On wrinkled penny-shaped cracks

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Abstract

A nominally flat penny-shaped crack is subjected to a static loading. A perturbation method is developed for calculating the stress-intensity factors, based on an asymptotic analysis of the governing hypersingular boundary integral equation for the crack-opening displacement. Comparisons with known exact solutions for an inclined flat elliptical crack and for a crack in the shape of a spherical cap are made. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A basic problem in classical fracture mechanics is the determination of the stress-intensity factors for a crack in an elastic solid. For three-dimensional problems, there are very few explicit solutions: one example is a penny-shaped crack under arbitrary loads. Recently, there has been some interest in calculating corrections to these stress-intensity factors when the crack is perturbed (Rice, 1989; Leblond and Mouchrif, 1996; Movchan et al., 1998). Taking the penny-shaped crack as the reference crack, two classes of perturbation emerge. First, the crack can be perturbed in its own plane. Such perturbations have been studied by Gao and Rice (1987) and by Martin (1996) for pressurized cracks (a scalar problem) and by Gao (1988) and Martin (1995) for shear-loaded cracks (a vector problem); comparisons with known exact solutions for flat elliptical cracks were made.

Second, the penny-shaped crack can be perturbed out of its own plane, giving a 'wrinkled' crack Ω. This is the problem studied here. Specifically, let us define Ω by

$$\Omega : z = \varepsilon f(x, y), \quad (x, y) \in D, \tag{1.1}$$

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where (x, y, z) are Cartesian coordinates, f is a given function, D is a flat circular disc in the xy-plane and ε is a small parameter. We have previously considered analogous scalar problems, such as calculating the added mass for potential flow past a wrinkled rigid disc (Martin, 1998a, b). We have also studied the plane-strain problem of a slightly curved crack (Martin, 2000). We use a similar method here, although the details are more complicated. Thus, we first reduce the boundary-value problem to a boundary integral equation over the crack Ω; we choose to use a hypersingular integral equation for the crack-opening displacement (COD). Next, we project this equation onto the disc D, leading to a two-dimensional hypersingular integral equation over D. At this stage, the analysis is exact. Then, we introduce Eq. (1.1), leading to a sequence of hypersingular integral equations for each term in the regular expansion of the COD in powers of ε. These equations are just the corresponding hypersingular integral equation for a penny-shaped crack, with various forcing functions. Such equations can be solved exactly. We do this, in detail, for two particular crack geometries, namely, an inclined flat elliptical crack and a shallow spherical-cap crack. We obtain agreement with the known exact solutions, correct to order ε. These are stringent tests of the theory, because the solution for the spherical-cap crack, in particular, is very complicated. Indeed, one may regard the asymptotic approximations as validating the exact solution!

The use of hypersingular integral equations leads to a simpler formulation than would follow from the use of regularized integral equations; see, for example, the equations derived by Le Van and Royer (1986). Shliapoberskii (1978) and Xu et al. (1994) have developed perturbation theories based on regularized integral equations for dislocation densities (tangential gradient of the COD). Shliapoberskii (1978) assumed that the crack edge $\partial\Omega$ was in the plane $z = 0$ (which is a severe restriction for three-dimensional problems) and did not give any explicit applications. Xu et al. (1994) were able to find the first-order correction for a semi-infinite crack.

To conclude this introduction, let us compare our approach with a more direct treatment, in which one tries to perturb the boundary-value problem itself. This would inevitably be a *singular* perturbation (unless the crack edge is fixed), because the perturbed fields are singular at the actual crack edge, not at the edge of the unperturbed crack. This leads to various technical complications, such as the introduction of a boundary layer near the crack edge. Our approach is less direct, perhaps, but it leads to *regular* perturbations: this happens because we work on the crack faces only, not within the solid, and, physically, we do not expect the COD to be much different for the perturbed and unperturbed cracks.

2. An integral equation

In this section, we derive an exact hypersingular integral equation for the COD across a crack Ω in a three-dimensional homogeneous isotropic elastic solid. We assume that Ω is modelled by a smooth open surface with a simply connected edge $\partial\Omega$. The basic ingredient is Kelvin's fundamental solution

$$G_{ij}(P, Q) = [16\pi\mu(1 - v)R_0]^{-1}\left\{(3 - 4v)\,\delta_{ij} + R_i R_j\right\}.$$

Here, $i, j = 1, 2, 3$, μ is the shear modulus, v is Poisson's ratio, the points P and Q have Cartesian coordinates (x_1, x_2, x_3) and (x'_1, x'_2, x'_3), respectively, δ_{ij} is the Kronecker delta,

$$R_0 = \{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2\}^{1/2} \tag{2.1}$$

is the distance between P and Q, and

$$R_i = \frac{\partial R_0}{\partial x_i} = \frac{x_i - x'_i}{R_0};$$

see, for example, Cruse (1969).

Consider a bounded cavity in an otherwise unbounded elastic solid; denote the surface of the cavity by S. Assume that the displacement components $u_i = o(1)$ and the stress components $\tau_{ij} = o(\mathscr{R}^{-1})$ as $\mathscr{R} \to \infty$, where $\mathscr{R}$ is distance from some fixed point in the vicinity of the cavity. Then, a familiar calculation yields the integral representation

$$u_i(P) = \int_S \{u_j(q)\, T_{ij}(P, q) - t_j(q)\, G_{ij}(P, q)\}\, \mathrm{d}s_q, \quad i = 1, 2, 3, \tag{2.2}$$

where summation over repeated subscripts is implied, $t_j(q) = n_k^q \tau_{jk}$, $\mathbf{n}(q) = (n_1^q, n_2^q, n_3^q)$ is the unit normal at $q \in S$ pointing into the solid,

$$T_{ij} = [8\pi(1 - v)R_0^2]^{-1}\{(1 - 2v)[R_j n_i^q - R_i n_j^q] - R_n^q[(1 - 2v)\delta_{ij} + 3R_i R_j]\}$$

and

$$R_n^q \equiv \frac{\partial R_0}{\partial n_q} = n_i^q \frac{\partial R_0}{\partial x'_i} = -n_i^q R_i.$$

Formula (2.2) is the basis for boundary-element methods in three-dimensional elastostatics.

Next, let the cavity S shrink to a crack Ω. Denote the two sides of the crack by Ω^+ and Ω^-, and let q^+ and q^- be the corresponding points on Ω^+ and Ω^-, respectively. Define $\mathbf{n}(q) = \mathbf{n}(q^+)$ so that $\mathbf{n}(q^-) = -\mathbf{n}(q)$. Also, define the COD by

$$[u_i(q)] = u_i(q^+) - u_i(q^-).$$

Assuming that the imposed stresses are continuous across Ω, we find that Eq. (2.2) reduces to

$$u_i(P) = \int_\Omega [u_j(q)]\, T_{ij}(P, q)\, \mathrm{d}s_q, \quad i = 1, 2, 3. \tag{2.3}$$

an integral representation for the displacement components at any point P in the solid.

We can use Hooke's law to compute the tractions $t_i(p)$ on Ω corresponding to u_i. The result is

$$\frac{1}{\mu} t_i(p) = \fint_\Omega [u_j(q)]\, S_{ij}(p, q)\, \mathrm{d}s_q, \quad i = 1, 2, 3, \ p \in \Omega. \tag{2.4}$$

In this formula, the cross on the integral sign means that the integral is to be interpreted as a Hadamard finite-part integral; for references, see, for example, Stephan (1986),

Martin and Rizzo (1989) or Martin et al. (1998). The kernel S_{ij} is given by

$$2\pi S_{ij} = R_0^{-3}\{A\mathcal{N}\delta_{ij} + An_i^q n_j^p + (1 - 3A)n_i^p n_j^q + \tfrac{3}{2}(1 - 2A)\mathcal{N}R_iR_j$$

$$- 3R_n^q[An_i^p R_j + \tfrac{1}{2}(1 - 2A)R_i n_j^p] + 3R_n^p[\tfrac{1}{2}(1 - 2A)n_i^q R_j + AR_i n_j^q]$$

$$+ 3R_n^p R_n^q[5(1 - A)R_iR_j - \tfrac{1}{2}(1 - 2A)\delta_{ij}]\}, \tag{2.5}$$

where $\mathcal{N} = n_i^p n_i^q$,

$$A = \frac{1 - 2v}{2(1 - v)} \quad \text{and} \quad R_n^p \equiv \frac{\partial R_0}{\partial n_p} = n_i^p \frac{\partial R_0}{\partial x_i} = n_i^p R_i.$$

(One can check that $S_{ij} = \mu^{-1} n_k^p S_{jik}$ where S_{kij} is given by Eq. (13) in Cruse, 1969.) All the terms inside the curly brackets are bounded as $p \to q$. Thus, Eq. (2.5) exhibits the expected non-integrable R_0^{-3} singularity that is typical of two-dimensional finite-part integrals. Note that

$$S_{ij}(p, q) = S_{ji}(q, p).$$

Once the traction components $t_i(p)$ are prescribed on Ω, Eq. (2.4) becomes a hypersingular boundary integral equation for the components of the COD, $[u_j]$. It is to be solved subject to the natural edge conditions,

$$[u_i(q)] = 0, \quad i = 1, 2, 3, \ q \in \partial\Omega. \tag{2.6}$$

3. Projection

The integral equation (2.4) is exact and it holds on the surface Ω. It is more convenient to write (2.4) on a fixed reference surface D. Thus, we suppose for the moment that the surface Ω is given by

$$\Omega : x_3 \equiv az = aF(x, y), \quad (x, y) \in D, \tag{3.1}$$

where D is the *unit disc* in the xy-plane, centred at the origin, a is a length-scale, and the (dimensionless) function F gives the shape of Ω. Explicitly, the integration point q and the field point p are specified by

$$q : (x_1', x_2', x_3') = (ax, ay, aF(x, y)), \quad (x, y) \in D$$

and

$$p : (x_1, x_2, x_3) = (ax_0, ay_0, aF(x_0, y_0)), \quad (x_0, y_0) \in D,$$

respectively. For the unit normal $\mathbf{n}(q)$, we have

$$\mathbf{n}(q) = \mathbf{N}(q)/N(q),$$

where

$$\mathbf{N}(q) = (-F_1, -F_2, 1) \quad \text{and} \quad N(q) = |\mathbf{N}(q)| = \sqrt{1 + F_1^2 + F_2^2}.$$

Here,

$$F_1 = \partial F/\partial x \quad \text{and} \quad F_2 = \partial F/\partial y \quad \text{evaluated at } (x, y). \tag{3.2}$$

There is a similar expression for $\mathbf{n}(p)$; for example, $\mathbf{N}(p) = (-F_1^0, -F_2^0, 1)$ where $F_1^0 = \partial F/\partial x$ and $F_2^0 = \partial F/\partial y$ evaluated at (x_0, y_0).

Let

$$R = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}$$

and

$$\Lambda = \{F(x, y) - F(x_0, y_0)\}/R,$$

whence $R_0 = aR\sqrt{1 + \Lambda^2}$. Also, define the angle Θ by

$$x - x_0 = R \cos \Theta \quad \text{and} \quad y - y_0 = R \sin \Theta,$$

leading to the following expressions:

$$R_1 = -\Gamma \cos \Theta, \quad R_2 = -\Gamma \sin \Theta, \quad R_3 = -\Gamma \Lambda,$$

$$N(q) R_n^q = -\Gamma(F_1 \cos \Theta + F_2 \sin \Theta - \Lambda)$$

and

$$N(p) R_n^p = \Gamma(F_1^0 \cos \Theta + F_2^0 \sin \Theta - \Lambda),$$

where $\Gamma = (1 + \Lambda^2)^{-1/2}$. These allow us to write the kernel S_{ij} in terms of (x, y), (x_0, y_0) and F. Thus, we define a new kernel matrix $\tilde{S}_{ij}(x_0, y_0; x, y)$ by

$$2\pi a^3 R^3 N(p) N(q) S_{ij} = \tilde{S}_{ij}(x_0, y_0; x, y), \tag{3.3}$$

it is given explicitly in Appendix A. Finally, if we multiply the integral equation (2.4) by $N(p)$, and note that $ds_q = a^2 N(q) dA$, where $dA = dx\, dy$, we obtain

$$\frac{1}{2\pi} \oint_D \tilde{u}_j(x, y) \tilde{S}_{ij}(x_0, y_0; x, y) \frac{dA}{R^3} = \tilde{t}_i(x_0, y_0), \quad i = 1, 2, 3, \ (x_0, y_0) \in D, \tag{3.4}$$

where

$$\tilde{u}_i(x, y) = a^{-1}[u_i(q)] \quad \text{and} \quad \tilde{t}_i(x_0, y_0) = \mu^{-1} N(p) t_i(p). \tag{3.5}$$

Eq. (3.4) is our basic hypersingular integral equation for a crack Ω, defined by Eq. (3.1). In fact, it is a system of three coupled scalar integral equations for $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{u}_3$. This system is to be solved subject to the edge conditions (2.6), which become, after projection,

$$\tilde{u}_i(x, y) = 0 \text{ when } x^2 + y^2 = 1, \quad i = 1, 2, 3.$$

The vector integral equation (3.4) is exact, and it could be solved numerically. Here, we concentrate on analytical results for wrinkled cracks, where Ω is approximately a penny-shaped crack.

In fact, our goal is to calculate the stress-intensity factors at a typical point q on the crack edge, $\partial\Omega$. To do this, we will need an orthonormal triad of vectors at q. Let θ parametrise $\partial\Omega$, so that q has position vector

$$\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, aF(\cos\theta, \sin\theta)).$$

Thus, a unit tangent to $\partial\Omega$ is $\mathbf{t}(\theta) = \mathbf{r}'(\theta)/|\mathbf{r}'(\theta)|$, where

$$a^{-1}\mathbf{r}'(\theta) = (-\sin\theta, \cos\theta, F_2\cos\theta - F_1\sin\theta),$$

whereas a unit normal vector is $\mathbf{n}(\theta) = \mathbf{N}(\theta)/|\mathbf{N}(\theta)|$ with $\mathbf{N}(\theta) = (-F_1, -F_2, 1)$. For the third unit vector, we take $\mathbf{s}(\theta) = \mathbf{S}(\theta)/|\mathbf{S}(\theta)|$, where

$$\mathbf{S}(\theta) = a^{-1}\mathbf{r}' \times \mathbf{N}$$

$$= ((1 + F_2^2)\cos\theta - F_1F_2\sin\theta, (1 + F_1^2)\sin\theta - F_1F_2\cos\theta, F_1\cos\theta + F_2\sin\theta).$$

In all of these expressions, F_1 and F_2 are evaluated at $(x, y) = (\cos\theta, \sin\theta)$.

4. The flat penny-shaped crack

For a penny-shaped crack, we have $F(x, y) = c$, a constant. Hence, $\Gamma = 1$ and all of F_1, F_1^0, F_2, F_2^0, Λ, Φ, Φ^0 and Ψ are zero. It follows from the expressions given in Appendix A that

$$\tilde{S}_{ij} = (1 - v)^{-1}S_{ij}^0,$$

where

$$S_{11}^0 = \tfrac{1}{4}(2 - v + 3v\cos 2\Theta),$$

$$S_{22}^0 = \tfrac{1}{4}(2 - v - 3v\cos 2\Theta),$$

$$S_{12}^0 = S_{21}^0 = \tfrac{3}{2}v\sin 2\Theta, \quad S_{33}^0 = \tfrac{1}{2}$$

and $S_{13}^0 = S_{23}^0 = S_{31}^0 = S_{32}^0 = 0$. Thus, we obtain the following integral equations:

$$\frac{1}{8\pi}\fint_D \{(2 - v + 3v\cos 2\Theta)\,\tilde{u}_1 + 3v\,\tilde{u}_2\sin 2\Theta\}\,\frac{\mathrm{d}A}{R^3} = (1 - v)\tilde{t}_1(x_0, y_0), \tag{4.1}$$

$$\frac{1}{8\pi}\fint_D \{3v\,\tilde{u}_1\sin 2\Theta + (2 - v - 3v\cos 2\Theta)\,\tilde{u}_2\}\,\frac{\mathrm{d}A}{R^3} = (1 - v)\tilde{t}_2(x_0, y_0), \tag{4.2}$$

$$\frac{1}{4\pi}\fint_D \tilde{u}_3\,\frac{\mathrm{d}A}{R^3} = (1 - v)\tilde{t}_3(x_0, y_0) \tag{4.3}$$

for $(x_0, y_0) \in D$. These integral equations are well known; see, for example, Martin (1995, 1996). They can be solved explicitly. (Note that the same equations hold when D is any planar region.) Observe that the 3×3 system (3.4) has decoupled into a 2×2 system for $\tilde{u}_1$ and $\tilde{u}_2$ (shear loading), (4.1) and (4.2), and a single scalar equation, (4.3), for $\tilde{u}_3$ (normal loading).

For a simple example, suppose that $\tilde{t}_1$, $\tilde{t}_2$ and $\tilde{t}_3$ are all constants. Then, the exact solution of Eqs. (4.1)–(4.3) is given by

$$\tilde{u}_\alpha = \frac{-8(1-v)}{\pi(2-v)}\tilde{t}_\alpha\sqrt{1-r^2} \quad\text{and}\quad \tilde{u}_3 = \frac{-4(1-v)}{\pi}\tilde{t}_3\sqrt{1-r^2}, \tag{4.4}$$

where $\alpha = 1, 2$, and $r = \sqrt{x^2+y^2}$.

For a second example (which we will require in Section 7), suppose that

$$(1-v)\tilde{t}_i(x_0, y_0) = A_i x_0 + B_i y_0, \quad i = 1, 2, 3.$$

For this linear loading, we have (Martin, 1995, 1996)

$$\tilde{u}_1(x, y) = \frac{4\sqrt{1-r^2}}{3\pi(2-v)}\left\{x[-(4-v)A_1 + vB_2] + \frac{y}{1-v}[vA_2 - (4-3v)B_1]\right\}, \tag{4.5}$$

$$\tilde{u}_2(x, y) = \frac{4\sqrt{1-r^2}}{3\pi(2-v)}\left\{\frac{x}{1-v}[-(4-3v)A_2 + vB_1] + y[vA_1 - (4-v)B_2]\right\} \tag{4.6}$$

and

$$\tilde{u}_3(x, y) = (4/\pi)(A_3 x + B_3 y)\sqrt{1-r^2}. \tag{4.7}$$

5. Wrinkled penny-shaped cracks

Suppose that

$$F(x, y) = \varepsilon f(x, y),$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda \quad\text{with } \lambda = \{f(x, y) - f(x_0, y_0)\}/R,$$

we find from the expressions given in Appendix A that

$$\tilde{S}_{ij} = (1-v)^{-1}\{S_{ij}^0 + \varepsilon S_{ij}^1 + O(\varepsilon^2)\}$$

as $\varepsilon \to 0$, where

$$S_{13}^1 = \tfrac{3}{2}\lambda\cos\Theta - \tfrac{1}{4}(2-v+3v\cos 2\Theta)f_1 - \tfrac{3}{4}vf_2\sin 2\Theta$$
$$- \tfrac{1}{4}\{1+2v+3(1-2v)\cos 2\Theta\}f_1^0 - \tfrac{3}{4}(1-2v)f_2^0\sin 2\Theta, \tag{5.1}$$

$$S_{23}^1 = \tfrac{3}{2}\lambda\sin\Theta - \tfrac{3}{4}vf_1\sin 2\Theta - \tfrac{1}{4}(2-v-3v\cos 2\Theta)f_2$$
$$- \tfrac{3}{4}(1-2v)f_1^0\sin 2\Theta - \tfrac{1}{4}\{1+2v-3(1-2v)\cos 2\Theta\}f_2^0, \tag{5.2}$$

$$S_{31}^1 = \tfrac{3}{2}\lambda\cos\Theta - \tfrac{1}{4}\{1+2v+3(1-2v)\cos 2\Theta\}f_1 - \tfrac{3}{4}(1-2v)f_2\sin 2\Theta$$
$$- \tfrac{1}{4}(2-v+3v\cos 2\Theta)f_1^0 - \tfrac{3}{4}vf_2^0\sin 2\Theta, \tag{5.3}$$

$$S_{32}^1 = \tfrac{3}{2}\lambda \sin\Theta - \tfrac{3}{4}(1-2v)f_1 \sin 2\Theta - \tfrac{1}{4}\{1+2v-3(1-2v)\cos 2\Theta\}f_2$$

$$- \tfrac{3}{4}vf_1^0 \sin 2\Theta - \tfrac{1}{4}(2-v-3v\cos 2\Theta)f_2^0 \tag{5.4}$$

and $S_{11}^1 = S_{12}^1 = S_{21}^1 = S_{22}^1 = S_{33}^1 = 0$, where f_j, f_j^0 are defined similar to F_j, F_j^0 (see Eq. (3.2)).

We expand $\tilde{t}_i$ similarly. Suppose that the prescribed tractions are defined in terms of a stress field $\tau_{ij}(x_1, x_2, x_3)$, so that

$$\tilde{t}_i(x_0, y_0) = \mu^{-1}(\tau_{i3} - F_1^0 \tau_{i1} - F_2^0 \tau_{i2}),$$

evaluated at $(x_1, x_2, x_3) = (ax_0, ay_0, aF(x_0, y_0))$. This is exact. From Taylor's theorem, we have

$$\tau_{ij}(ax_0, ay_0, \varepsilon a f_0) = \tau_{ij}^{(0)}(x_0, y_0) + \varepsilon\tau_{ij}^{(1)}(x_0, y_0) + \cdots,$$

where $f_0 \equiv f(x_0, y_0)$ and

$$\tau_{ij}^{(n)}(x_0, y_0) = \frac{(af_0)^n}{n!}\left[\frac{\partial^n}{\partial z^n}\tau_{ij}(ax_0, ay_0, z)\right]\Bigg|_{z=0}.$$

Hence, we deduce that

$$\tilde{t}_i(x_0, y_0) = t_i^0 + \varepsilon t_i^1 + \cdots, \tag{5.5}$$

where

$$t_i^0 = \mu^{-1}\tau_{i3}^{(0)} \quad\text{and}\quad t_i^1 = \mu^{-1}(\tau_{i3}^{(1)} - f_1^0 \tau_{i1}^{(0)} - f_2^0 \tau_{i2}^{(0)}). \tag{5.6}$$

We can easily deduce subsequent terms in Eq. (5.5), but if the given stress field is *constant*, then we have, exactly,

$$\tilde{t}_i(x_0, y_0) = \mu^{-1}(\tau_{i3} - \varepsilon f_1^0 \tau_{i1} - \varepsilon f_2^0 \tau_{i2}). \tag{5.7}$$

Next, we expand the crack-opening displacement $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, writing

$$\tilde{u}_i = u_i^0 + \varepsilon u_i^1 + \cdots. \tag{5.8}$$

Substituting into Eq. (3.4) then gives integral equations for u_i^m. Each equation is of the form

$$\frac{1}{2\pi}\oint_D u_j^m(x, y)\,S_{ij}^0(x_0, y_0; x, y)\frac{\mathrm{d}A}{R^3} = b_i^m(x_0, y_0), \quad i = 1, 2, 3, \ (x_0, y_0) \in D, \tag{5.9}$$

where the forcing function b_i^m is known. This is simply the system of integral equations for a flat penny-shaped crack; given b_i^m, we can solve it exactly. At leading order $(m = 0)$, we obtain

$$b_i^0 = (1-v)t_i^0,$$

so that u_i^0 does not depend on the crack's shape. Having found u_i^0, the first-order correction $(m = 1)$ is obtained using

$$b_i^1(x_0, y_0) = (1-v)t_i^1(x_0, y_0) - \frac{1}{2\pi}\oint_D u_j^0(x, y)S_{ij}^1(x_0, y_0; x, y)\frac{\mathrm{d}A}{R^3}, \tag{5.10}$$

explicitly, we have

$$b_\alpha^1(x_0, y_0) = (1 - v)t_\alpha^1(x_0, y_0) - \frac{1}{2\pi}\oint_D u_3^0(x, y)S_{\alpha3}^1(x_0, y_0; x, y)\frac{\mathrm{d}A}{R^3} \tag{5.11}$$

for $\alpha = 1, 2$, and

$$b_3^1(x_0, y_0) = (1 - v)t_3^1(x_0, y_0) - \frac{1}{2\pi}\oint_D (u_1^0 S_{31}^1 + u_2^0 S_{32}^1)\frac{\mathrm{d}A}{R^3}. \tag{5.12}$$

We can see that further terms in Eq. (5.8) can be found, in principle, by solving the flat-crack integral equation with different forcing functions. Here, we content ourselves by calculating the first-order corrections for particular geometries, under uniform loading. In fact, as the given stress field is assumed constant, we can immediately use Eq. (4.4) to give

$$u_\alpha^0(x, y) = -B\tau_{\alpha3}\sqrt{1 - r^2} \quad \text{and} \quad u_3^0(x, y) = -C\tau_{33}\sqrt{1 - r^2}, \tag{5.13}$$

where $\alpha = 1, 2$, $r = \sqrt{x^2 + y^2}$,

$$B = \frac{8(1 - v)}{\pi\mu(2 - v)} \quad \text{and} \quad C = \frac{4(1 - v)}{\pi\mu}. \tag{5.14}$$

The calculation of b_i^1 and thence u_i^1 will depend on the shape of Ω.

For calculations of the stress-intensity factors, we require approximations of the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{s}$, as defined at the end of Section 3. We find that

$$\mathbf{n}(\theta) = (-\varepsilon f_1, -\varepsilon f_2, 1),$$

$$\mathbf{s}(\theta) = (\cos\theta, \sin\theta, \varepsilon(f_1\cos\theta + f_2\sin\theta)),$$

$$\mathbf{t}(\theta) = (-\sin\theta, \cos\theta, \varepsilon(f_2\cos\theta - f_1\sin\theta)),$$

correct to first order in ε. (Note that these approximations satisfy $\mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot\mathbf{s} = 0$ exactly, whereas $\mathbf{s}\cdot\mathbf{t} = O(\varepsilon^2)$ as $\varepsilon \to 0$.) Hence, the components of the COD in each of the three directions are given as follows:

$$u_n \equiv \tilde{\mathbf{u}}\cdot\mathbf{n} \simeq u_3^0 + \varepsilon(u_3^1 - u_1^0 f_1 - u_2^0 f_2), \tag{5.15}$$

$$u_s \equiv \tilde{\mathbf{u}}\cdot\mathbf{s} \simeq u_1^0\cos\theta + u_2^0\sin\theta + \varepsilon\{(u_1^1 + u_3^0 f_1)\cos\theta + (u_2^1 + u_3^0 f_2)\sin\theta\}, \tag{5.16}$$

$$u_t \equiv \tilde{\mathbf{u}}\cdot\mathbf{t} \simeq u_2^0\cos\theta - u_1^0\sin\theta + \varepsilon\{(u_2^1 + u_3^0 f_2)\cos\theta - (u_1^1 + u_3^0 f_1)\sin\theta\}. \tag{5.17}$$

From these, we can extract the stress-intensity factors K_α, defined by

$$\tau_{n\alpha} \sim K_\alpha/\sqrt{2\pi\rho'} \quad \text{as } \rho' \to 0, \ \alpha = n, s, t,$$

where ρ' is distance from the crack edge $\partial\Omega$. For, it is known (see, for example, Rice, 1989, p. 32) that the stresses ahead of $\partial\Omega$ are related to the COD behind $\partial\Omega$ by

$$u_\alpha \sim (4/\mu)(1 - v)K_\alpha\sqrt{\tfrac{1}{2}\rho/\pi}, \quad \alpha = n, s$$

and

$$u_t \sim (4/\mu)K_t\sqrt{\tfrac{1}{2}\rho/\pi}$$

as $\rho \to 0$, where ρ is the distance from $\partial \Omega$. Then, we may easily calculate the stress-intensity factors from Eqs. (5.15)–(5.17).

6. Example 1: inclined elliptical crack

Suppose that Ω is a flat elliptical crack, lying in the plane $z = x \tan \gamma$. Let X and Y be Cartesian coordinates on this plane, so that

$$X = x \cos \gamma + z \sin \gamma, \quad Y = y \quad \text{and} \quad Z = z \cos \gamma - x \sin \gamma,$$

where Z is a coordinate perpendicular to the plane. Then, the ellipse Ω with $\partial \Omega$ given by

$$X^2 \cos^2 \gamma + Y^2 = 1$$

can be specified by

$$z = F(x, y) = x \tan \gamma, \quad (x, y) \in D.$$

For small inclinations of the ellipse to the plane $z = 0$, set $\varepsilon = \tan \gamma$ and $f(x, y) = x$. Thus, $f_1 = f_1^0 = 1$, $f_2 = f_2^0 = 0$ and $\lambda = \cos \Theta$, whence Eqs. (5.1)–(5.4) give

$$S_{13}^1 = S_{31}^1 = \tfrac{1}{4} v(-1 + 3 \cos 2\Theta) \quad \text{and} \quad S_{23}^1 = S_{32}^1 = \tfrac{3}{4} v \sin 2\Theta.$$

Next, we evaluate b_i^1, as given by Eqs. (5.11) and (5.12). From Eq. (5.7), we have $t_i^1 = -\mu^{-1} \tau_{i1}$ whereas u_i^0 is given by Eq. (5.13). Let us define operators $\mathscr{H}_0$, $\mathscr{H}_c$ and $\mathscr{H}_s$ by

$$\mathscr{H}_0 u = \frac{1}{4\pi} \oint_D \sqrt{1 - r^2} \, \frac{u}{R^3} \, \mathrm{d}A,$$

$$\mathscr{H}_c u = \frac{1}{4\pi} \oint_D \sqrt{1 - r^2} \, \frac{u \cos 2\Theta}{R^3} \, \mathrm{d}A,$$

$$\mathscr{H}_s u = \frac{1}{4\pi} \oint_D \sqrt{1 - r^2} \, \frac{u \sin 2\Theta}{R^3} \, \mathrm{d}A.$$

Then, we find that

$$b_1^1 = -\mu^{-1}(1 - v)\tau_{11} + \tfrac{1}{2} v C \tau_{33}(-\mathscr{H}_0 1 + 3\mathscr{H}_c 1)$$

$$= -\mu^{-1}(1 - v)(\tau_{11} - \tfrac{1}{2} v \tau_{33}),$$

$$b_2^1 = -\mu^{-1}(1 - v)\tau_{12} + \tfrac{3}{2} v C \tau_{33} \mathscr{H}_s 1 = -\mu^{-1}(1 - v)\tau_{12},$$

$$b_3^1 = -\mu^{-1}(1 - v)\tau_{13} + \tfrac{1}{2} v B\{\tau_{13}(-\mathscr{H}_0 1 + 3\mathscr{H}_c 1) + 3\tau_{23} \mathscr{H}_s 1\}$$

$$= -2(1 - v)^2[\mu(2 - v)]^{-1} \tau_{13},$$

where we have used (Martin, 1995, 1996)

$$\mathscr{H}_0 1 = -\tfrac{1}{4}\pi \quad \text{and} \quad \mathscr{H}_c 1 = \mathscr{H}_s 1 = 0. \tag{6.1}$$

Thus, b_i^1 is constant, $i = 1, 2, 3$, and so we can solve Eq. (5.9) for u_i^1, using Eq. (4.4). When the result is combined with the leading-order solution, (5.13), we obtain

$$\tilde{u}_1(x, y) = - B\{\tau_{13} - \varepsilon(\tau_{11} - \tfrac{1}{2}v\tau_{33})\}\sqrt{1 - r^2},$$

$$\tilde{u}_2(x, y) = - B(\tau_{23} - \varepsilon\tau_{12})\sqrt{1 - r^2},$$

$$\tilde{u}_3(x, y) = - \{C\tau_{33} - \varepsilon B(1 - v)\tau_{13}\}\sqrt{1 - r^2}.$$

These are our approximations for the components of the COD; they are correct to first order in ε. From them and Eqs. (5.15)–(5.17), we obtain

$$u_n(x, y) = - C(\tau_{33} - 2\varepsilon\tau_{13})\sqrt{1 - r^2}, \tag{6.2}$$

$$u_s(x, y) = - B\{[\tau_{13} + \varepsilon(\tau_{33} - \tau_{11})]\cos\theta + [\tau_{23} - \varepsilon\tau_{12}]\sin\theta\}\sqrt{1 - r^2}, \tag{6.3}$$

$$u_t(x, y) = B\{[\tau_{13} + \varepsilon(\tau_{33} - \tau_{11})]\sin\theta - [\tau_{23} - \varepsilon\tau_{12}]\cos\theta\}\sqrt{1 - r^2}, \tag{6.4}$$

correct to first order in ε. These can be used to calculate the stress-intensity factors at points on $\partial\Omega$ ($r = 1$) parametrized by θ. In particular, when $\theta = 0$, u_n, u_s and u_t agree with the known exact solutions for $[u_Z]$, $[u_X]$ and $[u_Y]$, respectively (see Appendix B).

7. Example 2: spherical-cap crack

Consider a crack in the shape of a spherical cap, given by

$$x_3 = c - \sqrt{c^2 - x_1^2 - x_2^2}, \quad x_1^2 + x_2^2 \le a^2,$$

where c is the radius of the sphere. The cap subtends a solid angle of $2\pi(1 - \cos\alpha)$ at the centre of the sphere, where $\sin\alpha = a/c$. In dimensionless variables, the cap is given by

$$z = F(x, y) = (c/a) - \sqrt{(c/a)^2 - x^2 - y^2}, \quad (x, y) \in D.$$

We shall consider a shallow spherical cap, given approximately by $z = \varepsilon f(x, y)$ with

$$f(x, y) = \tfrac{1}{2}(x^2 + y^2) = \tfrac{1}{2}r^2 \quad \text{and} \quad \varepsilon = a/c = \sin\alpha.$$

We have $f_1 = x$, $f_2 = y$ and $\lambda = \tfrac{1}{2}\{(x + x_0)\cos\Theta + (y + y_0)\sin\Theta\}$, whence

$$S_{13}^1 = (-1 + 3\cos 2\Theta)\mathscr{P}(x; x_0) + 3\sin 2\Theta\mathscr{P}(y; y_0),$$

$$S_{23}^1 = 3\sin 2\Theta\mathscr{P}(x; x_0) - (1 + 3\cos 2\Theta)\mathscr{P}(y; y_0),$$

$$S_{31}^1 = (-1 + 3\cos 2\Theta)\mathscr{P}(x_0; x) + 3\sin 2\Theta\mathscr{P}(y_0; y),$$

$$S_{32}^1 = 3\sin 2\Theta\mathscr{P}(x_0; x) - (1 + 3\cos 2\Theta)\mathscr{P}(y_0; y),$$

where

$$\mathscr{P}(X; X_0) = \tfrac{1}{8}\{(1 - 2v)X - (1 - 4v)X_0\}.$$

Next, we evaluate b_i^1, assuming constant applied stresses. From Eq. (5.7), we have

$$t_i^1(x_0, y_0) = -\mu^{-1}(x_0 \tau_{i1} + y_0 \tau_{i2}),$$

whereas u_i^0 is again given by (5.13). Making use of the operators $\mathscr{H}_0$, $\mathscr{H}_c$ and $\mathscr{H}_s$, defined in Section 6, we find that

$$b_1^1(x_0, y_0) = (1 - v)t_1^1 + 2C\tau_{33}\{(-\mathscr{H}_0 + 3\mathscr{H}_c)\mathscr{P}(x; x_0) + 3\mathscr{H}_s \mathscr{P}(y; y_0)\}$$

$$= -\mu^{-1}(1 - v)\{x_0[\tau_{11} + \tfrac{1}{4}(1 - 4v)\tau_{33}] + y_0 \tau_{12}\},$$

where we have used Eq. (6.1),

$$\mathscr{H}_0 x = -\tfrac{3}{8}\pi x_0 \quad \text{and} \quad \mathscr{H}_c x = \mathscr{H}_s y = -\tfrac{1}{16}\pi x_0.$$

Similarly, we obtain

$$b_2^1(x_0, y_0) = -\mu^{-1}(1 - v)\{x_0 \tau_{12} + y_0[\tau_{22} + \tfrac{1}{4}(1 - 4v)\tau_{33}]\}$$

and

$$b_3^1(x_0, y_0) = -\tfrac{3}{2}(1 - v)[\mu(2 - v)]^{-1}(x_0 \tau_{13} + y_0 \tau_{23}),$$

where we have also used

$$\mathscr{H}_0 y = -\tfrac{3}{8}\pi y_0 \quad \text{and} \quad \mathscr{H}_c y = -\mathscr{H}_s x = \tfrac{1}{16}\pi y_0.$$

Thus, $b_i^1(x_0, y_0)$ is linear in x_0 and y_0, for $i = 1, 2, 3$.

To find u_i^1, we solve Eq. (5.9), using Eqs. (4.5)–(4.7). We obtain

$$u_1^1(x, y) = -\tfrac{1}{6}B\{x[-(4 - v)\tau_{11} + v\tau_{22} - \tfrac{1}{2}(1 - 4v)(2 - v)\tau_{33}] - 4y\tau_{12}\}\sqrt{1 - r^2},$$

$$u_2^1(x, y) = -\tfrac{1}{6}B\{-4x\tau_{12} + y[v\tau_{11} - (4 - v)\tau_{22} - \tfrac{1}{2}(1 - 4v)(2 - v)\tau_{33}]\}\sqrt{1 - r^2}$$

and

$$u_3^1(x, y) = -\tfrac{3}{4}B(x\tau_{13} + y\tau_{23})\sqrt{1 - r^2},$$

where B is defined by Eq. (5.14)$_1$. When these results are combined with Eq. (5.13), we obtain our approximation to the COD, in Cartesian coordinates. We can also calculate the components of the COD with respect to the normal-tangential coordinates, u_n, u_s and u_t, as given by Eqs. (5.15)–(5.17); we obtain

$$u_n = \frac{-4(1 - v)}{\pi\mu}\sqrt{1 - r^2}\left\{\tau_{33} - \frac{\varepsilon}{2(2 - v)}(x\tau_{13} + y\tau_{23})\right\},$$

$$u_s = -B\sqrt{1 - r^2}\{\tau_{13}\cos\theta + \tau_{23}\sin\theta$$
$$- \tfrac{1}{6}\varepsilon r[(2 - v)\{\tau_{11} + \tau_{22} - \tfrac{1}{2}(5 + 4v)\tau_{33}\} + 2(\tau_{11} - \tau_{22})\cos 2\theta + 4\tau_{12}\sin 2\theta]\},$$

$$u_t = -B\sqrt{1 - r^2}\{\tau_{23}\cos\theta - \tau_{13}\sin\theta + \tfrac{1}{3}\varepsilon r[(\tau_{11} - \tau_{22})\sin 2\theta - 2\tau_{12}\cos 2\theta]\}.$$

As a special case, consider the axisymmetric problem of uniaxial tension at infinity, so that the only non-zero stress component in the prescribed loading is

$$\tau_{33} = -p_0,$$

where p_0 is a constant. Then, we obtain $u_t \equiv 0$, $u_n = \eta \sqrt{1 - r^2} + O(\varepsilon^2)$ and

$$u_s = -\tfrac{1}{6}\eta\varepsilon r(5 + 4v)\sqrt{1 - r^2} + O(\varepsilon^2),$$

where $\eta = -4(1 - v)p_0/(\pi\mu)$. This result agrees with the known exact result for a crack in the shape of a spherical cap (Martynenko and Ulitko, 1979; Martin, 2001).

Appendix A. The kernel after projection

The kernel S_{ij} is given by Eq. (2.5), and it is related to the projected kernel $\tilde{S}_{ij}$ by (3.3). Set

$$\Phi = F_1 \cos\Theta + F_2 \sin\Theta - \Lambda, \quad \Phi^0 = F_1^0 \cos\Theta + F_2^0 \sin\Theta - \Lambda,$$

$\Psi = F_1 F_1^0 + F_2 F_2^0$, $E_1 = \cos\Theta$ and $E_2 = \sin\Theta$. Then, $\tilde{S}_{ij}$ is given explicitly as follows:

$$\tilde{S}_{33} = \Gamma^3\{1 - A + A\Psi\} + \tfrac{3}{2}\Gamma^5\Lambda\{(1 - 2A)\Lambda(1 + \Psi) - \Phi - \Phi^0\}$$
$$- 3\Gamma^5\Phi\Phi^0\{5(1 - A)\Lambda^2\Gamma^2 - \tfrac{1}{2}(1 - 2A)\},$$

$$\tilde{S}_{\alpha\alpha} = \Gamma^3\{A + A\Psi + (1 - 2A)F_\alpha F_\alpha^0\}$$
$$+ \tfrac{3}{2}\Gamma^5 E_\alpha\{(1 - 2A)E_\alpha(1 + \Psi) + \Phi F_\alpha^0 + \Phi^0 F_\alpha\}$$
$$- 3\Gamma^5\Phi\Phi^0\{5(1 - A)E_\alpha^2\Gamma^2 - \tfrac{1}{2}(1 - 2A)\},$$

$$\tilde{S}_{\alpha\beta} = \tfrac{3}{4}(1 - 2A)\Gamma^5(1 + \Psi)\sin 2\Theta$$
$$+ 3\Gamma^5\Phi\{AF_\alpha^0 E_\beta + \tfrac{1}{2}(1 - 2A)F_\beta^0 E_\alpha\} + \Gamma^3\{AF_\alpha F_\beta^0 + (1 - 3A)F_\alpha^0 F_\beta\}$$
$$+ 3\Gamma^5\Phi^0\{\tfrac{1}{2}(1 - 2A)F_\alpha E_\beta + AF_\beta E_\alpha\} - \tfrac{15}{2}\Gamma^7(1 - A)\Phi\Phi^0\sin 2\Theta,$$

$$\tilde{S}_{\alpha 3} = -\Gamma^3\{AF_\alpha + (1 - 3A)F_\alpha^0\} + \Gamma^5 E_\alpha\{\tfrac{3}{2}(1 - 2A)(\Lambda - \Phi) - 3A\Phi^0\}$$
$$+ \Gamma^5\Lambda\{\tfrac{3}{2}(1 - 2A)(E_\alpha\Psi + \Phi^0 F_\alpha) + 3A\Phi F_\alpha^0\} - 15\Gamma^7(1 - A)\Lambda E_\alpha\Phi\Phi^0,$$

$$\tilde{S}_{3\alpha} = -\Gamma^3\{AF_\alpha^0 + (1 - 3A)F_\alpha\} + \Gamma^5 E_\alpha\{\tfrac{3}{2}(1 - 2A)(\Lambda - \Phi^0) - 3A\Phi\}$$
$$+ \Gamma^5\Lambda\{\tfrac{3}{2}(1 - 2A)(E_\alpha\Psi + \Phi F_\alpha^0) + 3A\Phi^0 F_\alpha\} - 15\Gamma^7(1 - A)\Lambda E_\alpha\Phi\Phi^0.$$

Appendix B. The flat elliptical crack

Consider the elliptical crack $(X/A_0)^2 + (Y/B_0)^2 \leq 1$ on the plane $Z = 0$, where $A_0 \geq B_0 > 0$. There are constant loads on the crack, given by

$$\tau_{XZ} = q_X, \quad \tau_{YZ} = q_Y \quad \text{and} \quad \tau_{ZZ} = p_0.$$

Then, the COD is given by

$$[u_X] = 2q_X\mu^{-1}(1-v)B_0(k^2/\Omega_1)\sqrt{1-\rho^2}, \tag{B.1}$$

$$[u_Y] = 2q_Y\mu^{-1}(1-v)B_0(k^2/\Omega_2)\sqrt{1-\rho^2}, \tag{B.2}$$

$$[u_Z] = -2p_0(\mu E)^{-1}(1-v)B_0\sqrt{1-\rho^2}, \tag{B.3}$$

where $\rho^2 = (X/A_0)^2 + (Y/B_0)^2$,

$$\Omega_1(k) = (v-k^2)E - vk'^2K,$$

$$\Omega_2(k) = vk'^2K - (k^2 + vk'^2)E,$$

$k' = B_0/A_0$, $k^2 = 1 - k'^2$ and the complete elliptic integrals are defined by

$$E(k) = \int_0^{\pi/2} (1-k^2\sin^2 t)^{1/2}\,dt \quad\text{and}\quad K(k) = \int_0^{\pi/2} (1-k^2\sin^2 t)^{-1/2}\,dt.$$

The results (B.1)–(B.3) are exact; they can be found in Martin (1986a, b), for example.

To compare with Example 1, we take $A_0 = \sec\gamma \sim 1 + \frac{1}{2}\varepsilon^2$ and $B_0 = 1$, whence $k \sim \varepsilon$, $\gamma \sim \varepsilon$ and $\rho \sim r$ as $\varepsilon \to 0$. Also, E and K have known power-series expansions; thus $E(k) \sim \frac{1}{2}\pi$ as $k \to 0$, whereas both k^2/Ω_1 and $k^2/\Omega_2 \sim 4/[\pi(v-2)]$ as $k \to 0$. Hence

$$[u_X] \sim -Bq_X\sqrt{1-r^2}, \quad [u_Y] \sim -Bq_Y\sqrt{1-r^2} \quad\text{and}\quad [u_Z] \sim -Cp_0\sqrt{1-r^2}, \tag{B.4}$$

as $\varepsilon \to 0$, where the constants B and C are defined by Eq. (5.14).

Now, we have to calculate the constants q_X, q_Y and p_0 in terms of the applied stresses τ_{ij}, which are defined in terms of the Cartesian coordinates $Ox_1x_2x_3$. The tractions on the crack faces are given by

$$n_j\tau_{ij} = -\tau_{i1}\sin\gamma + \tau_{i3}\cos\gamma = T_i,$$

say. Thus

$$p_0 = T_in_i = \tau_{11}\sin^2\gamma - \tau_{13}\sin 2\gamma + \tau_{33}\cos^2\gamma$$

and, as $x_2 = Y$,

$$q_Y = T_2 = -\tau_{12}\sin\gamma + \tau_{23}\cos\gamma.$$

For q_X, we need $\hat{\mathbf{X}} = (\cos\gamma, 0, \sin\gamma)$, a unit vector in the X-direction, so that

$$q_X = T_i\hat{X}_i = \tau_{13}\cos 2\gamma + \tfrac{1}{2}(\tau_{33} - \tau_{11})\sin 2\gamma.$$

For small $\gamma \sim \varepsilon$, we obtain

$$q_X = \tau_{13} + \varepsilon(\tau_{33} - \tau_{11}), \quad q_Y = \tau_{23} - \varepsilon\tau_{12} \quad\text{and}\quad p_0 = \tau_{33} - 2\varepsilon\tau_{13},$$

correct to first order in ε. When these approximations are substituted into Eq. (B.4), we obtain results in agreement with Eqs. (6.2)–(6.4), evaluated at $\theta = 0$.

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