 7]. In terms of components,

$$G_{ij} = \left(\delta_{ij} + \frac{1}{k^2}\frac{\partial^2}{\partial x_i \partial x_j}\right) G,$$

where $\boldsymbol{r}_P = (x_1, x_2, x_3)$. This leads to the representation

$$(\text{B.2}) \quad 2\boldsymbol{U}(P) = -\int_S \{\text{curl}_P \, \mathbf{G}_0(P, q; k) \cdot (\boldsymbol{n} \times \boldsymbol{U}) + \mathbf{G}_0(P, q; k) \cdot (\boldsymbol{n} \times \text{curl}\,\boldsymbol{U})\} \, ds_q$$

for $P \in B_i$; see [22, equation (2.15)], noting that

$$(\text{B.3}) \qquad \text{curl}_P \{\mathbf{G}_0(P, Q; k)\} \cdot \boldsymbol{a} = \text{curl}_P \{G(P, Q; k)\,\boldsymbol{a}\}$$
$$= (\text{grad}_P \, G) \times \boldsymbol{a} = \boldsymbol{a} \cdot \{\text{curl}_Q \, \mathbf{G}_0(Q, P; k)\},$$

where $\boldsymbol{a}$ is any vector that does not depend on P.

The representation (B.2) is identical to the Stratton–Chu formula (3.2). To see this, first note that (B.3) immediately gives

$$C\{\boldsymbol{n} \times \boldsymbol{U}\} = \int_S \text{curl}_P \, \mathbf{G}_0(P, q; k) \cdot (\boldsymbol{n} \times \boldsymbol{U}) \, ds_q.$$

From Maxwell's equations (that is, (2.5) with $\beta = 0$), we have $\text{curl}\,\boldsymbol{U} = k\boldsymbol{U}'$, so it remains to show that

$$(\text{B.4}) \qquad (F\boldsymbol{a})(P) = k^2 \int_S \mathbf{G}_0(P, q; k) \cdot \boldsymbol{a}(q) \, ds_q,$$

where $\boldsymbol{a}(q) = \boldsymbol{n} \times \boldsymbol{U}'$. Substituting for F and $\mathbf{G}_0$, we have to show that

$$(\text{B.5}) \qquad \text{curl}_P \text{curl}_P \{\boldsymbol{a}(q) \, G(P, q; k)\} = k^2 \boldsymbol{a} \, G + \text{grad}_P \{\text{div}_P (\boldsymbol{a}\,G)\}.$$

But $\text{curl}_P \{\boldsymbol{a}\,G\} = (\text{grad}_P \, G) \times \boldsymbol{a}(q)$ whence the left-hand side of (B.5) is $k^2 \boldsymbol{a} \, G + (\boldsymbol{a} \cdot \nabla_P)\,\text{grad}_P \, G$. The result follows by noting that, for any constant vector $\boldsymbol{e}$,

$$\boldsymbol{e} \cdot (\boldsymbol{a} \cdot \nabla_P)\,\text{grad}_P \, G = e_i a_j \frac{\partial}{\partial x_j}\frac{\partial G}{\partial x_i} = e_i \frac{\partial^2}{\partial x_i \partial x_j}(a_j G) = \boldsymbol{e} \cdot \text{grad}_P \{\text{div}_P (\boldsymbol{a}\,G)\}.$$

It is known that the two representations (3.2) and (B.2) are identical [9, section 1.4.2] but this fact is seldom exploited. We can use it to derive the chiral generalization of the Stratton–Chu formula from the known chiral analogue of (B.2).

Thus, for chiral media, the dyadic Green's function is given by [16, p. 60]

$$\mathbf{G}(P,Q) = \mathbf{G}_L(P,Q) + \mathbf{G}_R(P,Q),$$

where $\mathbf{G}_L$ is a left-handed field, $\mathbf{G}_R$ is a right-handed field,

$$\mathbf{G}_L(P,Q) = \tfrac{1}{2}k\gamma^{-2} \left\{ \gamma_L \, \mathbf{G}_0(P,Q;\gamma_L) + \mathrm{curl}_P \, \mathbf{I} \, G(P,Q;\gamma_L) \right\},$$
$$\mathbf{G}_R(P,Q) = \tfrac{1}{2}k\gamma^{-2} \left\{ \gamma_R \, \mathbf{G}_0(P,Q;\gamma_R) - \mathrm{curl}_P \, \mathbf{I} \, G(P,Q;\gamma_R) \right\},$$

and $\mathbf{G}_0$ is defined by (B.1); note that $\mathbf{G}(P,Q) = \mathbf{G}_0(P,Q;k)$ when $\beta = 0$. This leads to the representation

$$-2\boldsymbol{U}(P) = \int_S \mathrm{curl}_P \, \mathbf{G}(P,q) \cdot (\boldsymbol{n} \times \boldsymbol{U}) \, ds_q$$
$$- 2\gamma^2\beta \int_S \mathbf{G}(P,q) \cdot (\boldsymbol{n} \times \boldsymbol{U}) \, ds_q + \int_S \mathbf{G}(P,q) \cdot (\boldsymbol{n} \times \mathrm{curl}\,\boldsymbol{U}) \, ds_q,$$

which reduces to (B.2) when $\beta = 0$. Making use of (2.7), we can eliminate $\mathrm{curl}\,\boldsymbol{U}$ to give

$$-2\boldsymbol{U}(P) = \int_S \mathrm{curl}_P \, \mathbf{G}(P,q) \cdot (\boldsymbol{n} \times \boldsymbol{U}) \, ds_q$$
$$+ \int_S \mathbf{G}(P,q) \cdot \left\{ \frac{\gamma^2}{k}(\boldsymbol{n} \times \boldsymbol{U}') - \gamma^2\beta(\boldsymbol{n} \times \boldsymbol{U}) \right\} ds_q.$$

Write this formula as $-2\boldsymbol{U} = I_1 + I_2$ and then consider each integral separately. For I_1, we note that

$$\mathrm{curl}_P \, \mathbf{G}_L(P,q) \cdot \boldsymbol{a} = \tfrac{1}{2}k\gamma^{-2} \left(\gamma_L \, \mathrm{curl}_P + \mathrm{curl}_P \, \mathrm{curl}_P \right) \left\{ \boldsymbol{a} \, G(P,q;\gamma_L) \right\}$$

and

$$\mathrm{curl}_P \, \mathbf{G}_R(P,q) \cdot \boldsymbol{a} = \tfrac{1}{2}k\gamma^{-2} \left(\gamma_R \, \mathrm{curl}_P - \mathrm{curl}_P \, \mathrm{curl}_P \right) \left\{ \boldsymbol{a} \, G(P,q;\gamma_R) \right\}.$$

Hence, the first integral I_1 can be written as

$$I_1 = \tfrac{1}{2}k\gamma^{-2} \left(\gamma_L \, C_L\{\boldsymbol{n} \times \boldsymbol{U}\} + \gamma_R \, C_R\{\boldsymbol{n} \times \boldsymbol{U}\} \right)$$
$$+ \tfrac{1}{2}k\gamma^{-2} \left(F_L\{\boldsymbol{n} \times \boldsymbol{U}\} - F_R\{\boldsymbol{n} \times \boldsymbol{U}\} \right),$$

where C_α and F_α are defined by (3.5).

Next, consider the second integral I_2. We have

$$\int_S \mathbf{G}_L(P,q) \cdot \boldsymbol{a}(q) \, ds_q = \frac{k\gamma_L}{2\gamma^2} \int_S \mathbf{G}_0(P,q;\gamma_L) \cdot \boldsymbol{a}(q) \, ds_q$$
$$+ \frac{k}{2\gamma^2} \, \mathrm{curl} \int_S \boldsymbol{a}(q) \, G(P,q;\gamma_L) \, ds_q$$
$$= \tfrac{1}{2}k\gamma^{-2} \left\{ \gamma_L^{-1} F_L\boldsymbol{a} + C_L\boldsymbol{a} \right\},$$

using (B.4). Similarly

$$\int_S \mathbf{G}_R(P,q) \cdot \boldsymbol{a}(q) \, ds_q = \tfrac{1}{2}k\gamma^{-2} \left\{ \gamma_R^{-1} F_R\boldsymbol{a} - C_R\boldsymbol{a} \right\}.$$

Hence,

$$
\begin{aligned}
I_2 = {} & \tfrac{1}{2}\left(\gamma_L^{-1} F_L\{\boldsymbol{n}\times\boldsymbol{U'}\} + C_L\{\boldsymbol{n}\times\boldsymbol{U'}\}\right) \\
& + \tfrac{1}{2}\left(\gamma_R^{-1} F_R\{\boldsymbol{n}\times\boldsymbol{U'}\} - C_R\{\boldsymbol{n}\times\boldsymbol{U'}\}\right) \\
& - \tfrac{1}{2}k\beta\left(\gamma_L^{-1} F_L\{\boldsymbol{n}\times\boldsymbol{U}\} + C_L\{\boldsymbol{n}\times\boldsymbol{U}\}\right) \\
& - \tfrac{1}{2}k\beta\left(\gamma_R^{-1} F_R\{\boldsymbol{n}\times\boldsymbol{U}\} - C_R\{\boldsymbol{n}\times\boldsymbol{U}\}\right).
\end{aligned}
$$

Finally, grouping terms, using $k\gamma^{-2}\gamma_L - k\beta = k\gamma^{-2}\gamma_R + k\beta = 1$, we obtain (3.6), as required. This gives an alternative proof of (3.6).

Acknowledgment. The authors are grateful to the referees for their constructive comments, including the observation (2.11) leading to a simpler proof of (3.6).

REFERENCES

[1] H. AMMARI AND J. C. NÉDÉLEC, *Time-harmonic electromagnetic fields in chiral media*, in Modern Mathematical Methods in Diffraction Theory and its Applications in Engineering, E. Meister, ed., Methoden und Verfahren der mathematischen Physik 42, Peter Lang, Frankfurt am Main, 1997, pp. 174–202.

[2] H. AMMARI AND J. C. NÉDÉLEC, *Small chirality behavior of solutions to electromagnetic scattering problems in chiral media*, Math. Methods. Appl. Sci., 21 (1998), pp. 327–359.

[3] H. AMMARI, M. LAOUADI, AND J.-C. NÉDÉLEC, *Low frequency behavior of solutions to electromagnetic scattering problems in chiral media*, SIAM J. Appl. Math., 58 (1998), pp. 1022–1042.

[4] C. ATHANASIADIS AND I. G. STRATIS, *Electromagnetic scattering by a chiral obstacle*, IMA J. Appl. Math., 58 (1997), pp. 83–91.

[5] C. ATHANASIADIS AND I. G. STRATIS, *Low frequency electromagnetic scattering theory for a multi-layered chiral scatterer*, Methods Appl. Anal., to appear.

[6] S. BASSIRI, *Electromagnetic waves in chiral media*, in Recent Advances in Electromagnetic Theory, H. N. Kritikos, and D. L. Jaggard, eds., Springer, New York, 1990, pp. 1–30.

[7] C. F. BOHREN, *Light scattering by an optically active sphere*, Chemical Phys. Lett., 29 (1974), pp. 458–462.

[8] J. C. CHAO, Y. J. LIU, F. J. RIZZO, P. A. MARTIN, AND L. UDPA, *Regularized integral equations and curvilinear boundary elements for electromagnetic wave scattering in three dimensions*, IEEE Trans. Antennas and Propagation, AP-43 (1995), pp. 1416–1422.

[9] W. C. CHEW, *Waves and Fields in Inhomogeneous Media*, IEEE Press, New York, 1995.

[10] D. COLTON AND R. KRESS, *Integral Equation Methods in Scattering Theory*, John Wiley, New York, 1983.

[11] M. GARDNER, *The New Ambidextrous Universe*, 3rd ed., W. H. Freeman, New York, 1990.

[12] R. F. HARRINGTON, *Boundary integral formulations for homogeneous material bodies*, J. Electromagnet. Waves Appl., 3 (1989), pp. 1–15.

[13] D. S. JONES, *Methods in Electromagnetic Wave Propagation*, 2nd ed., Clarendon Press, Oxford, U.K., 1994.

[14] R. E. KLEINMAN AND P. A. MARTIN, *On single integral equations for the transmission problem of acoustics*, SIAM J. Appl. Math., 48 (1988), pp. 307–325.

[15] A. LAKHTAKIA, *Beltrami Fields in Chiral Media*, World Scientific, Singapore, 1994.

[16] A. LAKHTAKIA, V. K. VARADAN, AND V. V. VARADAN, *Time-harmonic Electromagnetic Fields in Chiral Media*, Lecture Notes in Phys. 335, Springer, Berlin, 1989.

[17] A. LAKHTAKIA, V. K. VARADAN, AND V. V. VARADAN, *Surface integral equations for scattering by PEC scatterers in isotropic chiral media*, Internat. J. Engrg. Sci., 29 (1991), pp. 179–185.

[18] I. V. LINDELL, *Exact image theory for vertical electromagnetic sources above a slightly chiral half space*, J. Electromagnet. Waves Appl., 10 (1996), pp. 1583–1594.

[19] I. V. LINDELL, A. H. SIHVOLA, S. A. TRETYAKOV, AND A. J. VIITANEN, *Electromagnetic Waves in Chiral and Bi-isotropic Media*, Artech House, Boston, 1994.

[20] P. A. MARTIN AND P. OLA, *Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle*, Proc. Roy. Soc. Edinburgh Sect. A, 123A (1993), pp. 185–208.

[21] C. MÜLLER, *Foundations of the Mathematical Theory of Electromagnetic Waves*, Springer, Berlin, 1969.

[22] G. Neave, *A uniquely solvable integral equation for the exterior electromagnetic scattering problem*, Quart. J. Mech. Appl. Math., 40 (1987), pp. 57–70.

[23] P. Ola, *Boundary integral equations for the scattering of electromagnetic waves by a homogeneous chiral obstacle*, J. Math. Phys., 35 (1994), pp. 3969–3980.

[24] R. G. Rojas, *Integral equations for the scattering by a three-dimensional inhomogeneous chiral body*, J. Electromagnet. Waves Appl., 6 (1992), pp. 733–750.

[25] R. G. Rojas, *Integral equations for EM scattering by homogeneous/inhomogeneous two-dimensional chiral bodies*, IEEE Proc. Microwaves, Antennas and Propagation, 141 (1994), pp. 385–392.

[26] L. H. Ruotanen and I. V. Lindell, *Exact image theory for horizontal sources above a slightly chiral half space*, J. Electromagnet. Waves Appl., 11 (1997), pp. 1749–1762.

[27] C.-T. Tai, *Dyadic Green Functions in Electromagnetic Theory*, 2nd ed., IEEE Press, New York, 1994.

[28] P. Werner, *On the exterior boundary value problem of perfect reflection for stationary electromagnetic wave fields*, J. Math. Anal. Appl., 7 (1963) 348–396.