Mathematical Methods in the Applied Sciences
Math. Meth. Appl. Sci., **22**, 1175–1188 (1999)
MOS subject classification: 35 Q 60; 45 F 05; 78 A 45

Electromagnetic Scattering by a Homogeneous Chiral Obstacle: Scattering Relations and the Far-Field Operator

C. Athanasiadis[1], P. A. Martin[2,*] and I. G. Stratis[1]

[1] *Department of Mathematics, University of Athens, Panepistimiopolis, GR 15784 Athens, Greece*
[2] *Department of Mathematics, University of Manchester, Manchester M13 9PL, U.K.*

Communicated by G. F. Roach

Time-harmonic electromagnetic waves are scattered by a homogeneous chiral obstacle. The reciprocity principle, the basic scattering theorem and an optical theorem are proved. These results are used to prove that if the chirality measure of the obstacle is real, then the far-field operator is normal. Moreover, it is shown that the eigenvalues of the far-field operator are the same as the eigenvalues of Waterman's T-matrix. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS chiral media; transmission problem; far-field operator; T-matrix

1. Introduction

In recent years, extensive work has been performed on various problems related to electromagnetic scattering in chiral media; see the monograph by Lakhtakia [15] and the references therein.

In this work we study the reciprocity principle, the general scattering theorem and an optical theorem for the far-field pattern corresponding to the scattering of plane time-harmonic electromagnetic waves by a homogeneous chiral obstacle. Such results for achiral obstacles have been proved by Twersky in [18]; analogous scattering relations for a piecewise homogeneous achiral obstacle have been proved by one of the present authors [3]. Reciprocity relations for scattering problems corresponding to different boundary conditions have been studied by Angell *et al.* [2], and by Colton and Kress in their book [9].

In section 2 we formulate the scattering problem and furnish an equivalent dimensionless transmission problem; the introduction of convenient notation allows a

* Correspondence to: P. A. Martin, Department of Mathematics, University of Manchester, Manchester M13 9PL, UK. E-mail: pamartin@man.ac.uk

Contract/grant sponsor: British Council
Contract/grant sponsor: University of Athens

CCC 0170–4214/99/141175–14$17.50 *Received 6 May 1998*

unified study of both electric and magnetic fields. In section 3 we prove the reciprocity principle, the basic scattering theorem and an optical theorem.

In section 4, we introduce Herglotz pairs and the far-field operator $\mathbb{F}$, by analogy with the ideas of Colton and Kress [10]. We establish some scattering relations and study the point spectrum of $\mathbb{F}$. Finally, in section 5, we show that the eigenvalues of $\mathbb{F}$ are the same as the eigenvalues of Waterman's T-matrix (apart from a factor of 4π). This result is general and does not depend on the composition of the obstacle, although the existence proof for the eigenvalues only works for loss-less obstacles.

2. Chiral media

A homogeneous isotropic chiral medium is characterized by three (complex) parameters. These are the electric permittivity ε, the magnetic permeability μ and the chirality measure β. Thus, we use the Drude–Born–Fedorov constitutive relations

$$\mathbf{D} = \varepsilon(\tilde{\mathbf{E}} + \beta \operatorname{curl} \tilde{\mathbf{E}}) \quad \text{and} \quad \mathbf{B} = \mu\,(\tilde{\mathbf{H}} + \beta \operatorname{curl} \tilde{\mathbf{H}}),$$

where $\tilde{\mathbf{E}}$ is the electric field, $\tilde{\mathbf{H}}$ is the magnetic field, $\mathbf{B}$ is the magnetic flux density and $\mathbf{D}$ is the electric flux density.

In a source-free region, we also have

$$\operatorname{curl} \tilde{\mathbf{E}} - \mathrm{i}\omega\, \mathbf{B} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \tilde{\mathbf{H}} + \mathrm{i}\omega \mathbf{D} = \mathbf{0},$$

where we have suppressed a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ throughout.

Let B_{i} denote a bounded three-dimensional domain with a smooth closed boundary, S, and connected exterior, B_{e}. B_{e} is filled with an achiral ($\beta = 0$) medium, with constant electromagnetic parameters ε_{e} and μ_{e}. B_{i} is filled with a chiral medium, with parameters ε, μ and β. A given electromagnetic field is incident upon the obstacle; it is partly scattered and partly transmitted into the obstacle. This leads to the following (dimensional) transmission problem.

Transmission Problem. Find electric fields $\tilde{\mathbf{E}}_{\mathrm{sc}}$ and $\tilde{\mathbf{E}}_{\mathrm{i}}$, and magnetic fields $\tilde{\mathbf{H}}_{\mathrm{sc}}$ and $\tilde{\mathbf{H}}_{\mathrm{i}}$, that satisfy Maxwell's equations in B_{e},

$$\operatorname{curl} \tilde{\mathbf{E}}_{\mathrm{sc}} - \mathrm{i}\mu_{\mathrm{e}}\omega\, \tilde{\mathbf{H}}_{\mathrm{sc}} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \tilde{\mathbf{H}}_{\mathrm{sc}} + \mathrm{i}\varepsilon_{\mathrm{e}}\omega\, \tilde{\mathbf{E}}_{\mathrm{sc}} = \mathbf{0}, \quad \text{in } B_{\mathrm{e}},$$

a modified form of these equations in B_{i},

$$\left.\begin{aligned} &\operatorname{curl} \tilde{\mathbf{E}}_{\mathrm{i}} - \mathrm{i}\mu\omega\,(\gamma/k)^2\, \tilde{\mathbf{H}}_{\mathrm{i}} - \beta\gamma^2\, \tilde{\mathbf{E}}_{\mathrm{i}} = \mathbf{0} \\ &\operatorname{curl} \tilde{\mathbf{H}}_{\mathrm{i}} + \mathrm{i}\varepsilon\omega\,(\gamma/k)^2\, \tilde{\mathbf{E}}_{\mathrm{i}} - \beta\gamma^2\, \tilde{\mathbf{H}}_{\mathrm{i}} = \mathbf{0} \end{aligned}\right\} \text{ in } B_{\mathrm{i}}, \tag{2.1}$$

and two transmission conditions on the interface,

$$\hat{\mathbf{n}} \times \tilde{\mathbf{E}}_{\mathrm{t}} = \hat{\mathbf{n}} \times \tilde{\mathbf{E}}_{\mathrm{i}} \quad \text{and} \quad \hat{\mathbf{n}} \times \tilde{\mathbf{H}}_{\mathrm{t}} = \hat{\mathbf{n}} \times \tilde{\mathbf{H}}_{\mathrm{i}} \quad \text{on } S,$$

where $\hat{\mathbf{n}}$ is the outward unit normal vector, and the total fields in B_{e} are given by

$$\tilde{\mathbf{E}}_{\mathrm{t}} = \tilde{\mathbf{E}}_{\mathrm{sc}} + \tilde{\mathbf{E}}_{\mathrm{inc}}, \quad \tilde{\mathbf{H}}_{\mathrm{t}} = \tilde{\mathbf{H}}_{\mathrm{sc}} + \tilde{\mathbf{H}}_{\mathrm{inc}}, \quad \text{in } B_{\mathrm{e}},$$

and $(\tilde{\mathbf{E}}_{\mathrm{inc}}, \tilde{\mathbf{H}}_{\mathrm{inc}})$ is the given incident field. In addition, the scattered fields $(\tilde{\mathbf{E}}_{\mathrm{sc}}, \tilde{\mathbf{H}}_{\mathrm{sc}})$ must satisfy a Silver–Müller radiation condition

$$\sqrt{\mu_{\mathrm{e}}}\, \hat{\mathbf{r}} \times \tilde{\mathbf{H}}_{\mathrm{sc}} + \sqrt{\varepsilon_{\mathrm{e}}}\, \tilde{\mathbf{E}}_{\mathrm{sc}} = o(r^{-1}) \text{ as } r \to \infty,$$

uniformly for all directions $\hat{\mathbf{r}} \in S^2$, where S^2 is the unit sphere.

We assume that the constants ε_{e} and μ_{e} are positive, whereas the constants ε, μ and β can be complex. The constant γ appearing in (2.1) is given by

$$\gamma^2 = k^2\,(1 - k^2\,\beta^2)^{-1}.$$

We always assume that $|k\beta| < 1$.

It has been proved, under different assumptions on the physical parameters, by Athanasiadis and Stratis [5], and by Ola [15], that this transmission problem has a unique solution. Ammari and Nédélec [1] have proved unique solvability for inhomogeneous obstacles, assuming that μ, ε and β are twice-continuously differentiable real functions of position everywhere in space, taking constant values outside a bounded region.

It is convenient to consider a dimensionless version of the transmission problem [4]. Thus, scale all lengths using a, a typical length scale for the chiral obstacle, and then put

$$\tilde{\mathbf{E}}_{\mathrm{sc}} = \sqrt{\mu_{\mathrm{e}}}\, \mathbf{E}_{\mathrm{sc}}, \quad \tilde{\mathbf{H}}_{\mathrm{sc}} = \sqrt{\varepsilon_{\mathrm{e}}}\, \mathbf{H}_{\mathrm{sc}}, \quad \tilde{\mathbf{E}}_{\mathrm{i}} = \sqrt{\mu}\mathbf{E}, \quad \tilde{\mathbf{H}}_{\mathrm{i}} = \sqrt{\varepsilon}\mathbf{H},$$

with similar scalings for $\tilde{\mathbf{E}}_{\mathrm{t}}$, $\tilde{\mathbf{H}}_{\mathrm{t}}$, $\tilde{\mathbf{E}}_{\mathrm{inc}}$ and $\tilde{\mathbf{H}}_{\mathrm{inc}}$. These scalings reduce the transmission problem to the following dimensionless problem (assuming that $\mathbf{E}_{\mathrm{inc}}$ and $\mathbf{H}_{\mathrm{inc}}$ are dimensionless).

Dimensionless Transmission Problem. Find electric fields $\mathbf{E}_{\mathrm{sc}}$ and $\mathbf{E}$, and magnetic fields $\mathbf{H}_{\mathrm{sc}}$ and $\mathbf{H}$, that satisfy Maxwell's equations in B_{e},

$$\operatorname{curl} \mathbf{E}_{\mathrm{sc}} - \mathrm{i}\,(k_{\mathrm{e}}a)\, \mathbf{H}_{\mathrm{sc}} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H}_{\mathrm{sc}} + \mathrm{i}\,(k_{\mathrm{e}}a)\, \mathbf{E}_{\mathrm{sc}} = \mathbf{0} \quad \text{in } B_{\mathrm{e}}, \tag{2.2}$$

a modified form of these equations in B_{i},

$$\left.\begin{aligned} &\operatorname{curl} \mathbf{E} - \mathrm{i}(ka)\,(\gamma/k)^2\, \mathbf{H} - \beta a \gamma^2\, \mathbf{E} = \mathbf{0},\\ &\operatorname{curl} \mathbf{H} + \mathrm{i}(ka)\,(\gamma/k)^2\, \mathbf{E} - \beta a \gamma^2\, \mathbf{H} = \mathbf{0} \end{aligned}\right\} \text{ in } B_{\mathrm{i}}, \tag{2.3}$$

and two transmission conditions on the interface,

$$\rho \hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{t}} = \hat{\mathbf{n}} \times \mathbf{E} \quad \text{and} \quad \delta \hat{\mathbf{n}} \times \mathbf{H}_{\mathrm{t}} = \hat{\mathbf{n}} \times \mathbf{H} \quad \text{on } S, \tag{2.4}$$

where the total fields in B_{e} are given by $\mathbf{E}_{\mathrm{t}} = \mathbf{E}_{\mathrm{sc}} + \mathbf{E}_{\mathrm{inc}}$ and $\mathbf{H}_{\mathrm{t}} = \mathbf{H}_{\mathrm{sc}} + \mathbf{H}_{\mathrm{inc}}$,

$$k_{\mathrm{e}} = \omega\sqrt{\mu_{\mathrm{e}}\varepsilon_{\mathrm{e}}}, \quad \rho = \sqrt{\mu_{\mathrm{e}}/\mu} \quad \text{and} \quad \delta = \sqrt{\varepsilon_{\mathrm{e}}/\varepsilon}.$$

In addition, the scattered fields $(\mathbf{E}_{\mathrm{sc}}, \mathbf{H}_{\mathrm{sc}})$ must satisfy a Silver–Müller radiation condition

$$\hat{\mathbf{r}} \times \mathbf{H}_{\mathrm{sc}} + \mathbf{E}_{\mathrm{sc}} = o(r^{-1}) \text{ as } r \to \infty.$$

Henceforth, we assume that all lengths have been scaled using a, and so we can set $a=1$.

It is also convenient to introduce fields $\mathbf{U}$ and $\mathbf{U}'$ (the dual of $\mathbf{U}$) as follows:

$$\text{if } \mathbf{U}=\mathbf{E} \text{ then } \mathbf{U}' = \mathrm{i}\, \mathbf{H}, \tag{2.5}$$

$$\text{if } \mathbf{U}=\mathbf{H} \text{ then } \mathbf{U}' = -\mathrm{i}\, \mathbf{E}. \tag{2.6}$$

Here, $(\mathbf{E}, \mathbf{H})$ are regarded as solutions of (2.3). With this notation, we can write (2.3) (with $a=1$) as

$$\operatorname{curl} \mathbf{U} = \gamma^2 \beta\, \mathbf{U} + (\gamma^2/k)\, \mathbf{U}'. \tag{2.7}$$

3. Scattering relations

The electromagnetic far-field pattern $(\mathbf{E}_\infty(\hat{\mathbf{r}}), \mathbf{H}_\infty(\hat{\mathbf{r}}))$ is defined in terms of the scattered electromagnetic field $(\mathbf{E}_{\mathrm{sc}}(\mathbf{r}), \mathbf{H}_{\mathrm{sc}}(\mathbf{r}))$ by the relations [9]

$$\mathbf{E}_{\mathrm{sc}}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}k_{\mathrm{e}}r}}{r}\, \mathbf{E}_\infty(\hat{\mathbf{r}}) + O(r^{-2}), \quad r\to\infty, \tag{3.1}$$

$$\mathbf{H}_{\mathrm{sc}}(\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}k_{\mathrm{e}}r}}{r}\, \mathbf{H}_\infty(\hat{\mathbf{r}}) + O(r^{-2}), \quad r\to\infty, \tag{3.2}$$

uniformly in all directions $\hat{\mathbf{r}}\in S^2$. For the (dimensionless) incident electromagnetic field, we take

$$\begin{aligned} &\mathbf{E}_{\mathrm{inc}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}) = \mathrm{i}\ (k_{\mathrm{e}}a)\ \mathbf{p}\ \mathrm{e}^{i(k_{\mathrm{e}}a)\,\hat{\mathbf{d}}\cdot\mathbf{r}},\\ &\mathbf{H}_{\mathrm{inc}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}) = \hat{\mathbf{d}}\times\mathbf{E}_{\mathrm{inc}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}) \end{aligned} \tag{3.3}$$

(with $a=1$) where the unit vector $\hat{\mathbf{d}}$ describes the direction of propagation and the complex vector $\mathbf{p}$ gives the polarization, and satisfies $\hat{\mathbf{d}}\cdot\mathbf{p}=0$. We shall indicate the dependence of the scattered field, of the total exterior and interior fields, and of the far-field pattern on the incident direction $\hat{\mathbf{d}}$ and the polarization $\mathbf{p}$, by writing $(\mathbf{E}_{\mathrm{sc}}(\mathbf{r},\ \hat{\mathbf{d}},\ \mathbf{p}),\ \mathbf{H}_{\mathrm{sc}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}))$, $(\mathbf{E}_{\mathrm{t}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}),\ \mathbf{H}_{\mathrm{t}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p}))$ and $(\mathbf{E}_\infty(\hat{\mathbf{r}};\ \hat{\mathbf{d}},\ \mathbf{p}),\ \mathbf{H}_\infty(\hat{\mathbf{r}};\ \hat{\mathbf{d}},\ \mathbf{p}))$, respectively.

Let us note that $(\mathbf{E}_{\mathrm{inc}}, \mathbf{H}_{\mathrm{inc}})$ and $(\mathbf{E}_{\mathrm{t}}, \mathbf{H}_{\mathrm{t}})$ satisfy (2.2) in B_{e}. In what follows we shall use the $\mathbf{U}$-notation, given by (2.5) and (2.6); the meaning of the symbols $\mathbf{U}_{\mathrm{sc}}$, $\mathbf{U}_{\mathrm{inc}}$, $\mathbf{U}_{\mathrm{t}}$ and $\mathbf{U}_\infty$ is clear. Moreover, we shall employ the Twersky [18] notation

$$\{\mathbf{U}_1, \mathbf{U}_2\}_S := \int_S \{(\hat{\mathbf{n}}\times\mathbf{U}_1)\cdot\mathbf{U}_2' - (\hat{\mathbf{n}}\times\mathbf{U}_2)\cdot\mathbf{U}_1'\}\ \mathrm{d}s. \tag{3.4}$$

We are in a position to state and prove the following reciprocity principle.

Theorem 1. *The far-field pattern* $\mathbf{U}_\infty$ *satisfies the reciprocity principle*

$$\mathbf{q}\cdot\mathbf{U}_\infty(\hat{\mathbf{r}};\ \hat{\mathbf{d}},\ \mathbf{p}) = \mathbf{p}\cdot\mathbf{U}_\infty(-\hat{\mathbf{d}};\ -\hat{\mathbf{r}},\ \mathbf{q}),$$

for all $\hat{\mathbf{d}}, \hat{\mathbf{r}}\in S^2$ *and* $\mathbf{p}, \mathbf{q}\in\mathbb{C}^3$ *with* $\mathbf{p}\cdot\hat{\mathbf{d}} = \mathbf{q}\cdot\hat{\mathbf{r}} = 0$.

Proof. Using the transmission conditions (2.4) and applying the second vector Green's theorem in B_i, we obtain

$$\{\mathbf{U}_t^{(1)}, \mathbf{U}_t^{(2)}\}_S = (\rho\delta)^{-1}\,\{\mathbf{U}^{(1)}, \mathbf{U}^{(2)}\}_S$$
$$= (\rho\delta)^{-1}\int_{B_i} [\operatorname{div}\,(\mathbf{U}^{(1)}\times\mathbf{U}^{(2)\prime}) - \operatorname{div}\,(\mathbf{U}^{(2)}\times\mathbf{U}^{(1)\prime})]\,\mathrm{d}v.$$

In view of (2.7), the volume integral vanishes whence $\{\mathbf{U}_t^{(1)}, \mathbf{U}_t^{(2)}\}_S = 0$. Substituting $\mathbf{U}_t = \mathbf{U}_{inc} + \mathbf{U}_{sc}$ into this identity gives

$$\{\mathbf{U}_{sc}^{(1)}, \mathbf{U}_{sc}^{(2)}\}_S + \{\mathbf{U}_{sc}^{(1)}, \mathbf{U}_{inc}^{(2)}\}_S + \{\mathbf{U}_{inc}^{(1)}, \mathbf{U}_{sc}^{(2)}\}_S + \{\mathbf{U}_{inc}^{(1)}, \mathbf{U}_{inc}^{(2)}\}_S = 0. \tag{3.5}$$

From Gauss' divergence theorem and the Maxwell equations (2.2) (with $a = 1$) we have

$$\{\mathbf{U}_{inc}^{(1)}, \mathbf{U}_{inc}^{(2)}\}_S = 0. \tag{3.6}$$

Using, in addition, the radiation condition we get

$$\{\mathbf{U}_{sc}^{(1)}, \mathbf{U}_{sc}^{(2)}\}_S = 0. \tag{3.7}$$

From (3.5)–(3.7), we have

$$\{\mathbf{U}_{inc}^{(1)}, \mathbf{U}_{sc}^{(2)}\}_S = \{\mathbf{U}_{inc}^{(2)}, \mathbf{U}_{sc}^{(1)}\}_S. \tag{3.8}$$

Also, from relations (6.24) in [9], taking into account that $\mathbf{q}\cdot\hat{\mathbf{r}} = 0$, we obtain

$$\mathbf{q}\cdot\mathbf{U}_\infty\,(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \mathbf{p}) = -\frac{\mathrm{i}}{4\pi}\{\mathbf{U}_{sc}\,(\cdot\,; \hat{\mathbf{d}}, \mathbf{p}), \mathbf{U}_{inc}\,(\cdot;\ -\hat{\mathbf{r}}, \mathbf{q})\}_S. \tag{3.9}$$

When this is combined with (3.8), we have

$$\mathbf{q}\cdot\mathbf{U}_\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \mathbf{p}) = -\frac{\mathrm{i}}{4\pi}\{\mathbf{U}_{sc}\,(\cdot; -\hat{\mathbf{r}}, \mathbf{q}), \mathbf{U}_{inc}(\cdot; \hat{\mathbf{d}}, \mathbf{p})\}_S$$
$$= \mathbf{p}\cdot\mathbf{U}_\infty\,(-\hat{\mathbf{d}};\ -\hat{\mathbf{r}}, \mathbf{q}),$$

which proves the theorem. □

We observe that the standard reciprocity relation for the achiral case, [9, p. 179], is also valid for the chiral case. For a discussion of the basic reciprocity theorems for electromagnetic wave fields in time-invariant configurations, we refer to [12, ch. 28]. See also [7, sections 3.4, 10.5].

We proceed with the following basic scattering theorem. In the sequel the overhead bar denotes complex conjugation.

Theorem 2. *The far-field pattern* $\mathbf{U}_\infty$ *satisfies the relation*

$$\mathbf{q}\cdot\overline{\mathbf{U}_\infty(\hat{\mathbf{r}};\ \hat{\mathbf{d}},\ \mathbf{p})} + \bar{\mathbf{p}}\cdot\ \mathbf{U}_\infty(\hat{\mathbf{d}}; \hat{\mathbf{r}}, \mathbf{q}) = -\frac{1}{2\pi}\int_{S^2}\overline{\mathbf{U}_\infty(\hat{\mathbf{r}}';\ \hat{\mathbf{d}},\ \mathbf{p})}\cdot\mathbf{U}_\infty\,(\hat{\mathbf{r}}'; \hat{\mathbf{r}}, \mathbf{q})\,\mathrm{d}s\,(\hat{\mathbf{r}}')$$
$$-\frac{1}{2\pi}\mathscr{B}(\hat{\mathbf{d}}, \mathbf{p}; \hat{\mathbf{r}}, \mathbf{q}) \tag{3.10}$$

for all $\hat{\mathbf{d}}, \hat{\mathbf{r}} \in S^2$ *and* $\mathbf{p}, \mathbf{q} \in \mathbb{C}^3$ *with* $\hat{\mathbf{d}} \cdot \mathbf{p} = \hat{\mathbf{r}} \cdot \mathbf{q} = 0$, *where*

$$\mathscr{B}(\hat{\mathbf{d}}, \mathbf{p}; \hat{\mathbf{r}}, \mathbf{q}) = \tfrac{1}{2}\,\mathrm{i}\,\{\overline{\mathbf{U}_{\mathrm{t}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{t}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S \tag{3.11}$$

$$= \mathrm{Im}\left(\frac{\gamma^2}{\rho\bar{\delta}k}\right)(\mathbf{U}_1', \mathbf{U}_2') + \mathrm{Im}\left(\frac{\gamma^2}{\bar{\rho}\delta k}\right)(\mathbf{U}_1, \mathbf{U}_2)$$

$$+ \mathrm{Im}\,(\beta\gamma^2)\left\{\frac{1}{\rho\bar{\delta}}(\mathbf{U}_1', \mathbf{U}_2) + \frac{1}{\bar{\rho}\delta}(\mathbf{U}_1, \mathbf{U}_2')\right\}, \tag{3.12}$$

$\mathbf{U}_1 = \mathbf{U}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})$, $\mathbf{U}_2 = \mathbf{U}(\cdot\,; \hat{\mathbf{d}}, \mathbf{p})$ *and*

$$(\mathbf{U}, \mathbf{V}) = \int_{B_{\mathrm{i}}} \mathbf{U} \cdot \bar{\mathbf{V}}\, \mathrm{d}v$$

denotes the inner product in $L^2(B_{\mathrm{i}})$.

Proof. By the relation $\mathbf{U}_{\mathrm{t}} = \mathbf{U}_{\mathrm{inc}} + \mathbf{U}_{\mathrm{sc}}$ and the bilinearity of the form (3.4) we obtain

$$\begin{aligned}\{\overline{\mathbf{U}_{\mathrm{t}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{t}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S &= \{\overline{\mathbf{U}_{\mathrm{inc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{inc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S \\ &+ \{\overline{\mathbf{U}_{\mathrm{inc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S \\ &+ \{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{inc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S \\ &+ \{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S\,\cdot\end{aligned} \tag{3.13}$$

From (3.9) we have

$$\{\overline{\mathbf{U}_{\mathrm{inc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S = 4\pi\mathrm{i}\bar{\mathbf{p}} \cdot \mathbf{U}_\infty(\hat{\mathbf{d}}; \hat{\mathbf{r}}, \mathbf{q}), \tag{3.14}$$

$$\{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{inc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S = 4\pi\mathrm{i}\mathbf{q} \cdot \overline{\mathbf{U}_\infty(\hat{\mathbf{r}};\ \hat{\mathbf{d}},\ \mathbf{p})}, \tag{3.15}$$

since $\overline{\mathbf{U}_{\mathrm{inc}}'(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p})} = -\,\mathbf{U}_{\mathrm{inc}}'(\mathbf{r}; -\hat{\mathbf{d}}, \bar{\mathbf{p}})$ and $\overline{\mathbf{U}_{\mathrm{inc}}(\mathbf{r};\ \hat{\mathbf{d}},\ \mathbf{p})} = -\,\mathbf{U}_{\mathrm{inc}}(\mathbf{r};\ -\hat{\mathbf{d}}, \bar{\mathbf{p}})$.

We consider a sphere S_R centred at the origin with radius R large enough to include the chiral scatterer in its interior. Applying the second vector Green's theorem to $\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})$ and $\mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})$ in the region exterior to S and interior to S_R, we obtain

$$\{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S = \{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,,\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_{S_R}. \tag{3.16}$$

For $R \to \infty$, we can use the asymptotic forms (3.1) and (3.2) for the scattered fields. Taking into account that $\hat{\mathbf{n}} \times \mathbf{U}_\infty' = \mathrm{i}\,\mathbf{U}_\infty$ we conclude that

$$\{\overline{\mathbf{U}_{\mathrm{sc}}(\cdot\,;\ \hat{\mathbf{d}},\ \mathbf{p})}, \mathbf{U}_{\mathrm{sc}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{q})\}_S = 2\mathrm{i}\int_{S^2} \overline{\mathbf{U}_\infty(\hat{\mathbf{r}}';\ \mathbf{d},\ \mathbf{p})} \cdot \mathbf{U}_\infty(\hat{\mathbf{r}}'; \hat{\mathbf{r}}, \mathbf{q})\, \mathrm{d}s(\hat{\mathbf{r}}'). \tag{3.17}$$

Substituting (3.14)–(3.17) in (3.13), we arrive at (3.10), with $\mathscr{B}$ given by (3.11). To obtain (3.12), we use the transmission conditions on S to give

$$\mathscr{B} = \frac{\mathrm{i}}{2\bar{\rho}\delta} \int_S \hat{\mathbf{n}} \cdot (\bar{\mathbf{U}}_2 \times \mathbf{U}_1') \, \mathrm{d}s + \frac{\mathrm{i}}{2\rho\bar{\delta}} \int_S \hat{\mathbf{n}} \cdot (\overline{\mathbf{U}_2'} \times \mathbf{U}_1) \, \mathrm{d}s.$$

Then, an application of the divergence theorem in B_{i}, together with (2.7) and $(\mathbf{U}')' = \mathbf{U}$, lead to the desired result. □

Note that $\mathscr{B} \equiv 0$ if ε, μ and β are all real.

We conclude this section by recalling the definition of the scattering cross-section, σ, and proving an optical theorem that connects the far-field pattern to σ. The scattering cross-section is defined as the ratio of the time average rate (over a period) at which energy is scattered by the obstacle, to the corresponding time average at which the energy of the incident wave crosses a unit area normal to the direction of propagation. The scattering cross-section has the dimensions of area and is a measure of the distrubance caused by the obstacle to the incident wave.

Theorem 3. *The following relation holds*:

$$\sigma = -4\pi \, \mathrm{Re} \, (\bar{\mathbf{p}} \cdot \mathbf{U}_\infty \, (\hat{\mathbf{d}}; \hat{\mathbf{d}}, \mathbf{p})) - \mathscr{B}(\hat{\mathbf{d}}, \mathbf{p}; \hat{\mathbf{d}}, \mathbf{p}). \tag{3.18}$$

Proof. Since B_{e} is achiral, it can be proved as in the standard theory [18] that

$$\sigma = \int_{S^2} |\mathbf{U}_\infty \, (\hat{\mathbf{r}}; \hat{\mathbf{d}}, \mathbf{p})|^2 \, \mathrm{d}s \, (\hat{\mathbf{r}}).$$

The proof of (3.18) now follows, using Theorem 2 with $\hat{\mathbf{r}} = \hat{\mathbf{d}}$ and $\mathbf{p} = \mathbf{q}$. □

We remark that, if ε and μ are positive, then

$$\mathscr{B}(\hat{\mathbf{d}}, \mathbf{p}; \hat{\mathbf{d}}, \mathbf{p}) = \frac{2k^2 \, \mathrm{Im}\,(\beta)}{\rho\delta\,|1 - k^2 \ \beta^2|^2} [k \, (\mathrm{Re} \, \beta) \, (\|\mathbf{U}'\|^2 + \|\mathbf{U}\|^2) + (1 + k^2 |\beta|^2) \, \mathrm{Re} \, \{(\mathbf{U}', \mathbf{U})\}], \tag{3.19}$$

where $\|\mathbf{U}\|^2 = (\mathbf{U}, \mathbf{U})$ gives the norm of $\mathbf{U}$ in $L^2\,(B_{\mathrm{i}})$. If, in addition, β is real we obtain directly from (3.18) and (3.19) that

$$\sigma = -4\pi \, \mathrm{Re} \, (\bar{\mathbf{p}} \cdot \mathbf{U}_\infty \, (\hat{\mathbf{d}}; \hat{\mathbf{d}}, \mathbf{p})),$$

which coincides with the classical optical theorem [3, 12, section 10.7, 13, p. 453, 18]. The scattering cross-section in this case (that is when $\varepsilon > 0$, $\mu > 0$ and β is real) can be calculated from the far-field pattern in the direction in which the incident wave is travelling (forward scattering). Comments on the physical meaning of the classical optical theorem in the achiral case can be found in [13, pp. 453–454].

4. Herglotz pairs

We consider the equations

$$\mathrm{curl} \, \mathbf{U} = k_{\mathrm{e}} \, \mathbf{U}' \quad \text{and} \quad \mathrm{curl} \, \mathbf{U}' = k_{\mathrm{e}} \, \mathbf{U}. \tag{4.1}$$

Any solution $(\mathbf{U}_g, \mathbf{U}'_g)$ of (4.1) of the form

$$\mathbf{U}_g(\mathbf{r}) = \mathrm{i}k_e \int_{S^2} \mathbf{g}(\hat{\mathbf{q}})\, \mathrm{e}^{\mathrm{i}k_e \hat{\mathbf{q}}\cdot\mathbf{r}}\, \mathrm{d}s(\hat{\mathbf{q}}), \tag{4.2}$$

$$\mathbf{U}'_g(\mathbf{r}) = -k_e \int_{S^2} \hat{\mathbf{q}} \times \mathbf{g}(\hat{\mathbf{q}})\, \mathrm{e}^{\mathrm{i}k_e \hat{\mathbf{q}}\cdot\mathbf{r}}\, \mathrm{d}s(\hat{\mathbf{q}}),$$

where $\mathbf{g} \in L^2_T(S^2) = \{\mathbf{f}: S^2 \to \mathbb{C}^3; \mathbf{f} \in L^2(S^2) \text{ and } \mathbf{f}(\hat{\mathbf{q}})\cdot\hat{\mathbf{q}} = 0\}$, is called a *Herglotz pair with kernel* $\mathbf{g}$ for (4.1). For the classical Maxwell equations, this notion is discussed in [10].

In the sequel we consider Herglotz pairs as incident fields. For $\mathbf{g} \in L^2_T(S^2)$, define $\mathbf{U}_{\mathrm{inc},\mathbf{g}}$ by (4.2), so that

$$\mathbf{U}_{\mathrm{inc},\mathbf{g}}(\mathbf{r}) = \int_{S^2} \mathbf{U}_{\mathrm{inc}}(\mathbf{r}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))\, \mathrm{d}s(\hat{\mathbf{q}}),$$

where $\mathbf{U}_{\mathrm{inc}}$ is defined by (3.3). By linearity, the corresponding scattered field $\mathbf{U}_{\mathrm{sc},\mathbf{g}}$ and far-field pattern $\mathbf{U}_{\infty,\mathbf{g}}$ are given by

$$\mathbf{U}_{\mathrm{sc},\mathbf{g}}(\mathbf{r}) = \int_{S^2} \mathbf{U}_{\mathrm{sc}}(\mathbf{r}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))\, \mathrm{d}s(\hat{\mathbf{q}})$$

and

$$\mathbf{U}_{\infty,\mathbf{g}}(\hat{\mathbf{r}}) = \int_{S^2} \mathbf{U}_{\infty}(\hat{\mathbf{r}}; \hat{\mathbf{q}}, \mathbf{g}(\hat{\mathbf{q}}))\, \mathrm{d}s(\hat{\mathbf{q}}),$$

respectively. Define $\mathbf{U}_{\mathrm{inc},\mathbf{h}}$, $\mathbf{U}_{\mathrm{sc},\mathbf{h}}$ and $\mathbf{U}_{\infty,\mathbf{h}}$ similarly, with $\mathbf{h} \in L^2_T(S^2)$. We have the following result.

Theorem 4. *The following scattering relations hold*:

$$\{\mathbf{U}_{\mathrm{sc},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{inc},\mathbf{h}}}\}_S = -4\pi\mathrm{i}\, \langle \mathbf{U}_{\infty,\mathbf{g}}, \mathbf{h} \rangle, \tag{4.3}$$

$$\{\mathbf{U}_{\mathrm{sc},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{sc},\mathbf{h}}}\}_S = -2\mathrm{i}\, \langle \mathbf{U}_{\infty,\mathbf{g}}, \mathbf{U}_{\infty,\mathbf{h}} \rangle. \tag{4.4}$$

Here, by definition,

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{S^2} \mathbf{f}(\hat{\mathbf{q}}) \cdot \overline{\mathbf{g}(\hat{\mathbf{q}})}\, \mathrm{d}s(\hat{\mathbf{q}})$$

denotes the inner product in $L^2(S^2)$.

The proof of (4.3) follows from (3.9), while the proof of (4.4) follows from (3.17) after some calculations.

We define the *far-field operator* $\mathbb{F}: L_T^2(S^2) \to L_T^2(S^2)$, corresponding to the far-field pattern $\mathbf{U}_\infty$, by

$$(\mathbb{F}\mathbf{h})(\hat{\mathbf{r}}) = \int_{S^2} \mathbf{U}_\infty(\hat{\mathbf{r}}; \hat{\mathbf{q}}, \mathbf{h}(\hat{\mathbf{q}}))\, \mathrm{d}s(\hat{\mathbf{q}}). \tag{4.5}$$

Note that $\mathbb{F}\mathbf{h}$ is the far-field pattern of our transmission problem corresponding to the incident wave $\mathbf{U}_{\mathrm{inc},\mathbf{h}}$, [10]: $\mathbb{F}\mathbf{h} = \mathbf{U}_{\infty,\mathbf{h}}$.

The far-field operator plays a central role in the dual space method for solving the achiral inverse electromagnetic scattering problem [9, p. 199]. Its acoustic analogue also features prominently in the method of Colton and Kirsch [8] and Kirsch [14].

Using the operator $\mathbb{F}$, we can restate Theorem 2 as follows:

Corollary 1.

$$\langle \mathbb{F}\mathbf{g}, \mathbf{h}\rangle + \langle \mathbf{g}, \mathbb{F}\mathbf{h}\rangle + \frac{1}{2\pi}\langle \mathbb{F}\mathbf{g}, \mathbb{F}\mathbf{h}\rangle = \frac{\mathrm{i}}{4\pi}\{\mathbf{U}_{\mathrm{t},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{t},\mathbf{h}}}\}_S. \tag{4.6}$$

This is obtained by setting $\mathbf{p} = \mathbf{g}(\hat{\mathbf{d}})$ and $\mathbf{q} = \mathbf{h}(\hat{\mathbf{r}})$ in Theorem 2, and then integrating over S^2 twice.

For the subsequent discussion, we need to compute the quantity $\{\mathbf{U}_{\mathrm{t},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{t},\mathbf{h}}}\}_S$ in terms of the fields in B_{i} and the physical parameters of the scatterer. From (3.11), we have

$$-\tfrac{1}{2}\mathrm{i}\,\{\mathbf{U}_{\mathrm{t}}(\cdot\,; \hat{\mathbf{d}}, \mathbf{g}(\hat{\mathbf{d}})), \overline{\mathbf{U}_{\mathrm{t}}(\cdot\,; \hat{\mathbf{r}}, \mathbf{h}(\hat{\mathbf{r}}))}\}_S = \overline{\mathscr{B}(\hat{\mathbf{d}}, \mathbf{g}(\hat{\mathbf{d}}); \hat{\mathbf{r}}, \mathbf{h}(\hat{\mathbf{r}}))}.$$

Integrating this over $\hat{\mathbf{d}} \in S^2$ and over $\hat{\mathbf{r}} \in \mathbf{S}^2$, using

$$\mathbf{U}_{\mathbf{g}} = \int_{S^2} \mathbf{U}_2\, \mathrm{d}s(\hat{\mathbf{d}}) \quad \text{and} \quad \mathbf{U}_{\mathbf{h}} = \int_{S^2} \mathbf{U}_1\, \mathrm{d}s(\hat{\mathbf{r}}),$$

gives

$$-\frac{\mathrm{i}}{2}\{\mathbf{U}_{\mathrm{t},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{t},\mathbf{h}}}\}_S = \mathrm{Im}\left(\frac{\gamma^2}{\rho\bar{\delta}k}\right)(\mathbf{U}'_{\mathbf{g}}, \mathbf{U}'_{\mathbf{h}}) + \mathrm{Im}\left(\frac{\gamma^2}{\bar{\rho}\delta k}\right)(\mathbf{U}_{\mathbf{g}}, \mathbf{U}_{\mathbf{h}})$$
$$+ \frac{1}{\rho\bar{\delta}}\,\mathrm{Im}\,(\beta\gamma^2)\,(\mathbf{U}_{\mathbf{g}}, \mathbf{U}'_{\mathbf{h}}) + \frac{1}{\bar{\rho}\delta}\,\mathrm{Im}\,(\beta\gamma^2)\,(\mathbf{U}'_{\mathbf{g}}, \mathbf{U}_{\mathbf{h}}). \tag{4.7}$$

If the physical parameters ε and μ are positive, then (4.7) takes the form

$$-\tfrac{1}{2}\mathrm{i}\,\{\mathbf{U}_{\mathrm{t},\mathbf{g}}, \overline{\mathbf{U}_{\mathrm{t},\mathbf{h}}}\}_S = (\mathrm{Im}\,\beta)\, A_{\mathbf{g},\mathbf{h}}, \tag{4.8}$$

where

$$A_{\mathbf{g},\mathbf{h}} = \frac{k^2}{\rho\delta\,|1 - k^2\beta^2|^2}\{2k(\mathrm{Re}\,\beta)\,[(\mathbf{U}'_{\mathbf{g}}, \mathbf{U}'_{\mathbf{h}}) + (\mathbf{U}_{\mathbf{g}}, \mathbf{U}_{\mathbf{h}})]$$
$$+ (1 + k^2|\beta|^2)\,[(\mathbf{U}_{\mathbf{g}}, \mathbf{U}'_{\mathbf{h}}) + (\mathbf{U}'_{\mathbf{g}}, \mathbf{U}_{\mathbf{h}})]\}.$$

Theorem 5. *Suppose that* $\varepsilon > 0$, $\mu > 0$ *and* $\operatorname{Im} \beta = 0$. *Then the far-field operator is normal and hence has a countable number of eigenvalues. These eigenvalues all lie on the circle*

$$|\lambda|^2 + 4\pi \operatorname{Re} \lambda = 0. \tag{4.9}$$

Proof. Since $\operatorname{Im} \beta = 0$, Corollary 1 and (4.8) give

$$\langle \mathbb{F}\mathbf{g}, \mathbf{h} \rangle + \langle \mathbf{g}, \mathbb{F}\mathbf{h} \rangle + \frac{1}{2\pi} \langle \mathbb{F}\mathbf{g}, \mathbb{F}\mathbf{h} \rangle = 0. \tag{4.10}$$

As in [3, 10], it follows that $\mathbb{F}$ is a normal operator, and hence has a countable number of eigenvalues. Now, let $\mathbb{F}\mathbf{h} = \lambda \mathbf{h}$, $\lambda \in \mathbb{C}$. Setting $\mathbf{g} = \mathbf{h} \neq \mathbf{0}$ in (4.10), we conclude that the eigenvalues lie on the circle (4.9). □

Note that, if we set $\mathbf{h} = \mathbf{g}$ and $\mathbb{F}\mathbf{g} = \lambda \mathbf{g}$ in (4.6), and assume that $\langle \mathbf{g}, \mathbf{g} \rangle = 1$, we obtain

$$|\lambda|^2 + 4\pi \operatorname{Re}\lambda = \tfrac{1}{2}\, \mathrm{i}\, \{\mathbf{U}_{\mathrm{t,\,g}}, \overline{\mathbf{U}_{\mathrm{t,\,g}}}\}_S,$$

so that the right-hand side must be real. Thus,

$$\begin{aligned} |\lambda|^2 + 4\pi \operatorname{Re} \lambda &= -\tfrac{1}{2} \operatorname{Im} \{\mathbf{U}_{\mathrm{t,\,g}}, \overline{\mathbf{U}_{\mathrm{t,g}}}\}_S \\ &= -\operatorname{Im} \int_S (\hat{\mathbf{n}} \times \mathbf{U}_{\mathrm{t,\,g}}) \cdot \overline{\mathbf{U}'_{\mathrm{t,\,g}}}\, \mathrm{d}s \\ &= \operatorname{Re} \int_S \hat{\mathbf{n}} \cdot (\mathbf{E}_{\mathrm{t,\,g}} \times \overline{\mathbf{H}_{\mathrm{t,\,g}}})\, \mathrm{d}s, \end{aligned}$$

which is recognized as the time-averaged flow of electromagnetic energy across S [13, p. 242]; this quantity vanishes as energy is neither created nor destroyed within B_{i} when ε, μ and β are all real.

Let us define an operator $\mathbb{R}$ by

$$(\mathbb{R}\mathbf{h})(\hat{\mathbf{r}}) = \mathbf{h}(\hat{\mathbf{r}}) + \frac{1}{2\pi} (\mathbb{F}\mathbf{h})(\hat{\mathbf{r}}).$$

Cho [7, section 10.5.2] calls $\mathbb{R}$ the 'dynamic scattering amplitude operator'. It follows from Theorem 5 that the eigenvalues of $\mathbb{R}$ are all on the unit circle. $\mathbb{R}$ is a unitary operator. Colton and Kress [10, Eq. (2.23)] noted a similar relation between the acoustic (scalar) far-field operator and the 'scattering operator' S [7, section 10.2]. In the next section, we relate the electromagnetic $\mathbb{F}$ to Waterman's T-matrix.

5. Connection with the *T*-matrix

In general, the scattering properties of an obstacle can be specified using its T-matrix. Thus, surround an obstacle by a sphere of radius d and centre O. Assume that the given incident field can be expanded as

$$\mathbf{U}_{\mathrm{inc}}(\mathbf{r}) = \sum_n a_n \hat{\psi}_n(\mathbf{r}) \quad \text{for } r < d,$$

where $\hat{\psi}_n$ are regular spherical vector wave functions. Similarly, we can write

$$\mathbf{U}_{\mathrm{sc}}(\mathbf{r}) = \sum_n f_n \psi_n(\mathbf{r}) \quad \text{for } r > d,$$

where ψ_n are outgoing spherical vector wave functions. To be specific, we shall use the notation and normalizations of Boström *et al.* [6, p. 173] in all that follows, with some deviations; in particular, we use $\hat{\psi}$ for their $\operatorname{Re}\psi$.

The known coefficients a_n are related to the unknown scattering coefficients f_n by

$$f_n = \sum_m T_{nm} a_m, \quad \text{or} \quad f = Ta$$

for brevity; see Ström [17, p. 161]. Many properties of the T-matrix are known; see Waterman [19] and Ström [17, p. 162]. In particular, T is symmetric, $T_{mn} = T_{nm}$, and, for loss-less obstacles, it satisfies

$$T^\dagger T + \operatorname{Re} T = 0, \tag{5.1}$$

where $(T^\dagger)_{mn} = \overline{T_{nm}}$. We shall return to (5.1) later.

In the formulae above, m and n are multi-indices. We shall write $n = \tau\nu$ where $\tau = 1, 2$ and $\nu = \sigma m l$ is another multi-index. Thus, for example,

$$\begin{aligned}\psi_{1\nu}(\mathbf{r}) &= [l(l+1)]^{-1/2} \operatorname{curl}\{\mathbf{r}\, h_l^{(1)}(k_e r)\, Y_{\sigma ml}(\hat{\mathbf{r}})\} \\ &= h_l^{(1)}(k_e r)\, \mathbf{A}_{1\nu}(\hat{\mathbf{r}}),\end{aligned}$$

where $h_l^{(1)}$ is a spherical Hankel function, $Y_{\sigma ml}$ is a normalized spherical harmonic [6, p. 171] and

$$\mathbf{A}_{1\nu}(\hat{\mathbf{r}}) = [l(l+1)]^{-1/2} \operatorname{curl}\{\mathbf{r}\, Y_{\sigma ml}(\hat{\mathbf{r}})\}$$

is a normalized real vector spherical harmonic. There are similar definitions for $\psi_{2\nu}$, $\mathbf{A}_{2\nu}$ and $\hat{\psi}_n$.

The functions $\{\mathbf{A}_n\}$ form a complete orthonormal basis for $L_T^2(S^2)$. Thus, we have

$$\mathbf{g}(\hat{\mathbf{r}}) = \sum_n g_n \mathbf{A}_n(\hat{\mathbf{r}}), \tag{5.2}$$

where $g_n = \langle \mathbf{g}, \mathbf{A}_n \rangle$ and $\hat{\mathbf{r}} \cdot \mathbf{g}(\hat{\mathbf{r}}) = 0$.

Let us now calculate $\mathbb{F}\mathbf{g}$. In the far field, we have

$$\psi_{\tau\nu}(\mathbf{r}) \sim \frac{e^{ik_e r}}{k_e r} (-i)^{l+2-\tau} \mathbf{A}_{\tau\nu}(\hat{\mathbf{r}}),$$

whence

$$(\mathbb{F}\mathbf{g})(\hat{\mathbf{r}}) = k_e^{-1} \sum_{\tau\nu} \mathbf{A}_{\tau\nu}(\hat{\mathbf{r}})(-\mathrm{i})^{l+2-\tau} f_{\tau\nu}.$$

Thus, we have the expansion

$$(\mathbb{F}\mathbf{g})(\hat{\mathbf{r}}) = \sum_n F_n \mathbf{A}_n(\hat{\mathbf{r}}), \tag{5.3}$$

where the coefficients $F_n = \langle \mathbb{F}\mathbf{g}, \mathbf{A}_n \rangle$ are given by

$$F = k_e^{-1} DTa, \tag{5.4}$$

D is a diagonal block matrix, given by

$$D_{\tau\sigma ml, \tau'\sigma'm'l'} = (-\mathrm{i})^{l+2-\tau} \delta_{\tau\tau'} \delta_{\sigma\sigma'} \delta_{mm'} \delta_{ll'},$$

δ_{ij} is the Kronecker delta, and we have used $f = Ta$.

Next, we calculate the incident-field coefficients a_n. We start with the coefficients a_n^p for a plane wave. Thus, we have

$$\mathbf{p} \exp(\mathrm{i}\, \mathbf{k} \cdot \mathbf{r}) = \sum_n a_n^p \hat{\psi}_n(\mathbf{r}),$$

where

$$a_n^p = 4\pi\, \mathrm{i}^{l+1-\tau}\, \mathbf{p} \cdot \mathbf{A}_n(\hat{\mathbf{k}}),$$

$\mathbf{k} = k_e \hat{\mathbf{k}}$ and $\mathbf{p} \cdot \mathbf{k} = 0$. In the literature, this formula is usually given only for $\mathbf{k}$ along the z-axis of the spherical coordinate system. We obtained it using the following 'connection formula',

$$4\pi\, \mathrm{i}^{l+1-\tau} \hat{\psi}_n(r\hat{\mathbf{k}}) = \int_{S^2} \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r})\, \mathbf{A}_n(\hat{\mathbf{r}})\, \mathrm{d}s(\hat{\mathbf{r}}),$$

which has been proved by Dassios and Rigou [11]. Then, setting $\mathbf{p} = \mathrm{i}k_e \mathbf{g}(\hat{\mathbf{k}})$, and integrating over S^2, we obtain

$$a_n = 4\pi\, k_e\, \mathrm{i}^{l+2-\tau} g_n,$$

where g_n are the coefficients in the expansion (5.2). Thus, in terms of the matrix D, we have

$$a = 4\pi k_e \bar{D} g.$$

When this is substituted into (5.4), we obtain (5.3) with

$$F = 4\pi\, DT\bar{D} g.$$

This gives a general representation for $\mathbb{F}$. The composition of the obstacle (chiral or achiral, homogeneous or inhomogeneous, perfectly conducting or otherwise) enters through its T-matrix.

We are interested in the eigenvalues of $\mathbb{F}$. We have

$$0 = \mathbb{F}\mathbf{g} - \lambda \mathbf{g} = \sum_n (F_n - \lambda g_n)\, \mathbf{A}_n$$

where, as $\bar{D} = D^{-1}$,

$$F - \lambda g = D(4\pi T - \lambda I)\, \bar{D} g.$$

Thus, we see that λ is an eigenvalue of $\mathbb{F}$ if and only if $\lambda/(4\pi)$ is an eigenvalue of T. This identifies the eigenvalues of the far-field operator as being precisely those of Waterman's T-matrix (apart from a factor of 4π).

For loss-less obstacles, we have the identity (5.1). Assume that

$$4\pi\ Tg = \lambda g.$$

We can normalise g with $gg^\dagger = 1$, where $g^\dagger = (\bar{g})^{\mathrm{T}}$. Hence, $4\pi\, g^\dagger\, T^\dagger = \bar{\lambda} g^\dagger$. From (5.1), noting the symmetry of T, we have

$$2g^\dagger\, T^\dagger\, Tg + g^\dagger (T + T^\dagger)\, g = 0.$$

Substituting for Tg and $g^\dagger T^\dagger$, we see that the eigenvalues λ lie on the circle (4.9), as before.

Acknowledgements

This research was supported by a grant from the British Council and the University of Athens.

References

1. Ammari, H. and Nédélec, J. C., 'Time-harmonic electromagnetic fields in chiral media', in: *Modern Mathematical Methods in Diffraction Theory and its Applications in Engineering* (E. Meister, ed.) *Methoden und Verfahren der Mathematischen Physik*, vol. 42, pp. 174–202, Peter Lang, Frankfurt am Main, 1997.
2. Angell, T. S., Colton, D. and Kress, R., 'Far field patterns and inverse scattering problems for imperfectly conducting obstacles', *Math. Proc. Camb. Philos. Soc*., **106**, 553–569 (1989).
3. Athanasiadis, C., 'Scattering theorems for time-harmonic electromagnetic waves in a piecewise homogeneous medium', *Math. Proc. Camb. Philos. Soc*., **123**, 179–190 (1998).
4. Athanasiadis, C., Martin, P. A. and Stratis, I. G., 'Electromagnetic scattering by a homogeneous chiral obstacle: boundary integral equations and low-chirality approximations', *SIAM J. Appl. Math*., to appear.
5. Athanasiadis, C. and Stratis, I. G., 'Electromagnetic scattering by a chiral obstacle', *IMA J. Appl. Math*., **58**, 83–91 (1997).
6. Boström, A. Kristensson, G. and Ström, S., 'Transformation properties of plane, spherical and cylindrical scalar and vector wave functions', in: *Field Representations and Introduction to Scattering* (V. V. Varadan, A. Lakhtakia, and V. K. Varadan, eds.), pp. 165–210. North-Holland, Amsterdam, 1991.
7. Cho, S. K., *Electromagnetic Scattering*, Springer, New York, 1990.
8. Colton, D. and Kirsch, A., 'A simple method for solving inverse scattering problems in the resonance region', *Inverse Problems*, **12**, 383–393 (1996).
9. Colton, D. and Kress, R., *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer, Berlin, 1992.

10. Colton, D. and Kress, R., 'Eigenvalues of the far field operator and inverse scattering theory', *SIAM J. Math. Anal.*, **26**, 601–615 (1995).
11. Dassios, G. and Rigou, Z., 'Elastic Herglotz functions', *SIAM J. Appl. Math.*, **55**, 1345–1361 (1995).
12. de Hoop, A. T., *Handbook of Radiation and Scattering of Waves*, Academic Press, London, 1995.
13. Jackson, J. D., *Classical Electrodynamics*, 2nd edn., Wiley, New York, 1975.
14. Kirsch, A., 'Characterization of the shape of a scattering obstacle using the spectral data of the far field operator', *Inverse Problems*, **14**, 1489–1512 (1998).
15. Lakhtakia, A., *Beltrami Fields in Chiral Media*, World Scientific, Singapore, 1994.
16. Ola, P., 'Boundary integral equations for the scattering of electromagnetic waves by a homogeneous chiral obstacle', *J. Math. Phys.*, **35**, 3969–3980 (1994).
17. Ström, S., 'The scattered field', in: *Field Representations and Introduction to Scattering* (V. V. Varadan, A. Lakhtakia and V. K. Varadan, eds.), pp. 143–164. North-Holland, Amsterdam, 1991.
18. Twersky, V., 'Multiple scattering of electromagnetic waves by arbitrary configurations', *J. Math. Phys.* (8), **3**, 589–598 (1967).
19. Waterman, P. C., 'Symmetry, unitarity, and geometry in electromagnetic scattering', *Phys. Rev. D*, **3**, 825–839 (1971).

