

On spherical-wave scattering by a spherical scatterer and related near-field inverse problems

C. ATHANASIADIS

*Department of Mathematics, University of Athens, Panepistimiopolis, GR 15784
Athens, Greece*

P. A. MARTIN

*Department of Mathematical and Computer Sciences, Colorado School of Mines,
Golden, CO 80401-1887, USA*

AND

I. G. STRATIS

*Department of Mathematics, University of Athens, Panepistimiopolis, GR 15784
Athens, Greece*

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A spherical acoustic wave is scattered by a bounded obstacle. A generalization of the ‘optical theorem’ (which relates the scattering cross-section to the far-field pattern in the forward direction for an incident plane wave) is proved. For a spherical scatterer, low-frequency results are obtained by approximating the known exact solution (separation of variables). In particular, a closed-form approximation for the scattered wavefield at the source of the incident spherical wave is obtained. This leads to the explicit solution of some simple near-field inverse problems, where both the source and coincident receiver are located at several points in the vicinity of a small sphere.

Keywords: near field inverse problems; optical theorem; small spherical scatterer; spherical acoustic waves.

1. Introduction

The interaction of an incident wavefield with a bounded three-dimensional obstacle is a classic problem in scattering theory. For an obstacle with a smooth boundary S , and time-harmonic waves, it is well known that the corresponding boundary-value problems for the Helmholtz equation can be reduced to boundary integral equations over S (Colton & Kress, 1983). This reduction is essentially independent of the form of the incident wavefield. Alternatively, for special geometries, such as a sphere, one can solve the boundary-value problem by the method of separation of variables; for this method to be effective, one has to be able to expand the given incident wavefield in terms of the appropriate separated solutions of the Helmholtz equation.

In fact, the vast majority of the literature is concerned with incident plane waves. However, in some recent papers, Dassios and his co-workers have studied incident waves generated by a point source in the vicinity of the scatterer; see, for example, Dassios & Kamvyssas (1995, 1997); Charalambopoulos & Dassios (1999); Dassios & Kleinman

(2000). As they note, such incident wavefields are more readily realized in practice, and they introduce an extra parameter (the distance of the source from the scatterer) which may be exploited in inverse problems.

The *standard inverse problem* is to recover the shape of an obstacle from a knowledge of the far-field pattern due to incident plane waves in various directions (Colton & Kress, 1992). In other words, the plane waves come in from infinity and the complex amplitude of the scattered spherical wave is measured at infinity. We shall consider a new class of *near-field inverse problem*: we generate waves from a point source in the vicinity of an obstacle and we measure the scattered field received *at the point-source location*. This may be regarded as a near-field version of attempting to determine the shape of an obstacle from back-scattered data.

There is some previous work on near-field inverse problems, in which the incident field is generated by point sources (or line sources in two dimensions) (Alves, 1998; Alves & Ribeiro, 1999). Coyle (2000) has used line sources to locate buried objects. See also the recent review article by Colton *et al.* (2000), where further references are given. Other near-field problems are discussed briefly by Colton & Kress (1992, p. 133). Point sources have been used by Potthast (1996, 1998) to solve the standard inverse problem (incident plane waves, far-field data).

We start with the direct problem for ‘small scatterers’, motivated by the work of Dassios & Kamvysas (1995). They developed a low-frequency theory for arbitrary smooth scatterers, which they then specialized to small spherical scatterers. Here, we note that if spheres are of primary interest, then one can solve the boundary-value problem for the Helmholtz equation exactly; this solution for point-source insonification is well known (Sengupta, 1969), and it is an exact Green’s function for related boundary-value problems. From this solution, one can easily extract the low-frequency results obtained in Dassios & Kamvysas (1995). These include an approximation for the scattering cross-section, σ .

For a scatterer of any shape and plane-wave incidence, it is known that σ satisfies a formula known as the ‘optical theorem’: this relates σ to the far-field pattern in the forward direction. Here, we derive an analogous formula for point-source incident wavefields. This new formula relates σ to the scattered field at the source point and a certain Herglotz wavefunction (Colton & Kress, 1992). Note that Herglotz functions are ubiquitous in the analysis of inverse problems.

We then consider inverse problems for a small sphere. Dassios & Kamvysas (1995) consider such problems, where one measures σ for various point-source locations. However, σ itself is a far-field quantity. Here, we investigate the possibility of using the scattered field *at the source point*, for various point-source locations. We find the magnitude of the scattered field at the source point, correct to order $(ka)^2$, where k is the acoustic wavenumber and a is the radius of the sphere. The asymptotics are interesting in themselves, because one has to sum an infinite number of contributions to calculate the total contribution at any given power of ka ; thus, for a sound-soft sphere and a point source at distance r_0 from the sphere centre, we find that the magnitude of the scattered field at the source point approaches $\tau/(1 - \tau^2)$ as $ka \rightarrow 0$, where $\tau = a/r_0$. Such results can then be used to solve various simple inverse problems.

It would be of interest to develop the theory of near-field inverse problems further. Note that, at a fixed frequency, the standard inverse problem involves four-dimensional data (directions of incident wave and observation) whereas the back-scattered version

gives two-dimensional data. The near-field problem discussed here is intermediate, as the location of the point source gives three-dimensional data.

2. Formulation

Consider a bounded three-dimensional obstacle B with a smooth closed boundary S , surrounded by a compressible fluid. Choose an origin O in the vicinity of B . We consider an incident spherical wavefield due to a point source at P_0 , with position vector \mathbf{r}_0 with respect to O . Following Dassios & Kamvyssas (1995), we take this incident field as

$$u_{\text{in}}(\mathbf{r}; \mathbf{r}_0) = \frac{r_0}{R} e^{ik(R-r_0)} = \frac{h_0(kR)}{h_0(kr_0)}, \tag{1}$$

where $r_0 = |\mathbf{r}_0|$, $R = |\mathbf{r} - \mathbf{r}_0|$ and $h_n(w) \equiv h_n^{(1)}(w)$ is a spherical Hankel function; note that $h_0(w) = e^{iw}/(iw)$. Here, we have suppressed a time dependence of $e^{-i\omega t}$.

The form (1) is convenient because

$$u_{\text{in}}(\mathbf{r}; \mathbf{r}_0) \sim \exp(-ik\hat{\mathbf{r}}_0 \cdot \mathbf{r}) \quad \text{as } r_0 \rightarrow \infty, \tag{2}$$

where $\hat{\mathbf{r}}_0 = \mathbf{r}_0/r_0$ is a unit vector, so that the incident field reduces to a plane wave propagating in the direction from P_0 towards O as the point source recedes to infinity. On the other hand, if we fix P_0 , we have

$$u_{\text{in}}(\mathbf{r}; \mathbf{r}_0) \sim f_{\text{in}}(\hat{\mathbf{r}}; \mathbf{r}_0) h_0(kr) \quad \text{as } r \rightarrow \infty,$$

where

$$f_{\text{in}}(\hat{\mathbf{r}}; \mathbf{r}_0) = ikr_0 \exp\{-ikr_0(1 + \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_0)\} \tag{3}$$

is the far-field pattern of the point-source incident wavefield. Thus, u_{in} satisfies the Sommerfeld radiation condition at infinity (with respect to \mathbf{r}).

We want to calculate the scattered field u , where u satisfies the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0,$$

everywhere in the fluid, the radiation condition at infinity and a boundary condition on S . For simplicity, we assume that the obstacle is sound-soft, so that the total field $u_t \equiv u + u_{\text{in}}$ vanishes on S . We assume that k^2 is a real constant.

The solution of this problem, $u_t(\mathbf{r}; \mathbf{r}_0)$, is the exact Green's function for the soft obstacle B . In general, it can be constructed by solving a boundary integral equation over S . It can then be used as a fundamental solution for more complicated problems; see Martin & Rizzo (1995) for more information.

The behaviour of the scattered waves in the far field is given by

$$u(\mathbf{r}; \mathbf{r}_0) \sim f(\hat{\mathbf{r}}; \mathbf{r}_0) h_0(kr) \quad \text{as } r \rightarrow \infty, \tag{4}$$

where f is the far-field pattern. From f , we can calculate the scattering cross-section σ_0 :

$$\sigma_0 = \frac{1}{k^2} \int_{S^2} |f(\hat{\mathbf{r}}; \mathbf{r}_0)|^2 ds(\hat{\mathbf{r}}), \tag{5}$$

where S^2 is the unit sphere.

For plane-wave incidence, it can be shown that the scattering cross-section, σ_p say, is related to f evaluated in the forward direction. This is sometimes known as the ‘optical theorem’. In the next section, we prove an analogous result for point-source incident fields.

In the sequel, we will sometimes write $u(\mathbf{r})$ for $u(\mathbf{r}; \mathbf{r}_0)$, for example, leaving the dependence on the point-source location as implicit.

3. An optical theorem for point-source insonification

Consider a volume B_r bounded internally by S and externally by a large sphere S_r centred at the origin with radius r large enough to include the scatterer B in its interior. We also exclude a small ball centred on the source point P_0 ; the boundary of this ball is a sphere S_ε of radius ε .

Apply Green’s theorem in B_r to u_t and \bar{u}_t , where the overbar denotes complex conjugation. As u_t and \bar{u}_t both satisfy the same Helmholtz equation in B_r , and they both satisfy homogeneous boundary conditions on S , we obtain

$$I(S_r) + I(S_\varepsilon) = 0, \quad (6)$$

where

$$I(S) = \int_S \left(u_t \frac{\partial \bar{u}_t}{\partial n} - \bar{u}_t \frac{\partial u_t}{\partial n} \right) dS$$

and $\partial/\partial n$ denotes normal differentiation in the outward direction with respect to B_r .

As $u_t \sim f_t h_0(kr)$ as $r \rightarrow \infty$, with $f_t \equiv f + f_{\text{in}}$, we find that

$$\begin{aligned} \lim_{r \rightarrow \infty} I(S_r) &= \frac{2}{ik} \int_{S^2} |f_t(\hat{\mathbf{r}}; \mathbf{r}_0)|^2 ds(\hat{\mathbf{r}}) \\ &= \frac{2}{ik} \left\{ k^2 \sigma_0 + \int_{S^2} |f_{\text{in}}|^2 ds(\hat{\mathbf{r}}) + 2 \operatorname{Re} \int_{S^2} f \bar{f}_{\text{in}} ds(\hat{\mathbf{r}}) \right\} \\ &= \frac{2}{ik} \left(k^2 \sigma_0 + 4\pi k^2 r_0^2 + 2kr_0 \operatorname{Im} \left\{ e^{ikr_0} \int_{S^2} f(\hat{\mathbf{r}}; \mathbf{r}_0) \exp(ik\hat{\mathbf{r}} \cdot \mathbf{r}_0) ds(\hat{\mathbf{r}}) \right\} \right) \end{aligned} \quad (7)$$

where we have used (3) and (5).

Next, consider $I(S_\varepsilon)$. From (1), we find that

$$u_t \frac{\partial \bar{u}_t}{\partial n} = \frac{r_0^2}{\varepsilon^3} + \frac{1}{\varepsilon^2} \{ ikr_0^2 + r_0 e^{ikr_0} u(\mathbf{r}_0; \mathbf{r}_0) \} + O(\varepsilon^{-1})$$

as $\varepsilon \rightarrow 0$. The most singular term, r_0^2/ε^3 , is real, so that the integrand in $I(S_\varepsilon)$ is $O(\varepsilon^{-2})$ as $\varepsilon \rightarrow 0$. As $dS = O(\varepsilon^2)$, we see that

$$\lim_{\varepsilon \rightarrow 0} I(S_\varepsilon) = 8\pi i r_0 (kr_0 + \operatorname{Im}\{e^{ikr_0} u(\mathbf{r}_0; \mathbf{r}_0)\}).$$

Combining this formula with (6) and (7), we obtain

$$\sigma_0 = \frac{2r_0}{k} \operatorname{Im} \left\{ e^{ikr_0} \left[2\pi u(\mathbf{r}_0; \mathbf{r}_0) - \int_{S^2} f(\hat{\mathbf{r}}; \mathbf{r}_0) \exp(ik\hat{\mathbf{r}} \cdot \mathbf{r}_0) ds(\hat{\mathbf{r}}) \right] \right\}. \quad (8)$$

This is the analogue of the optical theorem for a point-source incident wavefield. It shows that the scattering cross-section due to a point source at \mathbf{r}_0 is related to the scattered field at \mathbf{r}_0 and a Herglotz wavefunction with Herglotz kernel f .

It is of interest to examine the behaviour of (8) as the point source recedes to infinity. From (2), we know that $f(\widehat{\mathbf{r}}; \mathbf{r}_0) \sim f_p(\widehat{\mathbf{r}}; -\widehat{\mathbf{r}}_0)$ as $r_0 \rightarrow \infty$, where $f_p(\widehat{\mathbf{r}}; \widehat{\mathbf{p}})$ is the far-field pattern in the direction $\widehat{\mathbf{r}}$ due to a plane wave propagating in the direction $\widehat{\mathbf{p}}$. Hence

$$u(\mathbf{r}_0; \mathbf{r}_0) \sim f_p(\widehat{\mathbf{r}}_0; -\widehat{\mathbf{r}}_0) h_0(kr_0) \quad \text{as } r_0 \rightarrow \infty. \tag{9}$$

Next, consider the integral in (8) for large kr_0 . Choose spherical polar coordinates (θ, φ) on S^2 so that $\widehat{\mathbf{r}} \cdot \widehat{\mathbf{r}}_0 = \cos \theta$, and define

$$F(\theta; \mathbf{r}_0) = \int_0^{2\pi} f(\widehat{\mathbf{r}}; \mathbf{r}_0) d\varphi.$$

In particular, we have

$$F(0; \mathbf{r}_0) = 2\pi f(\widehat{\mathbf{r}}_0; \mathbf{r}_0) \quad \text{and} \quad F(\pi; \mathbf{r}_0) = 2\pi f(-\widehat{\mathbf{r}}_0; \mathbf{r}_0). \tag{10}$$

Hence

$$\begin{aligned} \int_{S^2} f(\widehat{\mathbf{r}}; \mathbf{r}_0) \exp(ik\widehat{\mathbf{r}} \cdot \mathbf{r}_0) ds(\widehat{\mathbf{r}}) &= \int_0^\pi F(\theta; \mathbf{r}_0) e^{ikr_0 \cos \theta} \sin \theta d\theta \\ &= \frac{i}{kr_0} \int_0^\pi F(\theta; \mathbf{r}_0) \frac{d}{d\theta} (e^{ikr_0 \cos \theta}) d\theta \\ &\sim \frac{i}{kr_0} \{F(\pi; \mathbf{r}_0) e^{-ikr_0} - F(0; \mathbf{r}_0) e^{ikr_0}\} \\ &\sim \frac{2\pi i}{kr_0} \{f_p(-\widehat{\mathbf{r}}_0; -\widehat{\mathbf{r}}_0) e^{-ikr_0} - f_p(\widehat{\mathbf{r}}_0; -\widehat{\mathbf{r}}_0) e^{ikr_0}\} \end{aligned}$$

for large r_0 . Thus, when we combine this formula with (9) in the right-hand side of (8), we see that two terms cancel, so that (8) reduces to

$$\sigma_p = -4\pi k^{-2} \operatorname{Re}\{f_p(-\widehat{\mathbf{r}}_0; -\widehat{\mathbf{r}}_0)\},$$

which is the standard optical theorem for plane-wave incidence; here

$$\sigma_p = \frac{1}{k^2} \int_{S^2} |f_p(\widehat{\mathbf{r}}; -\widehat{\mathbf{r}}_0)|^2 ds(\widehat{\mathbf{r}})$$

is the scattering cross-section for plane-wave incidence.

4. Exact Green’s function for a soft sphere

Dassios & Kamvyssas (1995) have developed a low-frequency asymptotic theory for point-source insonification of an arbitrary obstacle B . They then calculated explicit results from their theory for the special case of a sphere. Here we obtain these results more readily by approximating the known exact solution for point-source insonification of a sphere

(Sengupta, 1969, Section 10.2.1). (We note that Charalambopoulos & Dassios (1999) have recently obtained low-frequency results for a small ellipsoid.)

Consider a spherical scatterer of radius a . Take spherical polar coordinates (r, θ, φ) with the origin at the centre of the sphere so that the point source is at $r = r_0, \theta = 0$. Thus, the incident field u_{in} and the scattered field u are axisymmetric. From (1) and Abramowitz & Stegun (1965, equations 10.1.45 and 10.1.46), we have the expansion

$$u_{\text{in}}(r, \theta; r_0) = \sum_{n=0}^{\infty} (2n+1) j_n(kr) \mathcal{H}_n(kr_0) P_n(\cos \theta) \quad \text{for } r < r_0, \quad (11)$$

where $j_n(w)$ is a spherical Bessel function, $P_n(x)$ is a Legendre polynomial and

$$\mathcal{H}_n(w) = h_n(w)/h_0(w).$$

Note that $\mathcal{H}_0 = 1, \mathcal{H}_1 = w^{-1} - i$,

$$\mathcal{H}_n(w) \sim c_n w^{-n} \quad \text{as } w \rightarrow 0, \quad (12)$$

where $c_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ with $c_0 = 1$, and

$$\mathcal{H}_n(w) \sim (-i)^n \quad \text{as } w \rightarrow \infty. \quad (13)$$

The scattered field has a similar expansion to (11); taking the radiation condition into account, we have

$$u(r, \theta; r_0) = \sum_{n=0}^{\infty} (2n+1) A_n h_n(kr) \mathcal{H}_n(kr_0) P_n(\cos \theta) \quad \text{for } r \geq a,$$

where the coefficients A_n are determined from the boundary condition on $r = a$; for a sound-soft sphere,

$$A_n = -j_n(ka)/h_n(ka). \quad (14)$$

From (4) and (13), we find that the far-field pattern is given by

$$f(\hat{\mathbf{r}}; r_0) = \sum_{n=0}^{\infty} (2n+1)(-i)^n A_n \mathcal{H}_n(kr_0) P_n(\cos \theta) \quad (15)$$

and then (5) gives the scattering cross-section as

$$\sigma_0 = 4\pi k^{-2} \sum_{n=0}^{\infty} (2n+1) |A_n \mathcal{H}_n(kr_0)|^2. \quad (16)$$

We are also interested in the scattered waves received at the source point. This field is given by

$$u(\mathbf{r}_0) \equiv u(r_0, 0; r_0) = h_0(kr_0) \sum_{n=0}^{\infty} (2n+1) A_n [\mathcal{H}_n(kr_0)]^2. \quad (17)$$

All the formulae above are exact. In the rest of the paper, we assume that the sphere is small, $ka \ll 1$.

5. Far-field results for a small soft sphere

In the asymptotic results to follow, there are two parameters, namely

$$\kappa = ika \quad \text{and} \quad \tau = a/r_0.$$

We assume that $|\kappa| = ka \ll 1$ and note that the geometrical parameter τ satisfies $0 < \tau < 1$.

From (14), we have

$$A_n \sim \frac{(ka)^{2n+1}}{i(2n+1)c_n^2} \quad \text{as } ka \rightarrow 0, \text{ for } n = 0, 1, 2, \dots \quad (18)$$

Explicitly, with $\kappa = ika$,

$$\begin{aligned} A_0 &= -\kappa(1 - \kappa + \frac{2}{3}\kappa^2) + O(\kappa^4), \\ A_1 &= \frac{1}{3}\kappa^3 + O(\kappa^5), \\ A_2 &= -\frac{1}{45}\kappa^5 + O(\kappa^7) \end{aligned} \quad (19)$$

as $\kappa \rightarrow 0$.

Let us use these approximations to calculate f with an error of $O(\kappa^4)$. We have $A_0\mathcal{H}_0 = A_0$ and

$$\begin{aligned} A_1\mathcal{H}_1(kr_0) &= (\frac{1}{3}\kappa^3 + O(\kappa^5))(kr_0)^{-1} - i \\ &= \frac{1}{3}i\kappa^2(\tau - \kappa) + O(\kappa^4), \end{aligned}$$

with $\tau = a/r_0$. More generally, from (12) and (18), we have

$$A_n\mathcal{H}_n(kr_0) \sim \frac{(ka)^{n+1}\tau^n}{i(2n+1)c_n} \quad \text{as } ka \rightarrow 0, \quad (20)$$

so that we should retain A_2 . Thus, (15) gives the approximation

$$f(\hat{\mathbf{r}}; r_0) = -\kappa + \kappa^2[1 + \tau P_1(\cos \theta)] - \kappa^3[\frac{2}{3} + P_1(\cos \theta) + \frac{1}{3}\tau^2 P_2(\cos \theta)] + O(\kappa^4)$$

as $\kappa \rightarrow 0$, in agreement with Dassios & Kamvyssas (1995, equation (54)).

To obtain σ_0 to the same accuracy, (16) and (20) show that we only require A_0 and A_1 , giving the result

$$\sigma_0 = 4\pi a^2\{1 + \frac{1}{3}\kappa^2(1 - \tau^2)\} + O(\kappa^4)$$

as $\kappa \rightarrow 0$, in agreement with Dassios & Kamvyssas (1995, equation (55)).

6. Near-field results for a small soft sphere

The scattered field at the source point $u(\mathbf{r}_0)$ is given by (17). Let us evaluate $u(\mathbf{r}_0)$ for small ka . From (12) and (18) we have

$$A_n[\mathcal{H}_n(kr_0)]^2 \sim -(2n+1)^{-1}\kappa\tau^{2n},$$

so that *every* term in the infinite series for $u(\mathbf{r}_0)$ contributes to the leading-order behaviour. Specifically, we have

$$u(\mathbf{r}_0) \sim -\kappa h_0(kr_0) \sum_{n=0}^{\infty} \tau^{2n} = \frac{-\kappa h_0(kr_0)}{1 - \tau^2},$$

after summing the geometric series. In particular,

$$|u(\mathbf{r}_0)| \sim \frac{(a/r_0)}{1 - (a/r_0)^2} \quad \text{as } ka \rightarrow 0. \quad (21)$$

These results are uniformly valid in the geometrical parameter $\tau = a/r_0$.

Let us obtain higher-order terms, and calculate $|u(\mathbf{r}_0)|$ correct to $O((ka)^2)$. As $\mathcal{H}_0 = 1$, (19) gives

$$A_0[\mathcal{H}_0(kr_0)]^2 = -\kappa(1 - \kappa + \frac{2}{3}\kappa^2) + O(\kappa^4)$$

as $ka \rightarrow 0$. From Abramowitz & Stegun (1965, equation 10.1.16), we have

$$\mathcal{H}_n(w) = (-i)^n \sum_{j=0}^n (n + \frac{1}{2}, j) (-2iw)^{-j}, \quad (22)$$

where $(n + \frac{1}{2}, j) = [(n + j)! / (j! (n - j)!)]$. We have

$$(n + \frac{1}{2}, n) = 2^n c_n, \quad (n + \frac{1}{2}, n - 1) = 2^{n-1} c_n \quad \text{and} \quad (n + \frac{1}{2}, n - 2) = \frac{2^{n-2}(n-1)c_n}{2n-1},$$

where $c_n = 1 \cdot 3 \cdot 5 \cdots (2n - 1)$ and $c_0 = 1$. Hence

$$\mathcal{H}_n(w) \sim \frac{c_n}{w^n} \left(1 - iw - \frac{n-1}{2n-1} w^2 \right) \quad \text{as } w \rightarrow 0$$

for $n \geq 1$. (The right-hand side gives \mathcal{H}_1 and \mathcal{H}_2 exactly.) Squaring gives

$$[\mathcal{H}_n(w)]^2 \sim \frac{c_n^2}{w^{2n}} \left(1 - 2iw - \frac{4n-3}{2n-1} w^2 \right) \quad \text{as } w \rightarrow 0, n \geq 1. \quad (23)$$

Also, for $n \geq 1$, we have

$$A_n = -j_n(ka) / h_n(ka) \sim i j_n(ka) / y_n(ka) \quad \text{as } ka \rightarrow 0,$$

where Abramowitz & Stegun (1965, equations 10.1.2 and 10.1.3)

$$j_n(w) \sim \frac{w^n}{c_{n+1}} \left(1 - \frac{\frac{1}{2}w^2}{2n+3} \right) \quad \text{and} \quad y_n(w) \sim \frac{-c_n}{w^{n+1}} \left(1 + \frac{\frac{1}{2}w^2}{2n-1} \right),$$

whence

$$A_n \sim \frac{-i(ka)^{2n+1}}{(2n+1)c_n^2} \left\{ 1 - \frac{(2n+1)(ka)^2}{(2n+3)(2n-1)} \right\} \quad \text{as } ka \rightarrow 0, n \geq 1. \quad (24)$$

Combining (23) and (24) gives

$$A_n[\mathcal{H}_n(kr_0)]^2 \sim \frac{-\kappa \tau^{2n}}{2n+1} \left\{ 1 - 2\frac{\kappa}{\tau} + \frac{\kappa^2}{2n-1} \left(\frac{2n+1}{2n+3} + \frac{4n-3}{\tau^2} \right) \right\}$$

whence (17) gives

$$u(\mathbf{r}_0) = -\kappa h_0(kr_0) \left\{ \frac{1}{1-\tau^2} - \kappa \left(1 + \frac{2\tau}{1-\tau^2} \right) + \kappa^2 Q_1 + O(\kappa^3) \right\} \quad (25)$$

as $ka \rightarrow 0$, where

$$\begin{aligned} Q_1 &= \frac{2}{3} + \sum_{n=1}^{\infty} \frac{\tau^{2n}}{2n-1} \left(\frac{2n+1}{2n+3} + \frac{4n-3}{\tau^2} \right) \\ &= \frac{\tau^2-1}{2\tau^2} + 2 \sum_{n=0}^{\infty} \tau^{2n} + \frac{(1-\tau^2)^2}{2\tau^3} \sum_{n=0}^{\infty} \frac{\tau^{2n+1}}{2n+1} \\ &= \frac{\tau^2-1}{2\tau^2} + \frac{2}{1-\tau^2} + \frac{(1-\tau^2)^2}{4\tau^3} \log \left(\frac{1+\tau}{1-\tau} \right). \end{aligned}$$

Again, (25) is uniformly valid in τ . In particular, one can verify that letting $\tau \rightarrow 0$ ($r_0 \rightarrow \infty$) recovers the known result for plane-wave incidence.

From (25), we have

$$|u(\mathbf{r}_0)|^2 = \tau^2 \{ (1-\tau^2)^{-2} + (ka)^2 Q_2 \} + O((ka)^3)$$

as $ka \rightarrow 0$, where

$$\begin{aligned} Q_2 &= \left(1 + \frac{2\tau}{1-\tau^2} \right)^2 - \frac{2Q_1}{1-\tau^2} \\ &= 1 + \frac{1}{\tau^2} - \frac{4}{1+\tau} - \frac{1-\tau^2}{2\tau^3} \log \left(\frac{1+\tau}{1-\tau} \right). \end{aligned}$$

Hence

$$|u(\mathbf{r}_0)| = \frac{\tau}{1-\tau^2} \left\{ 1 + \frac{1}{2}(ka)^2(1-\tau^2)^2 Q_2 \right\} + O((ka)^3) \quad \text{as } ka \rightarrow 0. \quad (26)$$

7. Near-field inverse problems

Dassios and Kamvyssas have considered various inverse problems for small spherical scatterers, in which a sphere of unknown radius and location is identified from a knowledge of the scattered field due to a point source at several known locations. The knowledge required is one or two terms in the low-frequency asymptotic expansion of the scattering cross-section; this is a far-field quantity.

As an alternative, we can consider what is perhaps a more natural and realizable experiment, where one measures the magnitude of the scattered field at the source point, so that this is a genuine near-field experiment. From (21), we know that

$$|u(\mathbf{r}_0)| \sim \tau/(1-\tau^2) \quad \text{as } ka \rightarrow 0,$$

where $\tau = a/r_0$. So, following Dassios & Kamvyssas (1995), choose a Cartesian coordinate system $Oxyz$, and five point-source locations, namely $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, 2)$, which are (unknown) distances r_0, r_1, r_2, r_3 and r_4 , respectively, from the sphere centre. For each location, measure the leading-order term in the low-frequency expansion of $|u|$ at the source point, so that we know

$$m_i = \tau_i/(1 - \tau_i^2), \quad \tau_i = a/r_i, \quad i = 0, 1, 2, 3, 4.$$

Inverting this expression gives

$$\left(\frac{r_i}{a}\right)^2 = \frac{1}{\tau_i^2} = \frac{1 + 2m_i^2 + \sqrt{1 + 4m_i^2}}{2m_i^2}, \quad i = 0, 1, 2, 3, 4.$$

Thus, we have five measurements and six unknowns, r_0, r_1, r_2, r_3, r_4 and a . However, r_0, r_3 and r_4 are related, using the cosine rule (Dassios & Kamvyssas, 1995): $r_4^2 = 2 + 2r_3^2 - r_0^2$. This gives

$$2/a^2 = (r_4/a)^2 - 2(r_3/a)^2 + (r_0/a)^2,$$

so that we can obtain a from m_0, m_3 and m_4 . We can then locate the centre of the sphere at the intersection of the four spheres centred at the first four source points. Measurability and sensitivity aspects of similar algorithms are discussed in Dassios & Kamvyssas (1995).

Similar calculations can be made for a sound-hard sphere (Neumann boundary condition). Thus, we find that the coefficients A_n should be replaced by Sengupta (1969, Section 10.3.1)

$$A_n^N = \frac{-j'_n(ka)}{h'_n(ka)} \sim \frac{in(ka)^{2n+1}}{(2n+1)(n+1)c_n^2}$$

as $ka \rightarrow 0$, for $n \geq 1$, whilst $A_0^N \sim -\frac{1}{3}i(ka)^3$. Hence, we find, for example, that

$$u(\mathbf{r}_0) \sim \kappa h_0(kr_0) \sum_{n=1}^{\infty} \frac{n\tau^{2n}}{n+1} = \kappa h_0(kr_0) \left\{ \frac{1}{1-\tau^2} + \frac{1}{\tau^2} \log(1-\tau^2) \right\}$$

whence

$$|u(\mathbf{r}_0)| \sim \tau/(1-\tau^2) + \tau^{-1} \log(1-\tau^2) \quad \text{as } ka \rightarrow 0.$$

One can use this formula to solve similar inverse problems for a hard sphere.

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