PERGAMON

Journal of the Mechanics and Physics of Solids
49 (2001) 1155–1178

JOURNAL OF THE
MECHANICS AND
PHYSICS OF SOLIDS

www.elsevier.com/locate/jmps

Waves in wood: free vibrations of a wooden pole

P.A. Martin[a,*], J.R. Berger[b]

[a] *Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA*

[b] *Division of Engineering, Colorado School of Mines, Golden, CO 80401-1887, USA*

Received 26 May 2000; accepted 1 July 2000

Abstract

Elastic waves in materials with cylindrical orthotropy are considered, this being a plausible model for a wooden pole. For time-harmonic motions, the problem is reduced to some coupled ordinary differential equations. Previously, these have been solved using the method of Frobenius (power-series expansions). Here, Neumann series (expansions in Bessel functions of various orders) are used, motivated by the known classical solutions for homogeneous isotropic solids. This is shown to give an effective and natural method for wave propagation in cylindrically orthotropic materials. As an example, the frequencies of free vibration of a wooden pole are computed. The problem itself arose from a study of ultrasonic devices as used in the detection of rotten regions inside wooden telegraph (utility) poles and trees; some background to these applications is given. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Wood; Cylindrical orthotropy; Wave propagation; Neumann series

1. Introduction

> Wood is a unique material. …It is multicomponent, hygroscopic, anisotropic, inhomogeneous, discontinuous, inelastic, fibrous, porous, biodegradable, and renewable. (Bodig and Jayne, 1982, p. vii)

Despite these complications, there is a need to have physical models for the behaviour of wood. One application (which motivated the present study) is to model the use of ultrasonic devices for the detection of rotten regions inside telegraph poles, so as to predict the strength of in-service poles. Typically, these devices use stress waves through the pole cross-section, which we can take to be circular. In Appendix A, we

* Corresponding author. Tel.: +1-303-273-3895; fax: +1-303-273-3875.
E-mail address: pamartin@mines.edu (P.A. Martin).

PII: S0022-5096(00)00068-5

give some background information on this application, reviewing the sparse literature and discussing the problem of how to model the rotten region.

Traditionally, wood is modelled as an orthotropic elastic solid (Bodig and Jayne, 1982, Chapter 3). Thus, at any point P in a wooden pole, we can identify three mutually orthogonal directions, namely longitudinal (along the grain), radial and tangential. These can be taken to specify three symmetry planes at P, and this leads to the orthotropic model. Elastic waves in orthotropic materials are discussed in detail by Musgrave (1970, Chapter 9).

However, the local orthotropic description ignores one obvious characteristic of trees and poles — the presence of *annual rings*. Thus, Bucur (1995, p. 3) wrote: 'At the annual ring level the structure is again one of a layered composite built up with two layers corresponding to the *earlywood* and *latewood*'. Typically, the density of earlywood is about half that of latewood (Bucur, 1995, p. 151). Bodig and Jayne (1982, Section 10.3.2) give more details. In particular, the transition between earlywood and latewood is observed to be abrupt in certain pines and Douglas fir. In some other trees, the transition is more gradual.

The effect of this layered structure on wave-speed measurement is discussed by Bucur (1995, Section 4.3.2.4): 'The opinions of different authors are rather divergent'. However, it is clear that the curvature of the rings should be taken into account if wave propagation over significant distances is to be modelled.

The above considerations suggest that a pole could be modelled as a composite material with concentric layers of two different materials, giving an axisymmetric structure. For simplicity, each layer could be assumed to be homogeneous (constant material parameters) with continuity conditions across the interfaces between layers.

For the wood itself, an alternative formulation of the theory suggests itself, in which the wood is assumed to be *cylindrically anisotropic*. Thus, Bodig and Jayne (1982, p. 21) wrote that one 'might model [trees or poles] as homogeneous with cylindrical anisotropy due to the layered growth ring structure', although they did not go further. By definition, cylindrical anisotropy means that the elastic stiffnesses are constants when referred to cylindrical polar coordinates. Properties of materials with cylindrical anisotropy have been studied by several authors. See, for example, Love (1927, Section 114), Lekhnitskii (1968) and Ambartsumyan (1970); in particular, Ambartsumyan (1970, Chapter 4) has examined the free vibrations of thin plates with cylindrical anisotropy. Further references will be given below.

In this paper, we consider time-harmonic waves in wood, which we model as an elastic solid with (circular) cylindrical orthotropy. We consider wave motions that are independent of the axial coordinate (along the grain) as we are interested in propagation through a wooden pole. We set up the equations of motion and reduce them to a set of three ordinary differential equations for the radial dependence of the three components of the displacement vector. Explicit solutions are recovered for the anti-plane component (perpendicular to a cross-sectional plane) and for the in-plane components, $u(r)$ and $v(r)$, when the motion is axisymmetric. However, the main concern of the paper is how to calculate $u(r)$ and $v(r)$ when the motion is *not* axisymmetric.

The two components, $u(r)$ and $v(r)$, satisfy a pair of coupled, second-order, linear homogeneous ordinary differential equations. Previous workers have solved these equations using the method of Frobenius (power series in r); we recall this method in Section 7. This method is very effective for small r and for static problems (for which it gives explicit closed-form solutions) but it is inappropriate for wave propagation over significant distances. Thus, motivated by the classical solutions for a homogeneous isotropic elastic solid, we generalize the method of Frobenius and use *Neumann series* (expansions in series of Bessel functions of varying orders). This new method is described in detail in Section 8. As a simple application, we have computed the first few frequencies of free vibration of a wooden pole. Further applications are mentioned in Section 10.

Returning to the NDE problem, let us end this introduction with some concluding remarks taken from an extensive review of the literature on the development and use of NDE devices for assessing wooden objects:

> Many questions remain unanswered regarding the effectiveness of stress wave NDE techniques to evaluate members in complicated structures. No published work documents how wave behavior is affected by the varied boundary conditions found in wood structures. In addition, little information has been published on the relationship between excitation system characteristics and wave behavior. Research efforts in these two areas would advance state-of-the-art inspection techniques considerably. (Ross and Pellerin, 1994, p. 9).

2. Governing equations

Let $x_1 \equiv x$, $x_2 \equiv y$ and $x_3 \equiv z$ be Cartesian coordinates. Then, the governing equations of motion for an anisotropic elastic material are

$$\frac{\partial}{\partial x_j}\left(\tilde{C}_{ijk\ell}\frac{\partial}{\partial x_k}\tilde{u}_\ell\right) = \rho\frac{\partial^2}{\partial t^2}\tilde{u}_i,$$

where $\tilde{\mathbf{u}}$ is the displacement vector, ρ is the mass density, t is the time, $\tilde{C}_{ijk\ell}$ are the elastic stiffnesses and the summation convention holds. As usual, we assume that

$$\tilde{C}_{ijk\ell} = \tilde{C}_{jik\ell} = \tilde{C}_{k\ell ij} = \tilde{C}_{ij\ell k}.$$

Introduce cylindrical polar coordinates (r,θ,z), where $x = r\cos\theta$ and $y = r\sin\theta$. If the $\tilde{C}_{ijk\ell}$ depend on θ only, the material is said to be *angularly inhomogeneous*. If $C_{ijk\ell}$ denote the elastic stiffnesses referred to (r,θ,z), we have

$$C_{ijk\ell}(\theta) = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{\ell s}\tilde{C}_{pqrs}(\theta),$$

where

$$\Omega_{ij}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are interested in special materials for which $C_{ijk\ell}(\theta)$ are constant, so that

$$C_{ijk\ell}(\theta) = C_{ijk\ell}(0) = \tilde{C}_{ijk\ell}(0)$$

such materials are said to be *cylindrically anisotropic*.

Following the formulation of Ting (1996a) for static problems, we write the equations of motion as

$$\frac{\partial}{\partial r}(r\mathbf{t}_r) + \frac{\partial}{\partial \theta}\mathbf{t}_\theta + \mathbf{K}\mathbf{t}_\theta + r\frac{\partial}{\partial z}\mathbf{t}_z = \rho r \frac{\partial^2}{\partial t^2}\tilde{\mathbf{u}},$$

where

$$\mathbf{t}_r = \begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \\ \tau_{rz} \end{pmatrix}, \quad \mathbf{t}_\theta = \begin{pmatrix} \tau_{\theta r} \\ \tau_{\theta\theta} \\ \tau_{\theta z} \end{pmatrix}, \quad \mathbf{t}_z = \begin{pmatrix} \tau_{zr} \\ \tau_{z\theta} \\ \tau_{zz} \end{pmatrix}, \quad \tilde{\mathbf{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}, \tag{2.1}$$

τ_{ij} are the stress components and

$$\mathbf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ting (1996a, p. 2399) gives expressions for the traction vectors $\mathbf{t}_i$ in terms of $\tilde{\mathbf{u}}$.

In this paper, we are concerned with motions in a cross-sectional plane, so we assume that $\tilde{\mathbf{u}}$ does not depend on z. Thus, we find that two-dimensional motions are governed by

$$\begin{aligned} &r\mathbf{Q}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\tilde{\mathbf{u}}\right) + r(\mathbf{R}+\mathbf{R}^{\mathrm{T}})\frac{\partial^2}{\partial r\partial\theta}\tilde{\mathbf{u}} + \mathbf{T}\frac{\partial^2}{\partial\theta^2}\tilde{\mathbf{u}} \\ &\quad + r(\mathbf{R}\mathbf{K}+\mathbf{K}\mathbf{R}^{\mathrm{T}})\frac{\partial}{\partial r}\tilde{\mathbf{u}} + (\mathbf{T}\mathbf{K}+\mathbf{K}\mathbf{T})\frac{\partial}{\partial\theta}\tilde{\mathbf{u}} + \mathbf{K}\mathbf{T}\mathbf{K}\tilde{\mathbf{u}} = \rho r^2\frac{\partial^2}{\partial t^2}\tilde{\mathbf{u}}, \end{aligned} \tag{2.2}$$

generalizing (Ting, 1996a, Eq. (3.1)) to dynamic problems. The 3×3 matrices occurring here are given by Ting (1996a) as

$$\mathbf{Q} = \begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{pmatrix},$$

where $\mathbf{R}^{\mathrm{T}}$ is the transpose of $\mathbf{R}$, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses (Ting, 1996b, Section 2.3).

We look for time-harmonic solutions in the form

$$\tilde{\mathbf{u}}(r,\theta,t) = \mathrm{Re}_i\{\mathbf{u}_m(r)\mathrm{e}^{\mathrm{j}m\theta}\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{2.3}$$

where i and j are two non-interacting complex units, m is an integer, ω is the radian frequency, and Re_i denotes the real part with respect to i. Use of $\mathrm{e}^{\mathrm{j}m\theta}$ rather than $\cos m\theta$ and $\sin m\theta$ allows us to retain the nice matrix notation in what follows. Thus, from Eq. (2.2), we find that $\mathbf{u}_m(r)$ solves

$$r^2\mathbf{Q}\mathbf{u}_m'' + r(\mathbf{Q}+\mathbf{R}\mathbf{K}_m+\mathbf{K}_m\mathbf{R}^{\mathrm{T}})\mathbf{u}_m' + (\rho\omega^2 r^2 + \mathbf{K}_m\mathbf{T}\mathbf{K}_m)\mathbf{u}_m = \mathbf{0}, \tag{2.4}$$

where

$$\mathbf{K}_m = \mathbf{K} + \mathrm{j}m\mathbf{I} = \begin{pmatrix} \mathrm{j}m & -1 & 0 \\ 1 & \mathrm{j}m & 0 \\ 0 & 0 & \mathrm{j}m \end{pmatrix}.$$

If $m=0$ and $\omega=0$, (2.4) reduces to Eq. (3.2) of Ting (1996a).

Setting $\mathbf{u}_m=(u_m, v_m, w_m)$, Eq. (2.4) gives three coupled ordinary differential equations for the three components of $\mathbf{u}_m$.

3. Cylindrically orthotropic materials

Wood is usually modelled as an orthotropic material. For such materials, the non-trivial stiffnesses are (Ting, 1996b, pp. 36 and 45)

$$C_{11}=C_{1111}, \quad C_{12}=C_{1122}, \quad C_{13}=C_{1133},$$
$$C_{22}=C_{2222}, \quad C_{23}=C_{2233}, \quad C_{33}=C_{3333},$$
$$C_{44}=C_{2323}, \quad C_{55}=C_{1313}, \quad C_{66}=C_{1212}.$$

Thus, the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ simplify to

$$\mathbf{Q} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} C_{66} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}.$$

This shows that motions independent of z in a cylindrically orthotropic medium depend on only six of the nine stiffnesses, namely C_{11}, C_{12}, C_{22}, C_{44}, C_{55} and C_{66}. Note that, for isotropic materials, $C_{11}=C_{22}=\lambda+2\mu$, $C_{12}=\lambda$ and $C_{44}=C_{55}=C_{66}=\mu$, where λ and μ are the Lamé moduli. On the other hand, wood is typically highly anisotropic. Numerical values for a particular wood, Scots pine, are given in Appendix B: these show, for example, that $C_{11} \simeq 2C_{22}$ and $C_{44} \simeq 10C_{66}$.

Simple calculations give

$$\mathbf{R}\mathbf{K}_m + \mathbf{K}_m\mathbf{R}^{\mathrm{T}} = \begin{pmatrix} 0 & \mathrm{j}m(C_{66}+C_{12}) & 0 \\ \mathrm{j}m(C_{66}+C_{12}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{K}_m\mathbf{T}\mathbf{K}_m = \begin{pmatrix} -m^2C_{66}-C_{22} & -\mathrm{j}m(C_{66}+C_{22}) & 0 \\ \mathrm{j}m(C_{66}+C_{22}) & -C_{66}-m^2C_{22} & 0 \\ 0 & 0 & -m^2C_{44} \end{pmatrix}.$$

When these expressions are substituted in the 3×3 system (2.4), we find that it uncouples into a 2×2 system for u_m and v_m, and a single equation for w_m. We shall examine these in turn, but first let us recall the classical results for *isotropic* solids.

4. Isotropy

For a homogeneous isotropic elastic solid, the solutions of Navier's equation in cylindrical polar coordinates are well known. It is convenient to record these solutions here as they motivate some of the discussions to follow.

4.1. Anti-plane motions

The general solution is

$$w_m(r) = AJ_m(k_S r) + BY_m(k_S r),$$

where J_v and Y_v are Bessel functions, A and B are arbitrary constants and $k_S = \omega\sqrt{\rho/\mu}$ is the shear wavenumber.

4.2. In-plane motions

Let (u_m, v_m) be a solution pair. There are four independent solution pairs, so that the general solution is a linear combination of these four (for each m). The first two pairs are

$$\begin{aligned} u_m(r) &= J_{m-1}(k_P r) - J_{m+1}(k_P r) = 2J'_m(k_P r), \\ v_m(r) &= \mathrm{j}\,\{J_{m-1}(k_P r) + J_{m+1}(k_P r)\} = 2\mathrm{j}m(k_P r)^{-1}J_m(k_P r) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} u_m(r) &= J_{m-1}(k_S r) + J_{m+1}(k_S r) = 2m(k_S r)^{-1}J_m(k_S r), \\ v_m(r) &= \mathrm{j}\,\{J_{m-1}(k_S r) - J_{m+1}(k_S r)\} = 2\mathrm{j}\,J'_m(k_S r) \end{aligned} \tag{4.2}$$

where $k_P = \omega\sqrt{\rho/(\lambda + 2\mu)}$ is the compressional wavenumber. The second two pairs are obtained by replacing J_n in these expressions by Y_n. Further information can be found in textbooks on elastic waves; see, for example, (Achenbach, 1973) or (Graff, 1975).

The far-field (large r) properties of these solution pairs are of interest. We have

$$J_v(z) = \sqrt{2/(\pi z)}\cos(z - \tfrac{1}{2}\pi z - \tfrac{1}{4}\pi) + O(z^{-3/2}) \quad \text{as } z \to \infty. \tag{4.3}$$

Thus, for Eq. (4.1), we have $u_m(r) = O(r^{-1/2})$ but $v_m(r) = O(r^{-3/2})$ as $r \to \infty$: Eq. (4.1) represents a cylindrical compressional wave at infinity. On the other hand, Eq. (4.2) represents a cylindrical shear wave at infinity, as the angular component v_m is dominant.

5. Anti-plane motions

Returning to cylindrical orthotropy, we find that the single equation for the anti-plane component $w_m(r)$ is

$$r^2 C_{55} w''_m + rC_{55} w'_m + (\rho\omega^2 r^2 - m^2 C_{44})w_m = 0.$$

This is Bessel's equation. Two independent solutions are

$$J_\beta(\kappa_a r) \quad \text{and} \quad Y_\beta(\kappa_a r), \tag{5.1}$$

where

$$\beta = m\sqrt{C_{44}/C_{55}} \quad \text{and} \quad \kappa_a = \omega\sqrt{\rho/C_{55}}. \tag{5.2}$$

For isotropic materials, these reduce to $J_m(k_S r)$ and $Y_m(k_S r)$, as expected. Note that if β is not an integer, we can replace $Y_\beta(\kappa_a r)$ in Eq. (5.1) by $J_{-\beta}(\kappa_a r)$.

Solutions (5.1) were obtained in the time domain by Watanabe and Payton (1997). They were given erroneously by Yuan and Hsieh (1998, Eq. (16)).

6. In-plane motions: axisymmetric solutions

The ordinary differential equations for $u_m(r)$ and $v_m(r)$ are

$$\begin{aligned} &r^2 C_{11} u_m'' + r[C_{11} u_m' + \mathrm{j}m(C_{66} + C_{12}) v_m'] \\ &\quad + (\rho\omega^2 r^2 - m^2 C_{66} - C_{22}) u_m - \mathrm{j}m(C_{66} + C_{22}) v_m = 0, \end{aligned} \tag{6.1}$$

$$\begin{aligned} &r^2 C_{66} v_m'' + r[C_{66} v_m' + \mathrm{j}m(C_{66} + C_{12}) u_m'] \\ &\quad + (\rho\omega^2 r^2 - C_{66} - m^2 C_{22}) v_m + \mathrm{j}m(C_{66} + C_{22}) u_m = 0. \end{aligned} \tag{6.2}$$

For axisymmetric motions ($m = 0$), these equations simplify and uncouple

$$r^2 C_{11} u_0'' + r C_{11} u_0' + (\rho\omega^2 r^2 - C_{22}) u_0 = 0,$$

$$r^2 C_{66} v_0'' + r C_{66} v_0' + (\rho\omega^2 r^2 - C_{66}) v_0 = 0.$$

Typical solution pairs are

$$u_0(r) = J_\gamma(\kappa_1 r), \quad v_0(r) = 0 \tag{6.3}$$

and

$$u_0(r) = 0, \quad v_0(r) = J_1(\kappa r), \tag{6.4}$$

where

$$\gamma = \sqrt{C_{22}/C_{11}}, \quad \kappa_1 = \omega\sqrt{\rho/C_{11}} \quad \text{and} \quad \kappa = \omega\sqrt{\rho/C_{66}}. \tag{6.5}$$

For isotropic materials, these reduce to $\gamma = 1$, $\kappa_1 = k_P$ and $\kappa = k_S$, whence the solutions given in Section 4 (for $m = 0$) are recovered. Two other solution pairs can be obtained by replacing J_ν by Y_ν. These solutions have been given by Yuan and Hsieh (1998, Eqs. (19) and (26)).

Note that the pair (6.3) corresponds to a cylindrical compressional wave whereas the pair (6.4) represents a cylindrical shear wave. For Scots pine (Appendix B), $\gamma = 0.72$ and $\kappa/\kappa_1 = 3.9$, so that the compressional waves travel about four times faster than the shear waves.

7. In-plane motions: the method of Frobenius

For non-axisymmetric motions ($m \neq 0$), the situation is more complicated: we have been unable to find closed-form solutions, in general.

We begin with a slight simplification of notation. Thus, define dimensionless stiffnesses by

$$c_1 = C_{11}/C_{66}, \quad c_{12} = C_{12}/C_{66} \quad \text{and} \quad c_2 = C_{22}/C_{66}. \tag{7.1}$$

For isotropic materials, we have

$$c_1 = c_2 = c_{12} + 2 \quad \text{and} \quad c_{12} = \lambda/\mu.$$

On the other hand, for Scots pine (Appendix B), we have

$$c_1 = 23.0, \quad c_2 = 11.8 \quad \text{and} \quad c_{12} = 8.3.$$

Making use of Eqs. (7.1), (6.1) and (6.2) become

$$c_1(r^2 u_m'' + r u_m') + \mathrm{j}m(1 + c_{12}) r v_m' + (\kappa^2 r^2 - m^2 - c_2) u_m - \mathrm{j}m(1 + c_2) v_m = 0, \tag{7.2}$$

$$r^2 v_m'' + r v_m' + \mathrm{j}m(1 + c_{12}) r u_m' + (\kappa^2 r^2 - 1 - m^2 c_2) v_m + \mathrm{j}m(1 + c_2) u_m = 0, \tag{7.3}$$

where $\kappa^2 = \rho\omega^2/C_{66}$. Note that if (u_m, v_m) is a solution pair, then so is $(u_{-m}, -v_{-m})$. Thus, we can assume without loss of generality that $m > 0$.

A standard technique for solving ordinary differential equations is the method of Frobenius, in which one looks for solutions in the form of power series. In the present context, this approach has been used by many authors, including Chou and Achenbach (1972) and Yuan and Hsieh (1998). The method proceeds by writing

$$u_m(r) = \sum_{n=0}^{\infty} \hat{a}_n (\kappa r)^{2n+\alpha} \quad \text{and} \quad v_m(r) = \mathrm{j} \sum_{n=0}^{\infty} \hat{b}_n (\kappa r)^{2n+\alpha}, \tag{7.4}$$

where the coefficients $\hat{a}_n$, $\hat{b}_n$ and α are to be determined. It turns out that there is no loss of generality in using $(\kappa r)^{2n}$ rather than $(\kappa r)^n$.

Substitute expansions (7.4) in Eqs. (7.2) and (7.3), giving

$$\sum_{n=0}^{\infty} (\hat{a}_n A_{11} - \hat{b}_n A_{12}) (\kappa r)^{2n+\alpha} + \sum_{n=1}^{\infty} \hat{a}_{n-1} (\kappa r)^{2n+\alpha} = 0 \tag{7.5}$$

and

$$\sum_{n=0}^{\infty} (\hat{a}_n A_{21} + \hat{b}_n A_{22}) (\kappa r)^{2n+\alpha} + \sum_{n=1}^{\infty} \hat{b}_{n-1} (\kappa r)^{2n+\alpha} = 0, \tag{7.6}$$

where

$$A_{11}(n; m, \alpha) = (2n + \alpha)^2 c_1 - m^2 - c_2,$$

$$A_{12}(n; m, \alpha) = m\{(2n + \alpha)(1 + c_{12}) - (1 + c_2)\},$$

$$A_{21}(n;m,\alpha) = m\{(2n+\alpha)(1+c_{12}) + (1+c_2)\},$$

$$A_{22}(n;m,\alpha) = (2n+\alpha)^2 - 1 - m^2 c_2.$$

Next, we set the coefficient of $(\kappa r)^{2n+\alpha}$ equal to zero, for $n = 0, 1, 2, \ldots$; to leading order ($n = 0$), we obtain

$$\{\alpha^2 c_1 - m^2 - c_2\}\hat{a}_0 - m\{\alpha(1+c_{12}) - (1+c_2)\}\hat{b}_0 = 0, \tag{7.7}$$

$$m\{\alpha(1+c_{12}) + (1+c_2)\}\hat{a}_0 + \{\alpha^2 - 1 - m^2 c_2\}\hat{b}_0 = 0. \tag{7.8}$$

This is a homogeneous pair of equations for $\hat{a}_0$ and $\hat{b}_0$. For a non-trivial solution, we require that the determinant

$$\Delta_0(\alpha; m) = 0, \tag{7.9}$$

where

$$\begin{aligned} \Delta_n(\alpha; m) &= A_{11}A_{22} + A_{12}A_{21} \\ &= [(2n+\alpha)^2 c_1 - m^2 - c_2][(2n+\alpha)^2 - 1 - m^2 c_2] \\ &\quad + m^2\{(2n+\alpha)^2(1+c_{12})^2 - (1+c_2)^2\} \\ &= \Delta_0(2n+\alpha; m). \end{aligned}$$

Eq. (7.9) is the *indicial equation*; it is a quadratic in α^2,

$$\alpha^4 c_1 + \alpha^2\{m^2(2c_{12} - \mathscr{C}) - (c_1 + c_2)\} + (m^2 - 1)^2 c_2 = 0 \tag{7.10}$$

with discriminant

$$\begin{aligned} \mathscr{D}_0 &= m^4\mathscr{C}\{\mathscr{C} - 4(c_{12}+1)\} + (c_1 - c_2)^2 \\ &\quad + 2m^2\{\mathscr{C}(c_1 + c_2 + 4) + 2c_{12}(2c_{12} - c_1 - c_2)\}, \end{aligned}$$

where $\mathscr{C} = c_1 c_2 - c_{12}^2 > 0$. ($C_{66}^2\mathscr{C}$ is a principal minor of $C_{\alpha\beta}$, a positive definite matrix.)

It appears that $\mathscr{D}_0$ may be negative; in particular, this may happen when m is large. Thus, the two solutions for α^2 are either both real or they form a complex-conjugate pair. However, let us assume that the elastic stiffnesses are such that $\mathscr{D}_0 > 0$, so that there are two real solutions. This can be ensured, for all m, by supposing that

$$\mathscr{C} \geq 4(1 + c_{12}). \tag{7.11}$$

(Equality holds for isotropic solids, whereas $\mathscr{C} - 4(1 + c_{12}) \simeq 165$ for Scots pine; see Appendix B.) Then, as the product of these two solutions is $(c_2/c_1)(m^2 - 1)^2$, they must have the same sign. Moreover, their sum is

$$\{c_1 + c_2 + m^2(\mathscr{C} - 2c_{12})\}/c_1$$

so that both solutions will be positive for all m if $\mathscr{C} \geq 2c_{12}$, a condition that is implied by Eq. (7.11). Thus, with some restrictions on the elastic stiffnesses, we obtain two positive real solutions for α^2, and hence four real solutions for α, $\pm\alpha_1(m)$ and $\pm\alpha_2(m)$, say, where we have emphasized the dependence on m. We can regard m as a continuous variable: the (positive) solutions $\alpha_1(m)$ and $\alpha_2(m)$ vary continuously with m. We agree to identify them using

$$\lim_{m\to 0}\alpha_1(m) = \gamma = \sqrt{c_2/c_1} \quad \text{and} \quad \lim_{m\to 0}\alpha_2(m) = 1, \tag{7.12}$$

where these are the appropriate limiting values for the axisymmetric problem ($m = 0$); see Eq. (6.5). Some numerical values of $\alpha_1(m)$ and $\alpha_2(m)$ for Scots pine are given in Appendix B.

Once a value of α has been selected, we can choose one of $\hat{a}_0$ and $\hat{b}_0$ arbitrarily (but not zero) and then the other is defined by Eq. (7.7) or Eq. (7.8).

Subsequent coefficients in Eq. (7.4) are given by Eqs. (7.5) and (7.6); we find that

$$\Delta_n\hat{a}_n = -A_{22}\hat{a}_{n-1} - A_{12}\hat{b}_{n-1}, \tag{7.13}$$

$$\Delta_n\hat{b}_n = A_{21}\hat{a}_{n-1} - A_{11}\hat{b}_{n-1} \tag{7.14}$$

for $n = 1, 2, \ldots$, so that $\hat{a}_n$ and $\hat{b}_n$ do not depend on frequency ω. Note that if both α and the starting value ($\hat{a}_0$ or $\hat{b}_0$) are real, then all the coefficients $\hat{a}_n$ and $\hat{b}_n$ will be real.

For large n, we have

$$\Delta_n \sim (2n)^4 c_1, \quad A_{11} \sim (2n)^2 c_1 \quad \text{and} \quad A_{22} \sim (2n)^2,$$

whereas A_{12} and A_{21} are $O(n)$ as $n \to \infty$. Hence, Eqs. (7.13) and (7.14) give

$$\hat{a}_n \sim -(4n^2c_1)^{-1}\hat{a}_{n-1} \quad \text{and} \quad \hat{b}_n \sim -(4n^2)^{-1}\hat{b}_{n-1}. \tag{7.15}$$

It follows that the power series (7.4) are absolutely and uniformly convergent for all values of r; the coefficients decay very rapidly with n.

8. In-plane motions: Neumann series

The method of Frobenius is efficient, in that the recursive structure is simple: one can determine $\hat{a}_n$ and $\hat{b}_n$ directly from $\hat{a}_{n-1}$ and $\hat{b}_{n-1}$. However, the result is a power series in κr, which we may expect to be computationally useful only for moderate values of κr. Thus, we seek alternative expansions.

8.1. Motivation

For isotropic materials (Section 4), we know that both u_m and v_m can be written as linear combinations of just two Bessel functions. This suggests an alternative procedure (when $m \neq 0$) in which we use a generalization of the method of Frobenius: we expand u_m and v_m as *Neumann series* (Watson, 1944, Chapter 16)

$$u_m(r) = \sum_{n=0}^{\infty} a_n J_{2n+\alpha}(kr) \quad \text{and} \quad v_m(r) = \mathrm{j}\sum_{n=0}^{\infty} b_n J_{2n+\alpha}(kr), \tag{8.1}$$

where the coefficients a_n, b_n and α are to be determined. Note that the parameter k is to some extent at our disposal; we will discuss its choice below.

Further motivation comes from system, (7.2) and (7.3), itself. Assume that κr is large. Then, the standard theory of (systems of) ordinary differential equations (Coddington and Levinson, 1955, Chapter 5, Theorem 2.1; Ince, 1956, Section 19.1) shows that the asymptotic behavior of the solutions u_m and v_m is given by

$$e^{ikr} r^{\sigma} \left(\phi_0 + \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + \cdots \right) \quad \text{as } r \to \infty, \tag{8.2}$$

where k can take the values $\pm\kappa$ and $\pm\kappa/\sqrt{c_1} = \pm\kappa_1$, as defined by Eq. (6.5). The coefficients ϕ_0, ϕ_1, $\phi_2, \ldots$ and the exponent σ do not come out of the general theory, but they can be determined by substituting in Eqs. (7.2) and (7.3) (see Appendix C). It turns out that $\sigma = -\frac{1}{2}$, and then it is seen that Eq. (8.2) is consistent with the asymptotic behavior of $J_v(kr)$ and $Y_v(kr)$ as $kr \to \infty$; see Eq. (4.3). It also turns out that when $k^2 = \kappa^2$, the angular component v_m is asymptotically larger than u_m at infinity, whereas u_m is asymptotically larger than v_m when $k^2 = \kappa_1^2$; these properties are also seen in the isotropic solutions.

We have investigated two methods for finding the coefficients a_n and b_n. The first, the *direct method*, proceeds analogously to the method of Frobenius. The second, *indirect method* constructs a_n and b_n in terms of the Frobenius coefficients, $\hat{a}_n$ and $\hat{b}_n$. It turns out that the indirect method is better, because the recursion scheme for the method of Frobenius is simpler than the corresponding recursion scheme for a_n and b_n.

8.2. The direct method

We substitute expansions (8.1) in Eqs. (7.2) and (7.3). In doing this, the following results are useful. First, from Bessel's equation, we have

$$r^2 u_m'' + r u_m' = \sum_{n=0}^{\infty} a_n \left\{ (2n+\alpha)^2 - X^2 \right\} J_{2n+\alpha}(X),$$

where $X \equiv kr$. Also, making use of the recurrence relations for Bessel functions, we obtain

$$z J_v'(z) = v J_v(z) + 2 \sum_{n=1}^{\infty} (2n+v)(-1)^n J_{2n+v}(z),$$

whence

$$\begin{aligned} r u_m' &= \sum_{n=0}^{\infty} a_n X J_{2n+\alpha}'(X) \\ &= \sum_{n=0}^{\infty} a_n (2n+\alpha) J_{2n+\alpha}(X) + 2 \sum_{n=1}^{\infty} A_n (-1)^n (2n+\alpha) J_{2n+\alpha}(X), \end{aligned} \tag{8.3}$$

where we have interchanged the order of summation and defined A_n by

$$A_n = \sum_{\ell=0}^{n-1} (-1)^{\ell} a_{\ell} \quad \text{for } n = 1, 2, \ldots.$$

Similarly, we have

$$z^2 J_v(z) = 4\sum_{j=0}^{\infty}(-1)^j(2j+v+1)(2j+v+2)J_{2j+v+2}$$
$$-2\sum_{j=0}^{\infty}(-1)^j(2j+v+1)\,z\,J_{2j+v+3}$$
$$= 4\sum_{\ell=0}^{\infty}(-1)^{\ell}(2\ell+v+2)(\ell+1)(\ell+v+1)J_{2\ell+v+2},$$

whence

$$\sum_{n=0}^{\infty} a_n X^2 J_{2n+\alpha}(X) = 4\sum_{n=1}^{\infty}\tilde{A}_n(-1)^{n+1}(2n+\alpha)J_{2n+\alpha}(X),$$

where

$$\tilde{A}_n = \sum_{\ell=0}^{n-1}(-1)^{\ell}(n-\ell)(n+\ell+\alpha)\,a_{\ell}.$$

We define B_n and $\tilde{B}_n$ similarly, with a_ℓ replaced by b_ℓ.

A_n and $\tilde{A}_n$ can be calculated recursively. Thus, $A_1 = a_0$, $\tilde{A}_1 = (\alpha+1)a_0$,

$$A_{n+1} = A_n + (-1)^n a_n \quad \text{and} \quad \tilde{A}_{n+1} = \tilde{A}_n + (2n+1+\alpha)A_{n+1} \quad \text{for } n = 1, 2, \ldots.$$

Eqs. (7.2) and (7.3) give

$$0 = \sum_{n=0}^{\infty}(a_n A_{11} - b_n A_{12})J_{2n+\alpha}(X)$$
$$-4(q^2 - c_1)\sum_{n=1}^{\infty}\tilde{A}_n(-1)^n(2n+\alpha)J_{2n+\alpha}(X)$$
$$-2m(1+c_{12})\sum_{n=1}^{\infty}B_n(-1)^n(2n+\alpha)J_{2n+\alpha}(X) \tag{8.4}$$

and

$$0 = \sum_{n=0}^{\infty}(a_n A_{21} + b_n A_{22})J_{2n+\alpha}(X)$$
$$-4(q^2 - 1)\sum_{n=1}^{\infty}\tilde{B}_n(-1)^n(2n+\alpha)J_{2n+\alpha}(X)$$
$$+2m(1+c_{12})\sum_{n=1}^{\infty}A_n(-1)^n(2n+\alpha)J_{2n+\alpha}(X), \tag{8.5}$$

respectively, where $A_{ij}(n; m, \alpha)$ are defined exactly as before and

$$q^2 = \kappa^2/k^2 = \rho\omega^2/(k^2 C_{66})$$

is unspecified at the moment; see Section 8.4 below.

The next step is to set the coefficient of $J_{2n+\alpha}(X)$ equal to zero, for $n=0,1,2,\ldots$; to leading order ($n=0$), we obtain exactly the same indicial equation as before, namely (7.9). Subsequent coefficients in the Neumann series are given by Eqs. (8.4) and (8.5); we find that

$$\Delta_n a_n = A_{22} f_n + A_{12} g_n, \tag{8.6}$$

$$\Delta_n b_n = A_{11} g_n - A_{21} f_n \tag{8.7}$$

for $n=1,2,\ldots$, where

$$f_n = (2n+\alpha)(-1)^n\{4(q^2-c_1)\tilde{A}_n + 2m(1+c_{12})B_n\},$$
$$g_n = (2n+\alpha)(-1)^n\{4(q^2-1)\tilde{B}_n - 2m(1+c_{12})A_n\}.$$

These formulae determine a_n and b_n recursively in terms of a_j and b_j with $0 \le j < n$. Again, note that if α, k^2 and the starting value (a_0 or b_0) are real, then all the coefficients a_n and b_n will be real.

In Appendix D, we discuss two other issues. The first is the possibility of solutions where only a_0, a_1, b_0 and b_1 are non-zero. This is plausible because of the known results for homogeneous, isotropic materials. However, we have not found any solutions of this form, in general. Second, we reduce the pair of second-order equations to a single fourth-order equation. The resulting equation is rather complicated, so that it is not clear that much is gained by the reduction.

8.3. The indirect method

We can determine the coefficients in Eq. (8.1) from the coefficients $\hat{a}_n$ and $\hat{b}_n$ obtained by the method of Frobenius. The basic formula needed is a particular Neumann series due to Gegenbauer (Watson, 1944, Section 5.2)

$$(\tfrac{1}{2}z)^v = \sum_{\ell=0}^{\infty} \frac{(2\ell+v)\,\Gamma(\ell+v)}{\ell!} J_{2\ell+v}(z).$$

Thus, we have

$$\begin{aligned}
u_m(r) &= \sum_{j=0}^{\infty} \hat{a}_j\,(\kappa r)^{2j+\alpha} \\
&= \sum_{j=0}^{\infty} \hat{a}_j \left(\frac{2\kappa}{k}\right)^{2j+\alpha} \sum_{\ell=0}^{\infty} \frac{(2\ell+2j+\alpha)\,\Gamma(\ell+2j+\alpha)}{\ell!} J_{2\ell+2j+\alpha}(kr) \\
&= \sum_{j=0}^{\infty} \hat{a}_j \left(\frac{2\kappa}{k}\right)^{2j+\alpha} \sum_{n=j}^{\infty} \frac{(2n+\alpha)\,\Gamma(n+j+\alpha)}{(n-j)!} J_{2n+\alpha}(kr) \\
&= \sum_{n=0}^{\infty}(2n+\alpha)\,J_{2n+\alpha}(kr) \sum_{j=0}^{n} \hat{a}_j \left(\frac{2\kappa}{k}\right)^{2j+\alpha} \frac{\Gamma(n+j+\alpha)}{(n-j)!}
\end{aligned}$$

after changing the order of summation. Hence

$$a_n = (2n+\alpha)\sum_{j=0}^{n} \hat{a}_j \left(\frac{2\kappa}{k}\right)^{2j+\alpha} \frac{\Gamma(n+j+\alpha)}{(n-j)!} \tag{8.8}$$

so that a_n is given as a weighted sum of $\hat{a}_0$, $\hat{a}_1, \ldots, \hat{a}_n$. There is a similar formula for b_n.

8.4. The choice of k

In general, the method of Frobenius gives four solution pairs, corresponding to the four solutions of the indicial Eq. (7.10) for α, namely $\pm\alpha_1(m)$ and $\pm\alpha_2(m)$ (subject to Eq. (7.11)). (There will be situations involving double roots, or roots differing by an integer (examples are isotropy and $m=1$) but we do not consider such situations here.) Thus, the method of Frobenius gives the small-r behavior of the four solution pairs.

On the other hand, we know that the large-r behavior is given by Eq. (8.2), wherein $k=\pm\kappa$ or $k=\pm\kappa_1$ and $\sigma=-\frac{1}{2}$. Thus, we should choose

$$k=\kappa \quad (q^2=1) \quad \text{or} \quad k=\kappa_1 \quad (q^2=c_1) \tag{8.9}$$

in the Neumann series (8.1). (As $J_\nu(-z)=e^{\nu\pi i}J_\nu(z)$, we can assume that $k>0$ in Eq. (8.1).)

So, we have four values of α and two values of k: given a value for α, which value of k should we choose? In other words, how do we connect the small-r behavior to the large-r behavior? As a consequence of Eq. (7.12) and the explicit axisymmetric solutions (Section 6), we have the following answer:

$$\text{if } \alpha=\pm\alpha_1(m) \quad \text{choose } k=\kappa_1(q^2=c_1), \tag{8.10}$$

whereas

$$\text{if } \alpha=\pm\alpha_2(m) \quad \text{choose } k=\kappa(q^2=1). \tag{8.11}$$

This relies on a ‘homotopy’ argument: the solutions must retain their character as m varies, and we know their character explicitly when $m=0$. In addition, we may check, computationally, that Eq. (8.10) does indeed lead to a cylindrical compressional wave (so that Eq. (C.5) holds) whereas (8.11) should lead to a cylindrical shear wave (satisfying Eq. (C.6)).

9. Free vibrations of a solid wooden pole

Let us now give an application of the foregoing theory to find the frequency equation for a solid wooden pole of radius d. First, we consider the solution obtained using the Neumann series, and then we compare our results with similar calculations using the standard method of Frobenius. For a solid cylinder, the displacements must be bounded at $r = 0$, so we retain only the positive roots α of the indicial equation. Thus, the displacement is given by Eq. (2.3), wherein

$$\mathbf{u}_m(r) = (u_m, v_m, w_m) = A_1(\phi_1, \psi_1, 0) + A_2(\phi_2, \psi_2, 0) + B(0, 0, w) \tag{9.1}$$

$w(r) = J_\beta(\kappa_a r)$ with β and κ_a defined by Eq. (5.2), and ϕ_i and ψ_i are defined using Neumann series, as in Section 8: for $i = 1, 2$,

$$\phi_i(r) = \sum_{n=0}^{\infty} a_n^i J_{2n+\alpha_i}(k_i r), \tag{9.2}$$

$$\psi_i(r) = \mathrm{j} \sum_{n=0}^{\infty} b_n^i J_{2n+\alpha_i}(k_i r), \tag{9.3}$$

where $k_1 = \kappa_1 = \omega\sqrt{\rho/C_{11}}$ and $k_2 = \kappa = \omega\sqrt{\rho/C_{66}}$.

From Eq. (2.1) and Ting (1996a, Eq. (2.9)), we have an expression for the radial traction vector

$$\mathbf{t}_r(r, \theta, t) = \mathrm{Re}_i\{\mathbf{t}_m(r)\,\mathrm{e}^{\mathrm{j}m\theta}\,\mathrm{e}^{-\mathrm{i}\omega t}\},$$

where

$$\mathbf{t}_m = \mathbf{Q}\mathbf{u}_m' + r^{-1}\mathbf{R}\mathbf{K}_m\mathbf{u}_m.$$

For cylindrical orthotropy, we obtain

$$\mathbf{t}_m(r) = \begin{pmatrix} C_{11}u_m' + r^{-1}C_{12}[u_m + \mathrm{j}mv_m] \\ C_{66}[v_m' + r^{-1}(\mathrm{j}mu_m - v_m)] \\ C_{55}w_m' \end{pmatrix}.$$

The outer surface of the pole ($r = d$) is stress-free, so that we have the boundary condition $\mathbf{t}_m(d) = \mathbf{0}$. Thus, from Eqs. (9.1), (9.2) and (9.3), we obtain

$$\sum_{i=1}^{2} A_i\{dc_1\phi_i'(d) + c_{12}[\phi_i(d) + \mathrm{j}m\psi_i(d)]\} = 0, \tag{9.4}$$

$$\sum_{i=1}^{2} A_i\{\mathrm{j}m\phi_i(d) + d\psi_i'(d) - \psi_i(d)\} = 0 \tag{9.5}$$

and $J_\beta'(\kappa_a d) = 0$. This last equation is the frequency equation for anti-plane motions. This is uncoupled from the in-plane motions because of the orthotropy. Our primary interest is with the in-plane motions; for a non-trivial solution, we have to solve

$$\det D = 0, \tag{9.6}$$

where the elements of the 2×2 matrix D can be obtained from Eqs. (9.4) and (9.5). Note that $\phi_i'(d)$ and $\psi_i'(d)$ can be calculated using Eq. (8.3) or by using $zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$, for example.

Table 1
Convergence of the root of the frequency equation using standard Frobenius power series and the Neumann series for the second mode. The quantity tabulated is $\omega d\sqrt{\rho/C_{66}}$, which is dimensionless. N is the number of terms in each series

N	Frobenius	Neumann
2	4.31503	3.38430
4	3.34162	3.34291
6	3.35371	3.35377
8	3.35407	3.35408
10	3.35408	3.35408
12	3.35408	3.35408
20	3.35408	3.35408

We now consider the specific case of a wooden pole made of Scots pine; the elastic constants are given in Appendix B. Selecting $m=2$, for example, we retain the positive values of α, which are $\alpha_1 = 0.37183$ and $\alpha_2 = 5.77083$; see Table 2 for other roots. Then, we see that all of the elements of D are real. The coefficients for the Neumann series solutions were determined numerically using a combination of the direct and indirect methods detailed in the previous section. Specifically, we fix $\hat{a}_0 = 1$, calculate a_0 with Eq. (8.8), then determine the remaining a_n with the direct method of Eqs. (8.6) and (8.7). The coefficients for the standard Frobenius power series were found using Eqs. (7.13) and (7.14).

The coefficients in the Neumann series expansions become quite large numerically as n becomes large. This is due to the fact that the Bessel function $J_n(r)$ decays rapidly as $n \to \infty$; for example, $J_{50}(5) = 2.29 \times 10^{-45}$. To balance this, the Neumann series coefficients were scaled for numerical computations as $\bar{a}_n = (-1)^n(2q)^{-2n}a_n$ with a similar formula for the b_n. This scaling, which was obtained by a rough estimate of Eq. (8.8) combined with the estimate (7.15), was found to be adequate; it could be refined further.

All computations were performed in double precision. For the Neumann series computations, the gamma function subroutine DGAMMA and the Bessel function subroutine DBESJ from the SLATEC library were used. Solutions of Eq. (9.6) were sought for the first, second, and third in-plane modes. Of interest in our calculations was a comparison of the standard Frobenius power series with the Neumann series expansions. Focusing our attention on the first mode, the root of the frequency equation, (9.6), is shown in Table 1 as a function of the number of terms taken in the two methods. We note that the Neumann series results converge slightly faster than the power series results, with both methods yielding nearly identical results after about 8 terms in each series. Similar results were obtained for the higher-order modes.

Finally, we have computed roots of the frequency equation for the first three modes, using 20-term Neumann series. The values found for $\omega d\sqrt{\rho/C_{66}}$ are 3.35408, 7.25838 and 11.66891.

10. Discussion

In this paper, we have given a first account of the use of Neumann series to solve problems of wave propagation in materials with cylindrical anisotropy, including an application to the free vibrations of a wooden pole. It is clear that the method will extend to other related problems, such as longitudinal motions (waves along the pole), hollow poles, poles with concentric layers (giving a better model of a tree, taking into account the alternate layers of earlywood and latewood) and rotten cores (once plausible models for decayed wood have been developed; see Appendix A). These applications of the basic methodology are currently being made.

Acknowledgements

This work was partially supported by EPSRC (Grant reference GR/M76560). PAM thanks Lee Renforth (IPEC Ltd.) and John Coffey (Manchester TEC Ltd.) for bringing the problem of waves in wooden poles to his attention.

Appendix A. The NDE problem

The basic problem is to estimate the size of a rotten region inside a wooden telegraph pole. Here, we give some background information to aid the modelling.

In the United Kingdom, most poles are made from Scots pine (*Pinus sylvestris*). The engineering parameters (Young's moduli, shear moduli, density, Poisson's ratios) are known (Hearmon, 1948; Bucur, 1995, p. 40), assuming that the wood is an orthotropic elastic solid; see Appendix B.

Much is known about wave propagation through wood, and this knowledge is well surveyed in the book by Bucur (1995). For example, the various wave speeds are known for many woods. These speeds vary with temperature and with moisture content. It is also known that attenuation (damping) can be significant, especially for propagation in the cross-sectional plane. Thus, it may be appropriate to use a viscoelastic model. The simplest way to incorporate damping is to make the elastic constants complex; this approach is described by Bucur (1995, Section 4.4) and by Bodig and Jayne (1982, Section 6.6.4).

The literature on the ultrasonic non-destructive evaluation of wooden poles is scarce and scattered. Patton-Mallory and De Groot (1989) have described experiments where a high-frequency acoustic pulse was generated at one side of a piece of wood and recorded at the opposite side. The waveforms received through decayed wood and sound wood were compared. Their conclusions were: 'sound velocity …increased in a linear fashion with increasing decay degradation; waveform amplitude, measured as root mean square, decreased with decay degradation; the high-frequency components of the waveform were attenuated from very early stages of decay; and acousto-ultrasonic parameters were sensitive to the early stages of decay that caused considerable strength loss'. At the same conference, Anthony and Bodig (1989) described a similar method, termed 'stress wave spectral NDE'. This is 'based on the principle that stress waves propagate at different speeds and attenuate differently at various frequencies in inhomogeneous materials …By collecting a time record of vibration and converting it to a frequency spectrum, useful information … including attenuation and frequency shifts … can be derived for correlation with the strength of the structural component … using linear or nonlinear multiple regression techniques'. Related work is described by Bodig et al. (1982), Bucur (1995, Section 9.2.2) and Dolwin et al. (1999). In general, these NDE methods are somewhat unsatisfactory as they do not use a theoretical model to guide the experiments or help interpret the received waveforms.

Tomikawa et al. (1986) have used 'computed tomography'. This assumes that sound travels along straight rays (high-frequency approximation for isotropic materials) and that rays cannot pass through a rotten region. Nevertheless, some good results were obtained. Computed tomography has also been used by Habermehl et al. (1986) for inspecting live trees.

An older paper (not cited by Bucur (1995)) is (Makow, 1969), in which a pulse-echo method is used under laboratory conditions. Similar methods were used later by Okyere and Cousin (1980).

A.1. Modelling the rotten region

Another aspect of the NDE problem concerns the rotten region itself. The first question to ask is: what is rot? Bodig and Jayne (1982, Section 11.4) give a general discussion on the biological deterioration of wood and its effects; for more information, see the book by Cartwright and Findlay (1958). Most decay is caused by fungi. It turns out, and 'it is an interesting and hitherto incompletely explained fact that comparatively few of the fungi which occur in standing [live] trees are important as causes of decay in felled timber' (Cartwright and Findlay, 1958, p. 166). One particular fungus, *Lentinus lepideus*, is the usual culprit for causing decay in telegraph poles in the UK (Cartwright and Findlay, 1958, pp. 172 and 252); it is also widespread in Europe and North America.

It is clear that we have to use a reasonable model for the rotten region. The simplest model is to use a cavity. Thus, Makow (1969) claims that 'rot is an empty cavity with poorly defined walls. …Echoes from the walls will be difficult to receive unless the rot contains air rather than liquid'. Tomikawa et al. (1986) also regard rot as an empty cavity. Obviously, elastic waves cannot pass *through* a cavity, although they can be diffracted. On the other hand, there are reported measurements of elastic waves through rotten regions (Bucur, 1995, Section 9.2.2).

Various photographs of the cross-sections of decayed trees suggest a rotten core with a solid concentric outer shell. The decay tends to start from the center: 'heartwood of pine is vigorously attacked' (Cartwright and Findlay, 1958, p. 172). Thus, a realistic model is obtained by taking a central rotten region (with radius c) surrounded by sound wood (with outer radius a). Empirical evidence (Mattheck and Kubler, 1997, Fig. 104) shows that hollow trees are not liable to fail until $c/a \simeq 0.7$.

Concerning the mechanical properties of the decayed wood, we can say that the density is significantly reduced (Cartwright and Findlay, 1958, p. 53). Bodig and Jayne

(1982, p. 587) refer to ‘advanced decay’, a stage in which “the wood develops a soft or ‘punky’ texture”. Such decay ‘caused by any wood destroying fungus nearly always results in serious reduction in strength properties’. However, although experiments have been done showing the gross effects of decay on the strength of beams (Cartwright and Findlay, 1958, pp. 53–57; Bodig and Jayne, 1982, Section 11.4.3), nothing seems to be known about the elastic properties of the decayed wood itself.

In summary, it seems that we have various choices. We could consider ‘hollow rot’ (a cavity with traction-free boundary), but this is likely to scatter waves too strongly. We could consider ‘elastic rot’, where the rotten region is modelled as an elastic inclusion; however, we do not have adequate data for the relevant material parameters. Finally, we could place an impedance-type boundary condition on the boundary of the rotten region; the impedance matrix may be varied to model dissipation.

Appendix B. An example: Scots pine

As an example, we consider a wooden pole made of Scots pine. We require the elastic stiffnesses, C_{ij}. Hearmon (1948, Table 2) gives data for the corresponding compliances, S_{ij}, as follows. First, denote the 1, 2 and 3 directions by R(adial), T(angential) and L(ongitudinal). Then, we have

$$S_{11} = E_{\mathrm{R}}^{-1}, \quad S_{22} = E_{\mathrm{T}}^{-1}, \quad S_{33} = E_{\mathrm{L}}^{-1},$$

$$S_{12} = -\sigma_{\mathrm{RT}} E_{\mathrm{R}}^{-1} = -\sigma_{\mathrm{TR}} E_{\mathrm{T}}^{-1}, \quad S_{44} = G_{\mathrm{LT}}^{-1},$$

$$S_{13} = -\sigma_{\mathrm{RL}} E_{\mathrm{R}}^{-1} = -\sigma_{\mathrm{LR}} E_{\mathrm{L}}^{-1}, \quad S_{55} = G_{\mathrm{LR}}^{-1},$$

$$S_{23} = -\sigma_{\mathrm{TL}} E_{\mathrm{T}}^{-1} = -\sigma_{\mathrm{LT}} E_{\mathrm{L}}^{-1}, \quad S_{66} = G_{\mathrm{TR}}^{-1},$$

where the Young’s moduli E_i, the Poisson’s ratios σ_{ij} and the rigidity moduli G_{ij} are the quantities tabulated by Hearmon (1948); see also (Bucur, 1995, Table 4.1). Thus, in $(\mathrm{GPa})^{-1}$, we obtain

$$S_{11} = 0.91, \quad S_{12} = -0.62, \quad S_{13} = -0.035,$$

$$S_{22} = 1.75, \quad S_{23} = -0.026, \quad S_{33} = 0.061,$$

$$S_{44} = 1.47, \quad S_{55} = 0.86, \quad S_{66} = 15.15$$

with $S_{ij} = S_{ji}$. The remaining entries in the 6×6 matrix (S_{ij}) are zeros.

To obtain the stiffnesses, we invert the matrix (S_{ij}). This gives, in GPa,

$$C_{11} = 1.52, \quad C_{12} = 0.55, \quad C_{13} = 1.11,$$

$$C_{22} = 0.78, \quad C_{23} = 0.65, \quad C_{33} = 17.3,$$

$$C_{44} = 0.68, \quad C_{55} = 1.16, \quad C_{66} = 0.066.$$

As Hearmon (1948, p. 29) notes, ‘there is much more relative variation among the coefficients, i.e. wood is both weaker and more anisotropic than [many] other materials’; see also (Hearmon, 1961, p. 41). The density ρ is given as 550 kg/m^3.

Table 2
Roots of the indicial equation for Scots pine

m	0	1	2	3	4	5
α_1	0.72	0.00	0.37	0.67	0.95	1.22
α_2	1.00	3.08	5.77	8.54	11.33	14.13

From this data, we obtain

$$c_1 = 23.0, \quad c_2 = 11.8, \quad c_{12} = 8.3 \quad \text{and} \quad \gamma = 0.72.$$

The first few positive roots of the indicial equation (7.10) are given in Table 2. As a check, one can easily solve Eq. (7.10) asymptotically for large m; we find that $\alpha_1(m) \sim 0.25m$ and $\alpha_2(m) \sim 2.83m$, which give $\alpha_1(5) = 1.25$ and $\alpha_2(5) = 14.15$.

Bucur (1995, Chapter 5) gives further information on determining the elastic constants of wood.

Appendix C. Asymptotic solution of the in-plane system

We seek solutions of Eqs. (7.2) and (7.3) in the form

$$u_m = \mathrm{e}^{\mathrm{i}kr} r^\sigma \sum_{n=0} U_n r^{-n} \quad \text{and} \quad v_m = \mathrm{j}\mathrm{e}^{\mathrm{i}kr} r^\sigma \sum_{n=0} V_n r^{-n}$$

valid for large r. Here the coefficients U_n, V_n, σ and k are to be determined. We have

$$ru_m' = \mathrm{e}^{\mathrm{i}kr} r^\sigma \left\{ \mathrm{i}krU_0 + \sum_{n=0} [(\sigma - n)U_n + \mathrm{i}kU_{n+1}] r^{-n} \right\}$$

and

$$r^2 u_m'' + ru_m' = \mathrm{e}^{\mathrm{i}kr} r^\sigma \left\{ (\mathrm{i}kr)^2 U_0 + \mathrm{i}kr[(2\sigma + 1)U_0 + \mathrm{i}kU_1] \right.$$
$$\left. + \sum_{n=0} [(\sigma - n)^2 U_n + \mathrm{i}k(2\sigma - 2n - 1)U_{n+1} - k^2 U_{n+2}] r^{-n} \right\}.$$

Substitute in Eqs. (7.2) and (7.3), and then cancel $\mathrm{e}^{\mathrm{i}kr} r^\sigma$. The leading order terms are $O(r^2)$ as $r \to \infty$. Balancing these gives

$$(\kappa^2 - c_1 k^2)U_0 = 0 \quad \text{and} \quad (\kappa^2 - k^2)V_0 = 0,$$

whence

$$k^2 = \kappa^2 / c_1 = \kappa_1^2 \quad \text{and} \quad V_0 = 0 \tag{C.1}$$

or

$$k^2 = \kappa^2 \quad \text{and} \quad U_0 = 0, \tag{C.2}$$

where κ and κ_1 are defined by Eq. (6.5).

The linear terms in r give

$$(2\sigma+1)\mathrm{i}kc_1U_0 - \mathrm{i}km(1+c_{12})V_0 + (\kappa^2 - c_1k^2)U_1 = 0, \tag{C.3}$$

$$(2\sigma+1)\mathrm{i}kV_0 + \mathrm{i}km(1+c_{12})U_0 + (\kappa^2 - k^2)V_1 = 0. \tag{C.4}$$

Let us suppose that Eq. (C.1) holds, whence Eqs. (C.3) and (C.4) reduce to

$$(2\sigma+1)U_0 = 0 \quad \text{and} \quad \mathrm{i}kmc_1(1+c_{12})U_0 + \kappa^2(c_1-1)V_1 = 0,$$

respectively. The first of these implies that we can only obtain a non-trivial solution by taking $\sigma = -\frac{1}{2}$, and then the second equation determines V_1 in terms of U_0. Note that the corresponding solutions are such that

$$u_m = O(r^{-1/2}) \quad \text{and} \quad v_m = O(r^{-3/2}) \quad \text{as } r \to \infty. \tag{C.5}$$

Thus, the solution pairs constructed with $k^2 = \kappa_1^2$ are such that the radial component is dominant at infinity: they are cylindrical compressional waves. (Recall that $\kappa_1 = k_\mathrm{P}$ for isotropic materials.)

Alternatively, if Eq. (C.2) holds, Eqs. (C.3) and (C.4) reduce to

$$\mathrm{i}km(1+c_{12})V_0 - k^2(1-c_1)U_1 = 0 \quad \text{and} \quad (2\sigma+1)V_0 = 0,$$

respectively. As before, we obtain $\sigma = -\frac{1}{2}$ from the second equation whereas the first equation gives U_1 in terms of V_0. The corresponding solutions are such that

$$u_m = O(r^{-3/2}) \quad \text{and} \quad v_m = O(r^{-1/2}) \quad \text{as } r \to \infty. \tag{C.6}$$

Thus, the solution pairs constructed with $k^2 = \kappa^2$ are such that the angular component is dominant at infinity: they are cylindrical shear waves. (Recall that $\kappa = k_\mathrm{S}$ for isotropic materials.)

Appendix D. Further properties of the in-plane system

D.1. Two-term expansions

It is of interest to study the possibility of solutions of Eqs. (7.2) and (7.3) with

$$a_n = 0 \quad \text{and} \quad b_n = 0 \quad \text{for } n \geq 2 \tag{D.1}$$

as such solutions exist in the isotropic case (Section 4). Thus, with Eq. (D.1), we have

$$A_1 = a_0, \quad \tilde{A}_1 = (1+\alpha)a_0,$$

$$A_n = a_0 - a_1, \quad \tilde{A}_n = n(n+\alpha)(a_0 - a_1) + (1+\alpha)a_1$$

for $n \geq 2$, with similar expressions for B_n and $\tilde{B}_n$. Then, Eqs. (8.6) and (8.7), with Eq. (D.1), give $f_n = 0$ and $g_n = 0$ for $n \geq 2$; explicitly, these give

$$2(q^2 - c_1)\{n(n+\alpha)(a_0 - a_1) + (1+\alpha)a_1\} + m(1+c_{12})(b_0 - b_1) = 0,$$

$$2(q^2 - 1)\{n(n+\alpha)(b_0 - b_1) + (1+\alpha)b_1\} - m(1+c_{12})(a_0 - a_1) = 0.$$

We can satisfy the first of these for all $n \geq 2$ by choosing $q^2 = c_1$ (see Eq. (8.9)) and $b_1 = b_0$. The second equation then reduces to

$$2(c_1 - 1)(1 + \alpha)b_0 = m(1 + c_{12})(a_0 - a_1).$$

This equation, which does not depend on n, can be viewed as defining a_1 in terms of a_0 and b_0.

So, at this stage, all the coefficients have been specified in terms of a_0, say. However, we have not used Eqs. (8.6) and (8.7) with $n = 1$; we will have a solution only if these two equations are also satisfied, and, in general, they are not satisfied.

D.2. Fourth-order equations

The pair of coupled second-order equations, (7.2) and (7.3), can be uncoupled, resulting in a single fourth-order equation for u_m or v_m. Thus, define the operator $\mathscr{D} = r(\mathrm{d}/\mathrm{d}r)$, and then write Eqs. (7.2) and (7.3) as

$$c_1\mathscr{D}^2 u_m + e_1 \mathrm{j}\mathscr{D}v_m + (R - e_2)u_m - e_3\mathrm{j}v_m = 0, \tag{D.2}$$

$$\mathrm{j}\mathscr{D}^2 v_m - e_1\mathscr{D}u_m + (R - e_4)\mathrm{j}v_m - e_3 u_m = 0, \tag{D.3}$$

where $e_1 = m(1 + c_{12})$, $e_2 = m^2 + c_2$, $e_3 = m(1 + c_2)$, $e_4 = 1 + m^2 c_2$ and $R = \rho\omega^2 r^2$. Next, consider (D.2), $\mathscr{D}$(D.2), $\mathscr{D}^2$(D.2), (D.3) and $\mathscr{D}$(D.3). These five equations involve v_m, $\mathscr{D}v_m$, $\mathscr{D}^2 v_m$ and $\mathscr{D}^3 v_m$; once eliminated, we obtain an equation for $u_m(r)$,

$$\sum_{\ell=0}^{4} p_\ell(R)\mathscr{D}^\ell u_m(r) = 0. \tag{D.4}$$

The coefficients $p_\ell(R)$ are polynomials in R, defined as follows:

$$p_0(R) = p_{00} + p_{01}R + p_{02}R^2 + e_1^2 R^3, \quad p_1(R) = 2R(p_{11} + e_1^2 R),$$

$$p_2(R) = p_{20} + p_{21}R + p_{22}R^2, \quad p_3(R) = -2e_1^2 c_1 R, \quad p_4(R) = c_1(p_{40} + e_1^2 R)$$

with

$$p_{00} = (e_3^2 - e_1^2 e_4)(e_2 e_4 - e_3^2),$$

$$p_{01} = -2e_1^3 e_3 + e_1^2[e_4(e_4 + 2e_2 - 4) - e_3^2] + 2e_1 e_2 e_3 + e_3^2(4 - e_2 - e_4),$$

$$p_{02} = -e_1^2(e_2 + 2e_4) - (2e_1 - e_3)e_3,$$

$$p_{11} = -e_1^4 + e_1^2(e_2 - 2e_4) + 2e_3^2,$$

$$p_{20} = (e_3^2 - e_1^2 e_4)(e_1^2 - e_2 - e_4 c_1),$$

$$p_{21} = e_1^4 - e_1^2(2e_4 c_1 + e_2 + e_4) - 2e_1 e_3 c_1 + e_3^2(1 + c_1),$$

$$p_{22} = e_1^2(1 + c_1), \quad p_{40} = e_3^2 - e_1^2 e_4.$$

One can derive a similar equation for v_m or one can express v_m in terms of u_m and its first three derivatives.

The differential equation (D.4) has the same indicial equation as systems (7.2) and (7.3); note that $p_1(0) = p_3(0) = 0$.

It is not clear whether there are any advantages in working with (D.4) rather than systems (7.2) and (7.3). One gain is that the Wronskian can be calculated easily, using Abel's identity (Ince, 1956, Section 5.2; Coddington and Levinson, 1955, Chapter 3, Section 6).

References

Achenbach, J.D., 1973. Wave Propagation in Elastic Solids. North-Holland, New York.

Ambartsumyan, S.A., 1970. Theory of Anisotropic Plates. Technomic Publishing Co., Stamford, CT.

Anthony, R.W., Bodig, J., 1989. Nondestructive evaluation of timber structures for reliable performance. Proceedings of the Second Pacific Timber Engineering Conference, University of Auckland, Vol. 2, pp. 197–200.

Bodig, J., Goodman, J.R., Phillips, G.E., Landers, P.G., 1982. Reliability of wood transmission structures through nondestructive evaluation. IEEE Trans. Power Apparatus Systems PAS-101, 2409–2414.

Bodig, J., Jayne, B.A., 1982. Mechanics of Wood and Wood Composites. Van Nostrand Reinhold, New York.

Bucur, V., 1995. Acoustics of Wood. CRC Press, Boca Raton, FL.

Cartwright, K.St.G., Findlay, W.P.K., 1958. Decay of Timber and its Prevention, 2nd Edition. Her Majesty's Stationery Office, London.

Chou, F.-H., Achenbach, J.D., 1972. Three-dimensional vibrations of orthotropic cylinders. Proc. ASCE, J. Engng Mech. Div. 98, 813–822.

Coddington, E.A., Levinson, N., 1955. Theory of Ordinary Differential Equations. McGraw-Hill, New York.

Dolwin, J.A., Lonsdale, D., Barnett, J., 1999. Detection of decay in trees. Arboricultural J. 23, 139–149.

Graff, K.F., 1975. Wave Motion in Elastic Solids. University Press, Oxford.

Habermehl, A., Pramann, F.W., Ridder, H.-W., 1986. Untersuchung von Alleebäumen mit einem neuen Computer-Tomographie-Gerät. Neue Landschaft 31, 806–812.

Hearmon, R.F.S., 1948. Elasticity of Wood and Plywood. Forest Products Research Special Report No. 7, Dept. of Scientific & Industrial Research. His Majesty's Stationery Office, London.

Hearmon, R.F.S., 1961. An Introduction to Applied Anisotropic Elasticity. University Press, Oxford.

Ince, E.L., 1956. Ordinary Differential Equations. Dover, New York.

Lekhnitskii, S.G., 1968. Anisotropic Plates. Gordon & Breach, New York.

Love, A.E.H., 1927. A Treatise on the Mathematical Theory of Elasticity, 4th Edition. University Press, Cambridge.

Makow, D.M., 1969. Ultrasonic detection of defects in wood. Non-destructive Testing 2, 197–199.

Mattheck, C., Kubler, H., 1997. Wood: the Internal Optimization of Trees. Springer, Berlin.

Musgrave, M.J.P., 1970. Crystal Acoustics. Holden-Day, San Francisco.

Okyere, J.G., Cousin, A.J., 1980. On flaw detection in live wood. Mater. Evaluation 38, 3, 43–47.

Patton-Mallory, M., De Groot, R.C., 1989. Acousto-ultrasonics for evaluating decayed wood products. Proceedings of the Second Pacific Timber Engineering Conference, University of Auckland, Vol. 2, pp. 185–190.

Ross, R.J., Pellerin, R.F., 1994. Nondestructive testing for assessing wood members in structures: a review. General Technical Report FPL-GTR-70 (Rev.), US Dept. of Agriculture, Forest Service, Forest Products Laboratory, Madison, Wisconsin, 40 pp.

Ting, T.C.T., 1996a. Pressuring, shearing, torsion and extension of a circular tube or bar of cylindrically anisotropic material. Proc. Roy. Soc. A 452, 2397–2421.

Ting, T.C.T., 1996b. Anisotropic Elasticity. University Press, Oxford.

Tomikawa, Y., Iwase, Y., Arita, K., Yamada, H., 1986. Nondestructive inspection of a wooden pole using ultrasonic computed tomography. IEEE Trans. Ultrasonics Ferroelectrics Frequency Control UFFC-33, 354–358.

Watanabe, K., Payton, R.G., 1997. SH wave in a cylindrically anisotropic elastic solid. A general solution for a point source. Wave Motion 25, 197–212.

Watson, G.N., 1944. Theory of Bessel Functions, 2nd Edition. University Press, Cambridge.

Yuan, F.G., Hsieh, C.C., 1998. Three-dimensional wave propagation in composite cylindrical shells. Composite Struct. 42, 153–167.