Inverse Problems **15** (1999) 1393–1404. Printed in the UK PII: S0266-5611(99)01426-4

Inverting the Sachs–Wolfe formula: an inverse problem arising in early-universe cosmology

A Berera†§ and P A Martin‡‖

† Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA

‡ Department of Mathematics, University of Manchester, Manchester M13 9PL, UK

Received 29 January 1999, in final form 7 June 1999

Abstract. The (ordinary) Sachs–Wolfe effect relates primordial matter perturbations to the temperature variations $\delta T/T$ in the cosmic microwave background radiation; $\delta T/T$ can be observed in all directions around us. A standard but idealized model of this effect leads to an infinite set of moment-like equations: the integral of $P(k)j_\ell^2(ky)$ with respect to k ($0 < k < \infty$) is equal to a given constant, C_ℓ, for $\ell = 0, 1, 2, \ldots$. Here, P is the power spectrum of the primordial density variations, j_ℓ is a spherical Bessel function and y is a positive constant. It is shown how to solve these equations exactly for $P(k)$. The same solution can be recovered, in principle, if the first m equations are discarded. Comparisons with classical moment problems (where $j_\ell^2(ky)$ is replaced by k^ℓ) are made.

1. Introduction

Some might say that the ultimate inverse problem is to understand the origin of structure in the universe using information that is currently available: it is the central problem in early-universe cosmology [18, 27]. One of the available cosmological observables is the cosmic microwave background radiation (CMBR). It is known that the temperature of the CMBR is remarkably uniform in its spatial variation, with $\delta T/T \simeq 10^{-5}$–$10^{-4}$ [35], [18, section 1.5], implying that the expansion of the universe is largely isotropic. Conversely, inhomogeneities in the density of the universe lead to temperature anisotropies, so that these can be used as a sensitive test of theories of structure formation [36].

In these theories, the primary unknown function is $P(k)$, the power spectrum of the primordial density fluctuations. Under certain assumptions, it can be shown that $P(k)$ satisfies the following set of equations:

$$\int_0^\infty k^{-2} P(k) j_\ell^2(ky)\,\mathrm{d}k = C_\ell, \qquad \ell = 0, 1, 2, \ldots. \tag{1.1}$$

Here, y is a given positive constant, j_ℓ is a spherical Bessel function and the constants C_ℓ are given.

Experiments have provided estimates for a finite number of the C_ℓ. How can these be used to recover $P(k)$? This is a major question, but it is *not* our main concern here. We are interested, first, in an idealized problem: given an exact knowledge of all the C_ℓ, reconstruct P.

§ Present address: Department of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK.

‖ Present address: Department of Mathematical and Computational Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA.

0266-5611/99/061393+12$30.00

We show that this inverse problem can be solved exactly. This is a new result. Furthermore, we show that the same solution can be recovered if the first few (in fact, any finite number) of the C_ℓ are unknown.

What use is an exact inversion formula for an idealized problem? First, it reveals the ill-posed nature of the problem†. Thus, $P(k)$ is found to be given by the Fourier sine transform of a certain function $g_0(\lambda)$, which is defined by an infinite series with a finite interval of convergence. This means that techniques of analytic continuation are required, a process that can be very difficult numerically [12] and one that gives the problem its ill-posed character. Second, our inversion formula reveals some of the analytic structure. For example, the low-k behaviour of $P(k)$ is intimately related to the asymptotic behaviour of $g_0(\lambda)$ as $\lambda \to \infty$. Third, exact results can be used to test numerical algorithms designed for the finite-data problem.

The plan of the paper is as follows. In the next section, we sketch a derivation of the governing equations (1.1). Careful derivations can be found in the literature; references are given. Our aim is to motivate the study of (1.1) in a way that is accessible to non-specialists. Thus, we limit our discussion to perhaps the simplest model of the underlying physics. In section 3, we formulate two moment-like problems, called the basic problem and the reduced-data problem, in which we are given C_ℓ for $\ell \geqslant 0$ and $\ell \geqslant \ell_0$, respectively, where ℓ_0 is any fixed positive integer. Apart from the system (1.1), we also consider (in section 5) the analogous classical moment problem where $j_\ell^2(ky)$ is replaced by k^ℓ. For both cases, we apply a general method described in section 4. This replaces the infinite system of moment-like equations by a single integral equation. Application to the astrophysical problem is made in sections 6–9.

2. The governing equations

There are several physical causes of temperature variations in the CMBR (Kolb and Turner list five [18, p 383]). For large angular scales, the dominant contribution comes from variations in the gravitational potential; this is known as the *Sachs–Wolfe effect* [31]. An elementary discussion of the relevant equations can be found in [18 section 9.6.2; 25, section 6.4 and 28, 7, 6], whereas advanced, detailed, derivations that are up-to-date with the present state of computation can be found in [15, 20]. A brief derivation is sketched here; for a rigorous derivation, involving time-dependent processes within general relativity, see the cited books and papers.

Consider the standard case of a flat universe ($\Omega_0 = 1$) with zero cosmological constant ($\Lambda = 0$). The observed temperature fluctuations of the CMBR in the direction of the unit vector $\hat{\boldsymbol{y}}$ are

$$\frac{\delta T(\hat{\boldsymbol{y}})}{T} = \frac{T(\hat{\boldsymbol{y}}) - T_0}{T_0},$$

where T_0 is the mean temperature. The Sachs–Wolfe effect relates the observed $\delta T/T$ to the primordial density fluctuations in the early universe at a time when light and matter decoupled. For the microwave radiation, the whole universe at this time is referred to as the *last scattering surface* [18, p 74]. Since the time of last scattering to the present, the distance travelled by light is $y = 2c/H_0$, where c is the speed of light and H_0 is the Hubble constant. A present-day observer at a given point will receive light from a spherical shell of present-day radius y on the last scattering surface. Thus, it is natural to expand on that shell in terms of spherical harmonics $Y_n^m(\hat{\boldsymbol{y}})$,

$$\frac{\delta T(\hat{\boldsymbol{y}})}{T} = \sum_{\ell,m} a_{\ell m}(y) Y_\ell^m(\hat{\boldsymbol{y}}), \tag{2.1}$$

† For the definition of an ill-posed problem see, for example, [8, ch 4].

where $a_{\ell m}$ are dimensionless expansion coefficients.

We can also give a Fourier representation for $\delta T/T$,

$$\frac{\delta T(\hat{\boldsymbol{y}})}{T} = \mathcal{V}\int \Theta(\boldsymbol{k}) \exp{(\mathrm{i}y\boldsymbol{k}\cdot\hat{\boldsymbol{y}})}\,\mathrm{d}\boldsymbol{k}, \tag{2.2}$$

where $\Theta(\boldsymbol{k})$ is a Fourier amplitude. The factor $\mathcal{V} = V/(2\pi)^3$ where V is a (large) volume, so that Θ is dimensionless. (For simplicity, we have used a continuum description (Fourier transform) here rather than a discrete description (Fourier series) followed by an appropriate limit.)

We can compare (2.1) and (2.2), using the plane-wave expansion

$$\exp{(\mathrm{i}y\boldsymbol{k}\cdot\hat{\boldsymbol{y}})} = 4\pi\sum_{\ell,m}\mathrm{i}^l j_l(ky)Y_\ell^m(\hat{\boldsymbol{y}})\overline{Y_\ell^m(\hat{\boldsymbol{k}})},$$

where j_ℓ is a spherical Bessel function, $\boldsymbol{k} = k\hat{\boldsymbol{k}}$, $k = |\boldsymbol{k}|$ and the overbar denotes complex conjugation. We find that

$$a_{\ell m}(y) = 4\pi\mathrm{i}^\ell\mathcal{V}\int \Theta(\boldsymbol{k})j_\ell(ky)\overline{Y_\ell^m(\hat{\boldsymbol{k}})}\,\mathrm{d}\boldsymbol{k},$$

whence use of the addition theorem for Legendre polynomials gives

$$\sum_m |a_{\ell m}|^2 = 4\pi(2\ell+1)\mathcal{V}^2\iint \Theta(\boldsymbol{k})\overline{\Theta(\boldsymbol{k}')}j_\ell(ky)j_\ell(k'y)P_\ell(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}')\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'. \tag{2.3}$$

The Sachs–Wolfe effect relates $\delta T/T$, through physics on the last scattering surface, to the primordial density fluctuations at the time of last scattering, $\delta\rho/\rho$. The latter quantity has a Fourier representation similar to (2.2), namely

$$\frac{\delta\rho(\boldsymbol{x})}{\rho} = \mathcal{V}\int \Delta(\boldsymbol{k})\exp{(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})}\,\mathrm{d}\boldsymbol{k}.$$

It is usual to assume that $\Delta(\boldsymbol{k})$ is an independent Gaussian random variable of zero mean. Thus $\langle\Delta(\boldsymbol{k})\rangle = 0$ and

$$\langle\Delta(\boldsymbol{k})\overline{\Delta(\boldsymbol{k}')}\rangle = \mathcal{V}^{-1}P(k)\delta^3(\boldsymbol{k}-\boldsymbol{k}'),$$

where $P(k)$ is the *power spectrum* of the density fluctuations. Note that

$$P(k) = \mathcal{V}\int\langle\Delta(\boldsymbol{k})\overline{\Delta(\boldsymbol{k}')}\rangle\,\mathrm{d}\boldsymbol{k}'. \tag{2.4}$$

Then, the Sachs–Wolfe analysis shows that (the Fourier transforms of) $\delta T/T$ and $\delta\rho/\rho$ are related by

$$\Delta(\boldsymbol{k}) = \mathcal{A}k^2\Theta(\boldsymbol{k}), \tag{2.5}$$

where $\mathcal{A}$ is a real constant with the dimensions of area. (Roughly speaking, the temperature fluctuations are proportional to the perturbed gravitational potential, Φ say, and $\nabla^2\Phi$ is proportional to the density fluctuations; from [28], we have $\mathcal{A} = -2c^2/H_0^2 = -\frac{1}{2}y^2$.) So, if we take an ensemble average of (2.3) and define rotationally invariant dimensionless coefficients C_ℓ by

$$C_\ell \equiv \frac{1}{2\ell+1}\sum_m\langle|a_{\ell m}|^2\rangle,$$

we find that

$$\begin{aligned} C_\ell &= \frac{4\pi\mathcal{V}}{\mathcal{A}^2}\int k^{-4}P(k)j_\ell^2(ky)\,\mathrm{d}\boldsymbol{k} \\ &= (4\pi/\mathcal{A})^2\mathcal{V}\int_0^\infty k^{-2}P(k)j_\ell^2(ky)\,\mathrm{d}k. \end{aligned} \tag{2.6}$$

The set of coefficients $\{C_\ell\}$ is known as the *angular power spectrum*†.

The temperature fluctuations have been measured experimentally; for example, this was done by the COBE satellite (see [34, 35] and references therein). This gives an estimate of the angular power spectrum [7, section 7]: thus, we have values of C_ℓ for a range of ℓ-values. From these, we would like to extract the power spectrum, $P(k)$. This is a moment-like problem. In section 7, we give an explicit solution of an idealized problem, in which we know C_ℓ for all $\ell \geqslant 0$.

In practice, there are complications. First, the measured values of C_0 and C_1, the 'nuisance parameters' [7, p 175], are unreliable. We show that $P(k)$ can be recovered, nevertheless, using C_ℓ for all $\ell \geqslant \ell_0$, where ℓ_0 is any fixed positive integer.

Another complication is that we cannot actually measure C_ℓ because it is defined as an ensemble average: we always have a relative uncertainty (or *cosmic variance*) of $(2\ell+1)^{-1/2}$, even if the experimental measurements are perfect [2, 36, p 354, 7, p 160, 34, p 189]. Once inverted, these errors will lead to corresponding errors in the calculated $P(k)$. As the errors in C_ℓ decay as ℓ increases, one possibility is to use C_ℓ for $\ell \geqslant \ell_0$, where ℓ_0 is chosen sufficiently large, so as to recover P. One could then compute C_ℓ for $\ell < \ell_0$, and compare with the experimental estimates, giving the possibility of cross-validation. However, we defer discussions of computational aspects to a later paper.

Further complications come from the underlying physics. Thus, the Sachs–Wolfe effect typically is decomposed into two additive quantities, the 'ordinary' and the 'integrated' Sachs–Wolfe effects, of which (2.6) only represents the former. The methods developed in this paper are only applicable to the ordinary Sachs–Wolfe effect. In the idealized limit that the CMBR anisotropy is composed only of super-Hubble scale, adiabatic, density perturbations in a flat ($\Omega_0 = 1$) universe, the ordinary Sachs–Wolfe effect dominates. This ideal limit has an approximate validity in inflationary cosmology, which is a promising candidate for development as a fundamental theory of the early universe; for reviews, see, for example, [18, ch 8] or [25, ch 10]. The understanding of the CMBR anisotropy is still a developing area of research, although the general consensus is that there are substantial modifications to the idealized regime of the ordinary Sachs–Wolfe effect. We will not attempt to survey this vast body of research here, but the interested reader can begin with a few exemplary papers [7, 10, 13, 14, 16]. It is safe to say that, when applied to real data, there would be practical limitations to the methods developed in this paper. Nevertheless, they offer a new and very different perspective on the subject.

3. Formulation of two mathematical problems

Let us begin by considering the following moment-like problem.

Basic problem. *Given a set of numbers $\{C_n\}$ and a set of functions $\{\psi_n(k)\}$, with $n = 0, 1, 2, \ldots,$ find a function $f(k)$ (in a certain space) that satisfies*

$$\int_0^\infty f(k)\psi_n(k)\,\mathrm{d}k = C_n, \qquad \textit{for} \quad n = 0, 1, 2, \ldots. \tag{3.1}$$

† In general relativity, a density fluctuation is not a gauge-invariant quantity. In practical terms this means density fluctuations are not well-defined observables. For sub-Hubble scales, the gauge dependence of density fluctuations is negligible, so that any reasonable gauge choice is operationally acceptable. However for super-Hubble scales, the gauge dependence is acute. Properly, the power spectrum $P(k)$ in (2.4) must be defined with respect to 'gauge-invariant' quantities [5, 17] that in the sub-Hubble scale regime are readily identified as density fluctuations. This is a well-studied problem in the literature [10, 14, 19, 21, 24, 26, 29, 32]. For our purposes, however, these details are unimportant and we will be content with our slightly imprecise description, (2.4). What is important for our work is that the basic relation between $P(k)$ and the angular power spectrum C_ℓ is of the form (2.6).

We are mainly interested in one particular set $\{\psi_n\}$, given by

$$\psi_n(k) = j_n^2(ky), \tag{3.2}$$

where y is fixed. For a simpler model problem, we shall also consider the choice

$$\psi_n(k) = k^n, \tag{3.3}$$

which gives rise to a classical (Stieltjes) moment problem [4, 33].

We are also interested in the following related problem where the first m of (3.1) are omitted.

Reduced-data problem. *Given a set of numbers $\{C_n\}$ and a set of functions $\{\psi_n(k)\}$, find a function $f_m(k)$ that satisfies*

$$\int_0^\infty f_m(k)\psi_n(k)\,\mathrm{d}k = C_n, \qquad \textit{for} \quad n \geqslant m \geqslant 0. \tag{3.4}$$

Here, m is fixed and $f_0 \equiv f$.

It is important to note that we seek f_m in the *same* space as f. We are also using the same numbers $\{C_n\}$ and the same functions $\{\psi_n\}$ (for $n \geqslant m$) as in the basic problem: thus, C_n and ψ_n are independent of m.

As we saw in section 2, the reduced-data problem, with $\psi_n(k) = j_n^2(ky)$, arises in connection with the Sachs–Wolfe effect. In that context, the basic unknown function is the power spectrum $P(k) = k^2 f(k)$, apart from a constant factor; see (2.6).

4. A general method

A general method for treating moment-like problems is to replace them by an integral equation. Thus, choose a second set of functions $\{\phi_n(\lambda)\}$, multiply the nth equation of (3.4) by ϕ_n and sum over n. This gives

$$\int_0^\infty f_m(k)\mathcal{K}_m(k, \lambda)\,\mathrm{d}k = g_m(\lambda), \tag{4.1}$$

where

$$\mathcal{K}_m(k, \lambda) = \sum_{n=m}^\infty \psi_n(k)\phi_n(\lambda) \tag{4.2}$$

and

$$g_m(\lambda) = \sum_{n=m}^\infty C_n\phi_n(\lambda). \tag{4.3}$$

To be effective, one has to be able to solve the integral equation (4.1), perhaps by recognizing the left-hand side as a known integral transform; implicitly, this requires that the sum defining the kernel $\mathcal{K}_m$ can be evaluated in closed form. To make the method rigorous, one has to show that the sums in (4.2) and (4.3) are convergent, and that the implied interchange of summation and integration is justified. Note that, at this stage, λ is unspecified.

We will use this method below for the two sets $\{\psi_n\}$ given by (3.3) and (3.2). We remark that a similar method was used in [22, 23] to analyse the so-called *null-field equations*, a system of moment-like equations (involving Hankel functions) that can be used to solve the boundary-value problems for acoustic scattering by bounded obstacles.

5. Moment problems

We start with $\psi_n(k) = k^n$, giving the following simple reduced-data problem.

Moment problem $\mathcal{M}_m$. *Find a function $f_m(k)$ that satisfies*

$$\int_0^\infty f_m(k) k^n \, \mathrm{d}k = C_n \qquad \textit{for all} \quad n \geqslant m \geqslant 0. \tag{5.1}$$

We assume that f_m is a smooth function that has a Laplace transform.

We can rewrite (5.1) as

$$\int_0^\infty \{k^m f_m(k)\} k^n \, \mathrm{d}k = C_{n+m}, \qquad \text{for} \quad n = 0, 1, 2, \ldots.$$

Multiply by $\phi_n(\lambda) = (-\lambda)^n/n!$ and sum over n to give

$$\int_0^\infty \{k^m f_m(k)\} \mathrm{e}^{-\lambda k} \, \mathrm{d}k = g_m(\lambda), \tag{5.2}$$

where

$$g_m(\lambda) = \sum_{n=0}^{\infty} C_{n+m} \frac{(-\lambda)^n}{n!}. \tag{5.3}$$

Equation (5.2) gives the Laplace transform of $k^m f_m(k)$, so that we can invert to obtain f_m itself. This shows that f_m is obtainable uniquely from the given coefficients $\{C_n\}$ with $n \geqslant m$.

In the derivation above, we assumed that the series (5.3) is convergent. In fact, we only require convergence for $|\lambda| < \lambda_0$, say, and then define $g_m(\lambda)$ for larger $|\lambda|$ by analytic continuation. The assumption that $f_m(k)$ has a Laplace transform is enough to ensure that $k^m f_m(k)$ also has a Laplace transform, this being what is actually required in the derivation.

The connection between classical moment problems and the Laplace transform is well known; see, for example, [33, p 97] and [9, p 230]. For a connection with a Hankel transform (take $\phi_n(\lambda) = (-\lambda)^n/(n!)^2$), see [33, p 96].

How are f_m and $f_0 \equiv f$ related, for $m > 0$? From (5.3), we have

$$\begin{aligned} g_m(\lambda) &= \sum_{n=m}^{\infty} C_n \frac{(-\lambda)^{n-m}}{(n-m)!} = \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^m \sum_{n=m}^{\infty} C_n \frac{(-\lambda)^n}{n!} \\ &= \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^m \sum_{n=0}^{\infty} C_n \frac{(-\lambda)^n}{n!} = \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^m g_0(\lambda). \end{aligned}$$

We also have

$$\int_0^\infty \{k^m f_m(k)\} \mathrm{e}^{-\lambda k} \, \mathrm{d}k = \left(-\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^m \int_0^\infty f_m(k) \mathrm{e}^{-\lambda k} \, \mathrm{d}k.$$

So, integrating m times gives

$$\int_0^\infty f_m(k) \mathrm{e}^{-\lambda k} \, \mathrm{d}k = g_0(\lambda) + p_m(\lambda), \tag{5.4}$$

where $p_0 \equiv 0$ and

$$p_m(\lambda) = \sum_{\ell=0}^{m-1} a_\ell \lambda^\ell$$

is an arbitrary polynomial of degree $(m-1)$. But the left-hand side of (5.4) is a Laplace transform, and so it must vanish as $|\lambda| \to \infty$ (in a right-hand half-plane). $g_0(\lambda)$ has the same property, as it is the Laplace transform of f_0. Hence, the polynomial $p_m(\lambda)$ must be absent:

$$\int_0^\infty f_m(k)\mathrm{e}^{-\lambda k}\,\mathrm{d}k = g_0(\lambda) = \int_0^\infty f_0(k)\mathrm{e}^{-\lambda k}\,\mathrm{d}k.$$

Thus, $f_m(k) = f_0(k)$ for all m. This means that if we know, *a priori*, that f_m and f_0 are both in the *same* space (here, we assumed that they both have Laplace transforms), then they are equal: deleting the first m moments C_n does not represent a loss of information. Similar comparisons can be made between f_m and $f_{m'}$ with $m \neq m'$.

For a simple example, take $C_n = n!$ for $n \geqslant 0$, whence

$$g_0(\lambda) = \sum_{n=0}^{\infty}(-\lambda)^n = (1+\lambda)^{-1}$$

for $|\lambda| < 1$. Note that we can define $g_0(\lambda)$ for all $\lambda \neq -1$ by analytic continuation. By inspection, we have $f(k) = \mathrm{e}^{-k}$.

6. The Sachs–Wolfe effect

Here, we assume that $\psi_n(k) = j_n^2(ky)$, where j_n is a spherical Bessel function and y is fixed. Thus, we consider the following problem.

Sachs–Wolfe problem $\mathcal{P}_m$. *Find a function $f_m(k)$ that satisfies*

$$\int_0^\infty f_m(k)\,j_n^2(ky)\,\mathrm{d}k = C_n \qquad \textit{for all} \quad n \geqslant m \geqslant 0. \tag{6.1}$$

Here, m is fixed, and the constants C_n are given.

What conditions on f_m are appropriate? Let us assume that

$$f_m(k) \sim \begin{cases} k^\alpha & \text{as} \quad k \to 0, \\ k^{1-\beta} & \text{as} \quad k \to \infty. \end{cases}$$

Then, considering the integral on the left-hand side of (6.1), we see that we need

$$2m + \alpha + 1 > 0 \qquad \text{for convergence at } k = 0$$

and

$$\beta > 0 \qquad \text{for convergence at } k = \infty.$$

Note that convergence at $k = 0$ depends on the smallest value of n used, which is m. However, we want conditions that do not depend on m. So, given that the basic problem is $\mathcal{P}_0$, minimal conditions are

$$\alpha > -1 \qquad \text{and} \qquad \beta > 0. \tag{6.2}$$

Arguments based on causality imply that $P(k) = \mathrm{O}(k^4)$ as $k \to 0$ [1, 30], where $P(k) = k^2 f(k)$. Thus, the desired physical solution should have $\alpha \geqslant 2$. This has led to studies of model power spectra of the form

$$f(k) = \frac{(ky)^\alpha}{1 + (k/k_0)^{\alpha+\beta-1}}, \tag{6.3}$$

where $\alpha \geqslant 2$, $\beta > 0$ and k_0 is a constant [6].

Next, let us consider an exact solution [11, formula 6.574.2]:

$$\int_0^\infty k^{1-\beta} j_n^2(ky)\,\mathrm{d}k = \frac{\pi y^{\beta-2}\Gamma(\beta)\Gamma(n-\frac{1}{2}\beta+1)}{2^{\beta+1}\{\Gamma(\frac{1}{2}(\beta+1))\}^2\Gamma(n+\frac{1}{2}\beta+1)}. \tag{6.4}$$

This holds for $0 < \beta < 2n+2$. Note that the right-hand side is $\mathrm{O}(n^{-\beta})$ as $n \to \infty$.

This exact solution throws some light on the next question: how does the integral on the left-hand side of (6.1) behave as $n \to \infty$? If we split the range of integration at $k = 1$, say, we can easily see that the asymptotic contribution from integrating over $0 \leqslant k \leqslant 1$ is exponentially small. The dominant contribution comes from integrating over $k > 1$, so that the large-k behaviour of $f_m(k)$ can be used. Comparing with the exact solution given above shows that

$$C_n = \mathrm{O}(n^{-\beta}) \qquad \text{as} \quad n \to \infty.$$

Thus, there is a link between the large-n behaviour of the (given) coefficients C_n and the large-k behaviour of $f_m(k)$.

Another exact solution is [11, formula 6.577.1]:

$$\int_0^\infty \frac{k^2}{k^2+k_0^2} j_n^2(ky)\,\mathrm{d}k = \frac{\pi}{2y} I_{n+1/2}(k_0 y) K_{n+1/2}(k_0 y), \tag{6.5}$$

where $I_{n+1/2}$ and $K_{n+1/2}$ are modified Bessel functions and k_0 is a positive constant. Note that the right-hand side is $\mathrm{O}(n^{-1})$ as $n \to \infty$. This formula gives the angular power spectrum for the model (6.3), exactly, when $\alpha = 2$ and $\beta = 1$. Further formulae can be obtained by differentiation of (6.5) with respect to k_0.

7. The basic problem $\mathcal{P}_0$

We are going to apply the general method of section 4 to problem $\mathcal{P}_0$, which we restate here.

Problem $\mathcal{P}_0$**.** *Find a function $f(k)$ that satisfies*

$$\int_0^\infty f(k) j_n^2(ky)\,\mathrm{d}k = C_n \qquad \textit{for} \quad n = 0, 1, 2, \ldots. \tag{7.1}$$

We have to choose the set $\{\phi_n(\lambda)\}$. There are very few known sums involving products of Bessel functions. From Abramowitz and Stegun [3, formula 10.1.45], we have

$$\sum_{n=0}^\infty (2n+1) P_n(\cos\theta) j_n^2(ky) = \frac{\sin\{ky\sqrt{2-2\cos\theta}\}}{ky\sqrt{2-2\cos\theta}}, \tag{7.2}$$

where P_n is a Legendre polynomial and θ is unrestricted. If we define λ by

$$\lambda = y\sqrt{2-2\cos\theta} = 2y\sin\tfrac{1}{2}\theta, \tag{7.3}$$

we can rewrite (7.2) as

$$\mathcal{K}_0(k,\lambda) = \frac{\sin k\lambda}{k\lambda}, \tag{7.4}$$

where $\mathcal{K}_m$ is defined by (4.2) and

$$\phi_n(\lambda) = (2n+1) P_n(1 - \tfrac{1}{2}(\lambda/y)^2). \tag{7.5}$$

Note that, for convergence of the integral in (4.1), we require that λ is *real*. Moreover, we can obtain the formula (7.4) for all $\lambda > 0$ if we allow $\theta = \theta_R + \mathrm{i}\theta_I$ to be complex; specifically, we can suppose that θ lies on the L-shaped contour given by

$$\{0 < \theta_R \leqslant \pi \quad \text{with} \quad \theta_I = 0\} \qquad \text{and} \qquad \{0 \leqslant \theta_I < \infty \quad \text{with} \quad \theta_R = \pi\}.$$

So, multiplying (7.1) by $(2n+1)P_n(\cos\theta)$ and summing over n gives

$$\int_0^\infty k^{-1} f(k)\sin\lambda k\,\mathrm{d}k = \lambda g_0(\lambda), \qquad \text{for} \quad \lambda > 0, \tag{7.6}$$

where

$$g_0(\lambda) = \sum_{n=0}^\infty (2n+1)C_n P_n\left(1 - \tfrac{1}{2}(\lambda/y)^2\right). \tag{7.7}$$

This reduces the determination of $f(k)$ to the inversion of a Fourier sine transform:

$$f(k) = \frac{2k}{\pi}\int_0^\infty \lambda g_0(\lambda)\sin\lambda k\,\mathrm{d}\lambda. \tag{7.8}$$

The derivation assumes that $k^{-1}f(k)$ has a Fourier sine transform, which is consistent with the convergence conditions (6.2). It also assumes that the series (7.7) is convergent for some values of λ, and that $g_0(\lambda)$ can be defined for other values of λ by analytic continuation.

For an example, take C_n to be defined by the right-hand side of (6.4) for $n \geqslant 0$ with $0 < \beta < 2$. In order to calculate $g_0(\lambda)$, defined by (7.7), we use the following expansion (see the appendix for a derivation):

$$(1-x)^{-\gamma} = \sum_{n=0}^\infty (2n+1)c_n P_n(x) \qquad \text{for} \quad |x| < 1, \tag{7.9}$$

where $\gamma < 1$ and

$$c_n = 2^{-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(\gamma)}\frac{\Gamma(n+\gamma)}{\Gamma(n-\gamma+2)}. \tag{7.10}$$

Comparing with C_n, we set $\gamma = 1 - \frac{1}{2}\beta$ whence

$$c_n = (4/\pi)2^{-\beta/2}y^{2-\beta}\Gamma(\beta)\sin\left(\tfrac{1}{2}\pi\beta\right)C_n.$$

With $x = 1 - \frac{1}{2}(\lambda/y)^2$, we find that

$$g_0(\lambda) = \frac{\pi}{2}\frac{\lambda^{\beta-2}}{\Gamma(\beta)\sin\left(\frac{1}{2}\pi\beta\right)}. \tag{7.11}$$

Note that the series defining $g_0(\lambda)$ converges for $0 < \lambda < 2y$, but we can use (7.11) to define $g_0(\lambda)$ for all $\lambda > 0$. Hence (7.8) gives

$$f(k) = \frac{k}{\Gamma(\beta)\sin\left(\frac{1}{2}\pi\beta\right)}\int_0^\infty \lambda^{\beta-1}\sin\lambda k\,\mathrm{d}\lambda = k^{1-\beta},$$

in agreement with (6.4).

8. The reduced-data problem $\mathcal{P}_m$

For $m > 0$, we consider the reduced-data problem $\mathcal{P}_m$ (the Sachs–Wolfe problem) described in section 6.

Proceeding as for $\mathcal{P}_0$, we multiply (6.1) by $(2n+1)P_n(\cos\theta)$ and sum over n from $n = m$. This gives, using (7.3),

$$\int_0^\infty f_m(k)\left\{\frac{\sin\lambda k}{\lambda k} - \sum_{n=0}^{m-1}\phi_n(\lambda)j_n^2(ky)\right\}\mathrm{d}k = g_m(\lambda), \tag{8.1}$$

where $g_m(\lambda)$ and $\phi_n(\lambda)$ are defined by (4.3) and (7.5), respectively.

Next, multiply (8.1) by λ and write as

$$\int_0^\infty k^{-1} f_m(k)\sin\lambda k\,\mathrm{d}k - \lambda\sum_{n=0}^{m-1}\phi_n(\lambda)\int_0^\infty f_m(k)\,j_n^2(ky)\,\mathrm{d}k = \lambda g_m(\lambda). \quad (8.2)$$

Noting that $P_n(x)$ is a polynomial in x of degree n, we see that the second term on the left-hand side of (8.2) is a polynomial in λ of degree $(2m-1)$. So, applying the differential operator $\mathrm{d}^{2m}/\mathrm{d}\lambda^{2m}$ gives

$$\frac{\mathrm{d}^{2m}}{\mathrm{d}\lambda^{2m}}\int_0^\infty k^{-1} f_m(k)\sin\lambda k\,\mathrm{d}k = \frac{\mathrm{d}^{2m}}{\mathrm{d}\lambda^{2m}}(\lambda g_m(\lambda)) = \frac{\mathrm{d}^{2m}}{\mathrm{d}\lambda^{2m}}(\lambda g_0(\lambda)).$$

Integrating $2m$ times then gives

$$\int_0^\infty k^{-1} f_m(k)\sin\lambda k\,\mathrm{d}k = \lambda g_0(\lambda) + p_{2m}(\lambda), \quad (8.3)$$

where p_{2m} is an arbitrary polynomial of degree $(2m-1)$. But the left-hand side of (8.3) is the Fourier sine transform of $k^{-1}f_m(k)$. This is assumed to exist and, by the Riemann–Lebesgue lemma, it must vanish as $|\lambda| \to \infty$. $\lambda g_0(\lambda)$ has the same property, as it is the Fourier sine transform of $k^{-1}f$, by (7.6). Hence, the polynomial $p_{2m}(\lambda)$ must be absent. It follows that $f_m(k) = f(k)$ for all m.

The solution given above is correct but somewhat deceptive, for in the formulation of $\mathcal{P}_m$ we are not given C_n for $0 \leqslant n < m$. Thus, we cannot form the series (7.11) defining $g_0(\lambda)$. So, instead of (8.3), we obtain

$$\int_0^\infty k^{-1} f_m(k)\sin\lambda k\,\mathrm{d}k = \lambda g_m(\lambda) + p_{2m}(\lambda),$$

where the polynomial $p_{2n}(\lambda)$ is to be determined by the requirement that the right-hand side vanishes as $|\lambda| \to \infty$. Thus (just as for $\mathcal{P}_0$), we have to effect the analytic continuation of $g_m(\lambda)$. Numerical consequences remain to be explored.

9. Discussion

We have given an explicit formula for solving the basic problem $\mathcal{P}_0$. For the application we have in mind, all the constants C_n are *non-negative*: does it follow that the solution $f(k)$ is positive? In general, the answer is 'no'. We show this by giving an explicit counter-example. Let

$$f(k) = y\{(ky/\pi)^{1/2} - (2/\pi)A\}, \quad (9.1)$$

where A is a constant to be chosen. Note that if $A > 0$, $f(k) < 0$ for $k < 4A^2/(\pi y)$. (The various factors in (9.1) are merely inserted for algebraic convenience.) We observe that the f defined by (9.1) is a linear combination of two of the exact solutions (6.4), corresponding to $\beta = \frac{1}{2}$ and $\beta = 1$. Thus,

$$C_n = \frac{\pi\,\Gamma(n+\frac{3}{4})}{2^{3/2}\{\Gamma(\frac{3}{4})\}^2\Gamma(n+\frac{5}{4})} - \frac{A}{2n+1}.$$

We have to show that $C_n > 0$ for all $n \geqslant 0$; we do this using an inductive argument. Straightforward calculation shows that

$$C_{n+1} = \frac{4n+3}{4n+5}C_n + \frac{4(n+1)}{(4n+5)(2n+1)(2n+3)}A.$$

So, $C_n > 0$ for all $n \geqslant 0$ provided $A > 0$ and $C_0 > 0$. But

$$C_0 = 1 - A,$$

so we can take any A with $0 < A < 1$.

Acknowledgment

Financial support was provided to AB by the US Department of Energy.

Appendix

Here, we derive the expansion (7.9) with (7.10). From (7.9), orthogonality gives

$$c_n = \frac{1}{2}\int_{-1}^{1}(1-x)^{-\gamma}P_n(x)\,\mathrm{d}x$$
$$= \frac{(-1)^n}{2^{n+1}n!}\int_{-1}^{1}(1-x)^{-\gamma}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n\,\mathrm{d}x$$

using Rodigues' formula. Integrating by parts n times, using

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x)^{-\gamma} = \frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}(1-x)^{-\gamma-n}$$

and noting that the integrated terms vanish, gives

$$c_n = \frac{1}{2^{n+1}n!}\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}\int_{-1}^{1}(1-x^2)^n(1-x)^{-\gamma-n}\,\mathrm{d}x.$$

In the integral, put $x = 1-2t$; it becomes

$$2^{n+1-\gamma}\int_0^1 t^{-\gamma}(1-t)^n\,\mathrm{d}t = 2^{n+1-\gamma}\frac{n!\Gamma(1-\gamma)}{\Gamma(n-\gamma+2)},$$

using the integral definition of the beta function, and the result (7.10) follows.

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