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Perturbed cracks in two dimensions: An integral-equation approach

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Abstract. A nominally straight crack of finite length is subjected to plane-strain loadings. A perturbation method is developed for calculating the stress-intensity factors, based on an asymptotic analysis of the governing hypersingular boundary integral equation for the crack-opening displacement. Known exact results for a shallow circular-arc crack are recovered, correct to second order in the small geometrical parameter. The method will extend to three-dimensional problems as it does not make essential use of two-dimensional techniques.

1. Introduction

Slightly curved or kinked cracks is the title of a well known paper by Cotterell and Rice (1980), in which a nominally straight crack is subject to plane-strain loading. The problem is to calculate the stress-intensity factors, correct to first order in ε, where the perturbed crack is defined by

$$y = \varepsilon\, f(x), \qquad -a \le x \le a, \tag{1}$$

with

$$f(a) = f(-a) = 0; \tag{2}$$

here, x and y are Cartesian coordinates, f is a given function and ε is a small parameter. Cotterell and Rice (1980) found expressions for the stress-intensity factors, using the complex-variable techniques of Muskhelishvili (1953). Expressions correct to second order in ε were subsequently obtained by Wu (1994), using similar methods.

Wu (1994) claimed that the perturbation expansion in ε is regular, in the sense that the approximations obtained are uniformly valid throughout the solid; this was implicitly assumed by Cotterell and Rice (1980). Why is this so? The reason is (2): in two dimensions, one can always arrange that the two crack tips are *fixed*, independently of ε, implying that the places where the stresses are singular do not move as the crack is perturbed from the straight ($\varepsilon = 0$) reference position. In general, we do not have this luxury in three dimensions, where the edge of a non-planar crack (a simple closed curve) need not lie in a plane.

The discussion above motivates the present work. We reconsider the plane-strain problem for a slightly curved crack, using integral-equation methods. We begin by reformulating the boundary-value problem as a boundary integral equation; we choose to use a hypersingular integral equation for the crack-opening displacement (COD). Next, we parametrise the curve defining the crack, leading to a one-dimensional hypersingular integral equation on a finite interval. At this stage, the analysis is exact. Then, we introduce (1), leading to a sequence of hypersingular integral equations for each term in the regular expansion of the COD in powers of ε. We verify that the method yields approximations in agreement with Muskhelishvili's (1953) exact solution for a circular-arc crack under constant loads, correct to second order in ε.

Our method does not require that $f(x)$ satisfies (2). We obtain regular perturbation expansions because we work only on the crack faces, not within the solid. (Displacements and stresses off the crack can be obtained, if desired, by inserting the approximations to the COD into the integral representation (5).) Moreover, our method does not make any essential use of two-dimensional techniques (such as those based on functions of a complex variable). Thus, it is perhaps not surprising that our method extends to perturbed cracks in three dimensions. We have obtained analogous results for some scalar problems (potential flow past wrinkled discs) (Martin, 1998a, b), and will describe our results on three-dimensional crack problems elsewhere.

We note that Dreilich and Gross (1985) have briefly considered perturbed cracks, using a singular integral equation for the derivative of the COD. Numerical solutions of this integral equation were presented by Chen et al. (1991). In fact, there are many other papers on curved cracks in two dimensions (see, for example, Lin'kov and Mogilevskaya, 1990, 1994; Sur and Altiero, 1988; Zang and Gudmundson, 1988), but we are mainly interested here in asymptotic results.

Finally, let us make some remarks on the asymptotic method developed by Movchan, Gao and Willis (1998) for two-dimensional (and three-dimensional) crack perturbations. Their theory uses local expansions near the crack tips and Bueckner weight functions for the unperturbed reference crack. It is a form of singular-perturbation theory, with a boundary layer near each crack tip; this 'occurs due to relocation of the coordinate system from the actual crack tip to the end of the reference crack'(Movchan et al., 1998, Section 2.1). These technical difficulties do not arise when the problem is first reformulated as a boundary integral equation, as we do below.

2. An integral equation

In this section, we derive an exact hypersingular integral equation for the COD when a curved crack is loaded under plane-strain conditions. The basic ingredient is the well-known fundamental solution

$$G_{ij}(P, Q) = \frac{1}{8\pi\mu(1-\nu)} \left\{ (3-4\nu)\,\delta_{ij}\,\log\frac{1}{R} + \frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j} \right\}.$$

Here, $i, j = 1, 2$, μ is the shear modulus, ν is Poisson's ratio, the points P and Q have Cartesian coordinates (x_1, x_2) and (x_1', x_2'), respectively, δ_{ij} is the Kronecker delta, and

$$R = \left\{ (x_1 - x_1')^2 + (x_2 - x_2')^2 \right\}^{1/2} \tag{3}$$

is the distance between P and Q (see, for example Rizzo, 1967).

Consider a cavity in an otherwise unbounded homogeneous isotropic elastic solid. Assume that the surface of the cavity, S (a simple closed curve) is loaded in such a way that the displacement components $u_i = o(1)$ and the stress components $\tau_{ij} = o(\mathcal{R}^{-1})$ as $\mathcal{R} \to \infty$, where $\mathcal{R}$ is distance from some fixed point in the vicinity of the cavity. Then, a familiar calculation yields the integral representation

$$u_i(P) = \int_S \left\{ u_j(q)\, T_{ij}(P,q) - t_j(q)\, G_{ij}(P,q) \right\} \, \mathrm{d}s_q, \quad i = 1, 2, \tag{4}$$

where summation over repeated subscripts is implied, $t_j(q) = n_k^q \tau_{jk}$, $\boldsymbol{n}(q) = (n_1^q, n_2^q)$ is the unit normal at $q \in S$ pointing into the solid,

$$T_{ij} = [4\pi(1-\nu)R]^{-1}\{(1-2\nu)[R_j n_i^q - R_i n_j^q]$$
$$-R_n^q[(1-2\nu)\delta_{ij} + 2R_i R_j]\},$$

$$R_i = \frac{\partial R}{\partial x_i} = \frac{x_i - x_i'}{R} \quad \text{and} \quad R_n^q \equiv \frac{\partial R}{\partial n_q} = n_i^q \frac{\partial R}{\partial x_i'} = -n_i^q R_i.$$

The formula (4) is the basis for boundary-element methods in two-dimensional elastostatics.

Next, let the cavity S shrink to a crack Γ. Denote the two sides of the crack by Γ^+ and Γ^-, and let q^+ and q^- be corresponding points on Γ^+ and Γ^-, respectively. Define $\boldsymbol{n}(q) = \boldsymbol{n}(q^+)$ so that $\boldsymbol{n}(q^-) = -\boldsymbol{n}(q)$. Also, define the crack-opening displacement (COD) by

$$[u_i(q)] = u_i(q^+) - u_i(q^-).$$

We assume that the imposed stresses are continuous across Γ. Then, we find that (4) reduces to

$$u_i(P) = \int_\Gamma [u_j(q)]\, T_{ij}(P,q)\, \mathrm{d}s_q, \quad i = 1, 2, \tag{5}$$

an integral representation for the displacement components at any point P in the solid.

We can compute the tractions $t_i(p)$ on Γ corresponding to u_i, using Hooke's law. The result is

$$\frac{1}{\mu} t_i(p) = \times\!\!\!\!\!\!\int_\Gamma [u_j(q)]\, S_{ij}(p,q)\, \mathrm{d}s_q, \quad i = 1, 2, \quad p \in \Gamma. \tag{6}$$

In this formula, the cross on the integral sign means that the integral is to be interpreted as a Hadamard finite-part integral; for references, see, for example, Martin and Rizzo (1989) or Martin et al. (1998). The kernel S_{ij} is given by

$$\pi S_{ij} = R^{-2}\{A\mathcal{N}\delta_{ij} + A n_i^q n_j^p + (1-3A) n_i^p n_j^q + (1-2A)\mathcal{N} R_i R_j -$$
$$-R_n^q[2A n_i^p R_j + (1-2A) R_i n_j^p] + R_n^p[(1-2A) n_i^q R_j + 2A R_i n_j^q] \tag{7}$$
$$+R_n^p R_n^q[8(1-A)R_i R_j - (1-2A)\delta_{ij}]\}$$

$$= (1-A)R^{-2}\{n_i^p n_j^q + \mathcal{N} R_i R_j - R_n^q R_i n_j^p + R_n^p n_i^q R_j +$$
$$+R_n^p R_n^q (8R_i R_j - \delta_{ij})\}, \tag{8}$$

where $\mathcal{N} = n_i^p n_i^q$,

$$A = \frac{1-2\nu}{2(1-\nu)} \quad \text{and} \quad R_n^p \equiv \frac{\partial R}{\partial n_p} = n_i^p \frac{\partial R}{\partial x_i} = n_i^p R_i.$$

All the terms inside the curly brackets are bounded as $p \to q$; in fact, in this limit, $R_n^p \to 0$, $R_n^q \to 0$ and $\mathcal{N} \to 1$, assuming that Γ is a twice-differentiable curve. Thus, (8) exhibits the expected non-integrable R^{-2} singularity that is typical of one-dimensional finite-part integrals. From (8), one can also show that $S_{ij} = S_{ji}$.

Once the traction components $t_i(p)$ are prescribed on Γ, (6) becomes a hypersingular boundary integral equation for the components of the COD, $[u_j]$. It is to be solved subject to the natural end-point conditions,

$$[u_i(q)] = 0, \quad i = 1, 2, \quad \text{when } q \text{ is an end-point of } \Gamma. \tag{9}$$

(Note that (6) can also be used if Γ represents several disjoint cracks.)

3. Projection

Our basic integral equation, (6), is exact and it holds on the curve Γ. It is more convenient to write (6) on a fixed reference curve, which we take to be a straight line segment of length 2. We do this projection by simply parametrising the curve Γ. Thus, we suppose that the integration point $q = (x_1', x_2')$ is specified by

$$x_1' = ax, \quad x_2' = aF(x), \quad -1 \le x \le 1,$$

where a is a length-scale and the (dimensionless) function F specifies the shape of Γ. Similarly, the point $p = (x_1, x_2)$ is specified by

$$x_1 = ax_0, \quad x_2 = aF(x_0), \quad -1 \le x_0 \le 1.$$

For the unit normals $\boldsymbol{n}(q)$ and $\boldsymbol{n}(p)$, we have $\boldsymbol{n}(q) = \boldsymbol{N}(q)/N(q)$ where

$$\boldsymbol{N}(q) = (-F'(x), 1) \quad \text{and} \quad N(q) = |\boldsymbol{N}(q)| = \sqrt{1 + [F'(x)]^2},$$

with a similar expression for $\boldsymbol{n}(p)$.

From (3), we have $R = a|x - x_0| \sqrt{1 + \Lambda^2}$, where

$$\Lambda = \frac{F(x) - F(x_0)}{x - x_0}.$$

Next, we have

$$R_1 = \frac{\Omega}{\sqrt{1 + \Lambda^2}}, \qquad R_2 = \frac{\Omega\Lambda}{\sqrt{1 + \Lambda^2}},$$

$$N(q)\, R_n^q = \frac{-\Omega(\Lambda - F')}{\sqrt{1 + \Lambda^2}}, \qquad N(p)\, R_n^p = \frac{\Omega(\Lambda - F_0')}{\sqrt{1 + \Lambda^2}}$$

and $N(p)\, N(q)\, \mathcal{N} = 1 + F' F_0'$, where $F' \equiv F'(x)$, $F_0' \equiv F'(x_0)$ and

$$\Omega = (x_0 - x)/|x_0 - x|$$

so that $\Omega^2 = 1$. These expressions allow us to write the kernel S_{ij} in terms of x, x_0 and F. Thus, we define a new kernel matrix $\tilde{S}_{ij}$ by

$$2\pi(1 - \nu)a^2(x - x_0)^2 N(p)\, N(q)\, S_{ij} = \tilde{S}_{ij}(x_0, x); \tag{10}$$

it is given explicitly as follows:

$$\tilde{S}_{11} = (1+\Lambda^2)^{-3}\{1-6\Lambda^2+\Lambda^4-(3-6\Lambda^2-\Lambda^4)F'F_0'+ \\ +2\Lambda(3-\Lambda^2)(F'+F_0')\},$$

$$\tilde{S}_{22} = (1+\Lambda^2)^{-3}\{1+6\Lambda^2-3\Lambda^4+(1-6\Lambda^2+\Lambda^4)F'F_0'- \\ -2\Lambda(1-3\Lambda^2)(F'+F_0')\},$$

$$\tilde{S}_{12} = \tilde{S}_{21} = (1+\Lambda^2)^{-3}\{2\Lambda(1-3\Lambda^2)-2\Lambda(3-\Lambda^2)F'F_0'- \\ -(1-6\Lambda^2+\Lambda^4)(F'+F_0')\}.$$

Finally, if we multiply the integral equation (6) by $2(1-\nu)N(p)$, and note that $\mathrm{d}s_q = aN(q)\,\mathrm{d}x$, we obtain

$$\frac{1}{\pi}\,⨍_{-1}^{1} \tilde{u}_j(x)\,\tilde{S}_{ij}(x_0,x)\,\frac{\mathrm{d}x}{(x-x_0)^2} = \tilde{t}_i(x_0), \quad i=1,2, \tag{11}$$

for $-1 < x_0 < 1$, where

$$\tilde{u}_i(x) = a^{-1}[u_i(q)] \quad \text{and} \quad \tilde{t}_i(x_0) = 2(1-\nu)\mu^{-1}N(p)\,t_i(p). \tag{12}$$

Equation (11) is our basic hypersingular integral equation for a curved crack Γ under plain-strain loading. In fact, it is a pair of coupled scalar integral equations for $\tilde{u}_1$ and $\tilde{u}_2$. This pair is to be solved subject to the end-point conditions (9), which become

$$\tilde{u}_i(1) = \tilde{u}_i(-1) = 0, \quad i=1,2,$$

after projection. The (vector) integral equation (11) is exact, and it can be solved numerically. However, our focus will be on slightly curved cracks, where Γ is approximately straight.

4. The straight crack

For a straight crack, we have

$$F(x) = mx + c,$$

where m and c are constants. Hence, $F' = F_0' = \Lambda = m$. Elementary calculations then give

$$\tilde{S}_{11} = \tilde{S}_{22} = 1 \quad \text{and} \quad \tilde{S}_{12} = \tilde{S}_{21} = 0,$$

so that the 2×2 system of scalar integral equations decouples into

$$(\mathcal{H}\tilde{u}_i)\,(x_0) = \tilde{t}_i(x_0), \quad i=1,2, \quad -1 < x_0 < 1,$$

where the operator $\mathcal{H}$ is defined by

$$(\mathcal{H}u)\,(x_0) = \frac{1}{\pi}\,⨍_{-1}^{1} \frac{u(x)}{(x-x_0)^2}\,\mathrm{d}x, \quad -1 < x_0 < 1. \tag{13}$$

The hypersingular integral equation $\mathcal{H}u = b$ can be solved explicitly. When the end-point conditions $u(1) = u(-1) = 0$ are imposed, the solution is given by

$$u(x_0) = \frac{1}{\pi}\int_{-1}^{1} b(x)\,\log\left(\frac{|x - x_0|}{1 - xx_0 + \sqrt{(1-x^2)(1-x_0^2)}}\right)\,\mathrm{d}x$$

for $-1 < x_0 < 1$, provided that b satisfies certain conditions (Martin, 1992). (For example, it is sufficient that b be continuous with integrable end-point singularities.) In practice, it is often simpler to use the following formula (Kaya and Erdogan, 1987; Martin, 1992):

$$\frac{1}{\pi}\,⨎_{-1}^{1} \frac{\sqrt{1-x^2}\,U_n(x)}{(x-x_0)^2}\,\mathrm{d}x = -(n+1)\,U_n(x_0), \quad n = 0, 1, 2, \ldots .$$

Here, $U_n(x)$ is a Chebyshev polynomial of the second kind, defined by $U_n(\cos\theta) = [\sin{(n+1)\theta}]/\sin\theta$. Thus, if

$$b(x) = \sum_{n=0} b_n\,U_n(x),$$

then

$$u(x) = -\sqrt{1-x^2}\,\sum_{n=0}(n+1)^{-1} b_n\,U_n(x)$$

is the unique solution of $\mathcal{H}u = b$, subject to $u(1) = u(-1) = 0$. Note that, as the Chebyshev polynomials are orthogonal with respect to the weight $\sqrt{1-x^2}$, we have

$$b_n = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,U_n(x)\,b(x)\,\mathrm{d}x,$$

so that the coefficients b_n are known in terms of the given function b.

5. Slightly curved cracks

Suppose that

$$F(x) = \varepsilon\, f(x),$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda \quad \text{with} \quad \lambda = \{f(x) - f(x_0)\}/(x - x_0),$$

we find that

$$\tilde{S}_{11} = 1 + \varepsilon^2 S_{11}^2 + O(\varepsilon^4),$$

$$\tilde{S}_{22} = 1 + \varepsilon^2 S_{22}^2 + O(\varepsilon^4),$$

$$\tilde{S}_{12} = \tilde{S}_{21} = \varepsilon S_{12}^1 + O(\varepsilon^3)$$

as $\varepsilon \to 0$, where

$$S_{22}^2 = 3\lambda^2 + f' f_0' - 2\lambda(f' + f_0'),$$

$$S_{11}^2 = -3S_{22}^2, \qquad S_{12}^1 = 2\lambda - f' - f_0',$$

$f' \equiv f'(x)$ and $f_0' \equiv f'(x_0)$. Note that $S_{12}^1(x_0, x)$ and $S_{22}^2(x_0, x)$ are both $O((x - x_0)^2)$ as $|x - x_0| \to 0$. These kernels are also symmetric in x and x_0.

We expand $\tilde{t}_i$ similarly. Suppose that the prescribed tractions are defined in terms of a stress field $\tau_{ij}(x_1, x_2)$, so that

$$\tilde{t}_i(x_0) = B\{\tau_{i2}(ax_0, aF_0) - F_0'\tau_{i1}(ax_0, aF_0)\},$$

exactly, where $F_0 \equiv F(x_0)$ and $B = 2(1 - \nu)/\mu$. From Taylor's theorem, we have

$$\tau_{ij}(ax_0, \varepsilon a f_0) = \tau_{ij}^{(0)}(x_0) + \varepsilon\tau_{ij}^{(1)}(x_0) + \varepsilon^2\tau_{ij}^{(2)}(x_0) + \cdots,$$

where $f_0 \equiv f(x_0)$ and

$$\tau_{ij}^{(n)}(x_0) = \frac{(af_0)^n}{n!}\left[\frac{\partial^n}{\partial y^n}\tau_{ij}(ax_0, y)\right]\Bigg|_{y=0}.$$

Hence, we deduce that

$$\tilde{t}_i(x_0) = t_i^0 + \varepsilon t_i^1 + \varepsilon^2 t_i^2 + \cdots, \tag{14}$$

where

$$t_i^0 = B\tau_{i2}^{(0)} \quad \text{and} \quad t_i^n = B\left(\tau_{i2}^{(n)} - f_0'\tau_{i1}^{(n-1)}\right) \tag{15}$$

for $n = 1, 2, \ldots$. In particular, if the given stress field is *constant*, then

$$\tilde{t}_i(x_0) = B(\tau_{i2} - \varepsilon f_0'\tau_{i1})$$

exactly.

Next, we expand the crack-opening displacement $\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2)$, writing

$$\tilde{u}_i = u_i^0 + \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \cdots.$$

Substituting in (11) then gives integral equations for u_i^j. These are all uncoupled scalar equations of the form

$$\mathscr{H}u_i^j = b_i^j, \quad i = 1, 2, \quad j = 0, 1, 2, \ldots,$$

where $\mathscr{H}$ is defined by (13) and b_i^j is known. For $j = 0$, we obtain

$$b_i^0 = t_i^0, \quad i = 1, 2,$$

so that the leading-order approximation is obtained by solving the integral equations for a straight crack. Having found u_i^0, the first-order approximation is obtained using

$$b_1^1(x_0) = t_1^1(x_0) - \frac{1}{\pi}\int_{-1}^{1} u_2^0(x)\, S_{12}^1\, \frac{\mathrm{d}x}{(x - x_0)^2}$$

and

$$b_2^1(x_0) = t_2^1(x_0) - \frac{1}{\pi}\int_{-1}^{1} u_1^0(x)\, S_{12}^1\, \frac{\mathrm{d}x}{(x - x_0)^2}.$$

At second order, we have

$$b_1^2(x_0) = t_1^2(x_0) + \frac{1}{\pi}\int_{-1}^{1}\left\{3u_1^0(x)\,S_{22}^2 - u_2^1(x)\,S_{12}^1\right\}\,\frac{\mathrm{d}x}{(x-x_0)^2}$$

and

$$b_2^2(x_0) = t_2^2(x_0) - \frac{1}{\pi}\int_{-1}^{1}\left\{u_2^0(x)\,S_{22}^2 + u_1^1(x)\,S_{12}^1\right\}\,\frac{\mathrm{d}x}{(x-x_0)^2}.$$

6. An example: quadratic cracks

As a simple example, let us take a quadratic crack,

$$f(x) = a_0 + a_1 x + a_2 x^2,$$

under uniform loading; here, a_0, a_1 and a_2 are constants. Hence $f' = 2a_2 x + a_1$, $\lambda = a_2(x + x_0) + a_1$, $S_{12}^1 = 0$ and $S_{22}^2 = -a_2^2(x - x_0)^2$. From (15), we have

$$t_i^0 = B\tau_{i2}, \quad t_i^1(x) = -B(2a_2 x + a_1)\tau_{i1} \quad \text{and} \quad t_i^2 = 0.$$

Hence

$$u_i^0(x) = -B\tau_{i2}\sqrt{1-x^2} \quad \text{and} \quad u_i^1(x) = B(a_2 x + a_1)\tau_{i1}\sqrt{1-x^2},$$

where we have noted that the solution of $(\mathscr{H}u)(x_0) = 2ax_0 + b$ is $u(x) = -(ax+b)\sqrt{1-x^2}$. For the second-order solution, we have

$$b_2^2(x_0) = \frac{a_2^2}{\pi}\int_{-1}^{1} u_2^0(x)\,\mathrm{d}x = -\tfrac{1}{2}Ba_2^2\tau_{22}$$

and $b_1^2 = \frac{3}{2}Ba_2^2\tau_{12}$, whence u_i^2 is proportional to u_i^0. So, combining all these results, we find that

$$u_1(x) = -B\left\{\tau_{12} - \varepsilon(a_2 x + a_1)\tau_{11} + \tfrac{3}{2}\varepsilon^2 a_2^2\tau_{12}\right\}\sqrt{1-x^2}$$

and

$$u_2(x) = -B\left\{\tau_{22} - \varepsilon(a_2 x + a_1)\tau_{12} - \tfrac{1}{2}\varepsilon^2 a_2^2\tau_{22}\right\}\sqrt{1-x^2},$$

correct to second order in ε.

The quantities of most interest are the stress-intensity factors, K_1 and K_2. We define these by

$$u_n(x) \sim -BK_1\sqrt{2\rho} \quad \text{and} \quad u_t(x) \sim -BK_2\sqrt{2\rho} \quad \text{as } \rho \to 0, \tag{16}$$

where

$$\rho = \sqrt{(1-x)^2 + \varepsilon^2[f(1) - f(x)]^2}$$

is distance from the edge at $x = 1$, and u_n and u_t are the normal and tangential components, respectively, of the crack-opening displacement, $\tilde{\boldsymbol{u}}$. The slope of the crack near $x = 1$ is $\varepsilon f'(1)$, so that if θ is the inclination of the crack to the x-axis at $x = 1$, then

$$\sin\theta = \varepsilon f'(1) \quad \text{and} \quad \cos\theta = 1 - \tfrac{1}{2}\varepsilon^2[f'(1)]^2,$$

with an error of $O(\varepsilon^3)$ as $\varepsilon \to 0$. Thus, near $x = 1$,

$$\begin{aligned} u_n &= u_2\cos\theta - u_1\sin\theta \\ &\sim u_2^0 + \varepsilon\{u_2^1 - u_1^0 f'(1)\} + \varepsilon^2\{u_2^2 - \tfrac{1}{2}u_2^0[f'(1)]^2 - u_1^1 f'(1)\} \\ &\sim -B\sqrt{2(1-x)}\{\tau_{22} - \varepsilon(3a_2 + 2a_1)\tau_{12} \\ &\quad + \varepsilon^2[(a_2+a_1)(2a_2+a_1)\tau_{11} - \tfrac{1}{2}(5a_2^2 + 4a_2a_1 + a_1^2)\tau_{22}]\} \end{aligned}$$

and

$$\begin{aligned} u_t &= u_1\cos\theta + u_2\sin\theta \\ &\sim u_1^0 + \varepsilon\{u_1^1 + u_2^0 f'(1)\} + \varepsilon^2\{u_1^2 - \tfrac{1}{2}u_1^0[f'(1)]^2 + u_2^1 f'(1)\} \\ &\sim -B\sqrt{2(1-x)}\{\tau_{12} + \varepsilon[(2a_2+a_1)\tau_{22} - (a_2+a_1)\tau_{11}] \\ &\quad - \varepsilon^2\tau_{12}[\tfrac{5}{2}a_2^2 + 5a_2a_1 + \tfrac{3}{2}a_1^2]\}. \end{aligned}$$

We now compare these expressions with (16), noting that

$$\begin{aligned} \sqrt{\rho} &= \sqrt{1-x}\left\{1 + \varepsilon^2[a_2(1+x) + a_1]^2\right\}^{1/4} \\ &\sim \sqrt{1-x}\left\{1 + \tfrac{1}{4}\varepsilon^2(2a_2+a_1)^2\right\}. \end{aligned}$$

Hence, we find that

$$\begin{aligned} K_1 = {} & \tau_{22} - \varepsilon(3a_2+2a_1)\tau_{12} \\ & + \varepsilon^2\left\{(a_2+a_1)(2a_2+a_1)\tau_{11} - \left(\tfrac{7}{2}a_2^2 + 3a_2a_1 + \tfrac{3}{4}a_1^2\right)\tau_{22}\right\} \end{aligned}$$

and

$$\begin{aligned} K_2 = {} & \tau_{12} + \varepsilon\{(2a_2+a_1)\tau_{22} - (a_2+a_1)\tau_{11}\} \\ & - \varepsilon^2\tau_{12}\left\{\tfrac{7}{2}a_2^2 + 6a_2a_1 + \tfrac{7}{4}a_1^2\right\}. \end{aligned}$$

A special case of this geometry is a shallow circular arc, given by

$$x_2 = aF(x_1) = c - \sqrt{c^2 - x_1^2}, \quad -a < x_1 < a,$$

where c is the radius of the arc. The arc subtends an angle of 2α at the centre of the circle, where $\sin\alpha = a/c$. For a shallow arc, we suppose that c/a is large, with a fixed. Hence, we see that

$$f(x) = \tfrac{1}{2}x^2 \quad \text{with} \quad \varepsilon = a/c = \sin\alpha,$$

where $\alpha = \varepsilon + O(\varepsilon^3)$ as $\varepsilon \to 0$. Thus, the previous calculation, with $a_2 = \frac{1}{2}$ and $a_1 = 0$, gives

$$K_1 = \tau_{22} - \tfrac{3}{2}\varepsilon\tau_{12} + \varepsilon^2\left(\tfrac{1}{2}\tau_{11} - \tfrac{7}{8}\tau_{22}\right)$$

and

$$K_2 = \tau_{12} + \varepsilon(\tau_{22} - \tfrac{1}{2}\tau_{11}) - \tfrac{7}{8}\varepsilon^2\tau_{12},$$

with an error of $O(\varepsilon^3)$ as $\varepsilon \to 0$. This agrees with Muskhelishvili's exact solution (Muskhelishvili, 1953, Section 124a), as re-expressed by Cotterell and Rice (1980, p. 158].

7. Discussion

We have presented a perturbation method for calculating the crack-opening displacement for a crack that is nominally straight and subjected to plane-strain loading. The method is general and takes proper account of the edge behaviour; we do not have to assume that the two crack tips lie on the x-axis. At each perturbation order, we have to solve two uncoupled scalar integral equations of the form $\mathcal{H}u = b$, where $\mathcal{H}$ is the basic one-dimensional hypersingular operator; such equations can be solved exactly. For any given crack shape f, the main difficulty comes from the evaluation of the forcing terms (b_i^j) for each integral equation. Nevertheless, the method does generalise to three-dimensional problems, such as non-planar perturbations of a penny-shaped crack under mixed-mode loadings; this work will be described elsewhere.

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