Acoustic scattering by inhomogeneous spheres

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(Received 6 March 2001; revised 19 December 2001; accepted 20 February 2002)

Acoustic scattering problems are considered when the material parameters (density ρ and speed of sound c) are spherically symmetric functions of position. Explicit separated solutions are derived (i) when $\rho(r)=\exp(\beta r)$ and c^{-2} is a linear function of r^{-1}, and (ii) when $\rho(r)=\exp(-\beta r^2)$ and c^{-2} is a linear function of r^2. In both cases, the radial parts of the solutions are given in terms of Coulomb wave functions or Whittaker functions; these are well-studied special functions, closely related to confluent hypergeometric functions. Two problems are discussed in detail: scattering by an inhomogeneous sphere embedded in a homogeneous fluid, and scattering by a homogeneous sphere with a concentric inhomogeneous coating.
[DOI: 10.1121/1.1470502]

PACS numbers: 43.20.Fn [ANN]

I. INTRODUCTION

Sound propagation in inhomogeneous media continues to be of interest. Much is known about layered, stratified media, where the properties of the media (density ρ and speed of sound c) depend on one coordinate, z, say, where x, y, and z are Cartesian coordinates (Brekhovskikh, 1960). Such situations have obvious application to underwater acoustics.

For time-harmonic motions of frequency ω the governing equation is

$$\rho \operatorname{div}(\rho^{-1} \operatorname{grad} p)+k^2 p=0, \tag{1}$$

where $k=\omega/c$ and p is the acoustic pressure. If the density is constant, Eq. (1) reduces to

$$\nabla^2 p+k_0^2 n(\boldsymbol{r})p=0, \tag{2}$$

where $\nabla^2 \equiv \operatorname{div}\operatorname{grad}$, $n(\boldsymbol{r})=[c_0/c(\boldsymbol{r})]^2$ is the (square of the) *refractive index* at position $\boldsymbol{r}$, $k_0=\omega/c_0$, and c_0 is a constant sound speed. We assume that $n(\boldsymbol{r})\to 1$ as $r\to\infty$ in all directions; this excludes layered media, of course.

There is a considerable literature on Eq. (2), especially in the context of quantum mechanics; some of this will be mentioned below. In addition, several point-source solutions (Green's functions) are known for various functional forms of $n(\boldsymbol{r})$; see Li *et al.* (1990) for a review.

For scattering problems in acoustics, there are essentially three cases, depending on properties of n. First, suppose that $[1-n(\boldsymbol{r})]$ has compact support, so that $n(\boldsymbol{r})\equiv 1$ for $r=|\boldsymbol{r}|>a$, say. Suppose further that $n(\boldsymbol{r})$ is smooth for all $\boldsymbol{r}$ in three-dimensional space. Then, one can reduce the scattering problem to an integral equation over D, the finite region in which $n(\boldsymbol{r})\not\equiv 1$. One such is the Lippmann–Schwinger equation (see, for example, Ahner, 1977; Colton and Kress, 1992, Sec. 8.2; and Newton, 1982, Sec. 10.3). Asymptotic approximations are available for $k_0 a \ll 1$ (Ahner, 1977; Kriegsmann *et al.*, 1983); Kriegsmann *et al.* (1983) also discuss Eq. (1) briefly.

Second, we could have situations in which $[1-n(\boldsymbol{r})]$ does *not* have compact support, but is such that $n(\boldsymbol{r})\to 1$ as $r\to\infty$. The corresponding scattering problems are uncommon in acoustics.

Third, we could have $n(\boldsymbol{r})\equiv 1$ outside D with n discontinuous across ∂D, the boundary of D. The corresponding scattering problem will require transmission conditions across the interface ∂D. If the material in D is actually homogeneous, so that $n(\boldsymbol{r})=n_1$, a constant, for all $\boldsymbol{r}\in D$, boundary integral equations over ∂D can be used; see Kleinman and Martin (1988) for a review.

In this paper, we are mainly concerned with this third class of problem: acoustic scattering by a bounded inhomogeneity embedded in an unbounded homogeneous medium. We begin (Sec. II) with a derivation of the partial differential equation (1): this equation governs the acoustic pressure in an otherwise stationary but inhomogeneous compressible fluid. We give this derivation because some textbook discussions are flawed. We then suppose that the inhomogeneity is *spherically symmetric*, so that ρ and c are assumed to be given functions of the spherical polar coordinate r (only). Such problems have been studied by several authors. For problems of the first type [smooth n with $n(r)\equiv 1$ for $r>a$], Ahner (1977) has given low-frequency expansions. For the second type [smooth n, with $n(r)\to 1$ rapidly as $r\to\infty$], Colton (1978) has used so-called "transformation operators," which map solutions of the Helmholtz equation

$$(\nabla^2+k_0^2)u=0, \tag{3}$$

into solutions of Eq. (2); see also Colton and Kress (1978). Colton and Kress (1979) have extended this approach to consider related transmission problems; see also Sleeman (1980). Frisk and DeSanto (1970) have exploited the notion of a Jost function from quantum mechanics so as to obtain approximate solutions of Eq. (2).

We consider acoustic scattering by an inhomogeneous sphere of a radius a. The medium in $r>a$ is homogeneous, with density ρ_0 and sound speed c_0. For time-harmonic motions, the acoustic pressure p_0 is governed by Eq. (3). Inside

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the sphere, $r<a$, the governing equation is Eq. (1). Across $r=a$, we impose continuity of pressure and normal velocity, the latter condition being equivalent to continuity of $\rho^{-1}(\partial p/\partial r)$. This gives a *transmission problem*. If the interior is homogeneous, with $\rho=\rho_1$ and $c=c_1$ constants, the problem can be solved exactly, by separation of variables. We will show that this method can be extended to certain functional forms for $\rho(r)$ and $c(r)$. We also consider scattering by a sphere with a homogeneous core and an inhomogeneous coating.

There are many papers on analogous electromagnetic scattering problems. For an early treatment, see the paper by Wyatt (1962). One popular technique is to replace the inhomogeneous sphere by many concentric layers, and then to use a simple approximation to the refractive index within each layer; see, for example, Perelman (1996). Kai and Massoli (1994) have reported the results of computations with as many as 10 000 layers.

In this paper, we consider the following specific functional forms

(i) $\rho(r)=\rho_1 e^{\beta r}$ and $[k(r)]^2=k_1^2+\alpha r^{-1}$,

(ii) $\rho(r)=\rho_1 e^{-\beta r^2}$ and $[k(r)]^2=k_1^2+\alpha r^2$.

Here, ρ_1, β, k_1^2, and α are adjustable parameters. For both (i) and (ii), explicit solutions of Eq. (1) are derived. The radial parts of these solutions are given in terms of known special functions, namely Coulomb wave functions and Whittaker functions. These solutions then permit the explicit solution of various scattering problems for inhomogeneous spheres. Such solutions can serve as benchmarks for numerical methods, but they also have intrinsic interest.

II. GOVERNING EQUATIONS

In the linear theory of acoustics for an inhomogeneous medium, the basic equation is (Morse and Ingard, 1986, p. 408)

$$\rho\,\mathrm{div}(\rho^{-1}\,\mathrm{grad}\,P)=c^{-2}(\partial^2 P/\partial t^2), \tag{4}$$

where $P(\boldsymbol{r},t)$ is the acoustic pressure at position $\boldsymbol{r}$ and time t, $\rho(\boldsymbol{r})$ is the density, and $c(\boldsymbol{r})$ is the speed of sound. For time-harmonic motions, with $P=\mathrm{Re}\{pe^{-i\omega t}\}$, we obtain Eq. (1).

According to Pierce (1990), Eq. (4) was first given by Bergmann (1946). However, the *derivation* of Eq. (4) does not seem to be well known. In particular, Eq. (4) cannot be derived without mentioning the entropy.

A. Derivation of Bergmann's equation

The exact equations for the motion of an inviscid compressible fluid are (Batchelor, 1967, Sec. 3.6; Ostashev, 1997, Sec. 2.1.1)

$$\frac{\mathrm{D}\tilde{\rho}}{\mathrm{D}t}+\tilde{\rho}\,\mathrm{div}\,\tilde{\boldsymbol{v}}=0, \tag{5}$$

$$\tilde{\rho}\frac{\mathrm{D}\tilde{\boldsymbol{v}}}{\mathrm{D}t}+\mathrm{grad}\,\tilde{P}=\boldsymbol{0}, \tag{6}$$

$$\frac{\mathrm{D}\tilde{S}}{\mathrm{D}t}=0. \tag{7}$$

Here, $\tilde{\rho}$ is the density, $\tilde{\boldsymbol{v}}$ is the velocity, $\tilde{P}$ is the pressure, and $\tilde{S}$ is the entropy per unit mass; all these quantities may depend on $\boldsymbol{r}$ and t. They will be related to the quantities in Eq. (4) later. The material derivative is defined by $(\mathrm{D}f/\mathrm{D}t)(\boldsymbol{r},t)=\partial f/\partial t+(\tilde{\boldsymbol{v}}\cdot\nabla)f$. In writing Eq. (6), we have assumed that there are no body forces. Equation (7) means that the flow is *isentropic* (Batchelor, 1967, p. 156).

We also require an equation of state. As usual, we suppose that

$$\tilde{P}=\tilde{P}(\tilde{\rho},\tilde{S}). \tag{8}$$

It follows that $\mathrm{grad}\,\tilde{P}=\tilde{c}^2\,\mathrm{grad}\,\tilde{\rho}+\tilde{h}\,\mathrm{grad}\,\tilde{S}$, where

$$\tilde{c}^2(\tilde{\rho},\tilde{S})=(\partial\tilde{P}/\partial\tilde{\rho})_{\tilde{S}} \quad\text{and}\quad \tilde{h}(\tilde{\rho},\tilde{S})=(\partial\tilde{P}/\partial\tilde{S})_{\tilde{\rho}}.$$

The temperature $\tilde{T}$ satisfies [Batchelor, 1967, Eq. (3.6.6)]

$$\tilde{\rho}\frac{\mathrm{D}\tilde{T}}{\mathrm{D}t}=\nu\tilde{T}\frac{\mathrm{D}\tilde{P}}{\mathrm{D}t},$$

where ν is the ratio of the specific heat at constant pressure to the coefficient of thermal expansion. Thus

$$\tilde{T}^{-1}\frac{\mathrm{D}\tilde{T}}{\mathrm{D}t}=\frac{\nu\tilde{c}^2}{\tilde{\rho}}\frac{\mathrm{D}\tilde{\rho}}{\mathrm{D}t}=-\nu\tilde{c}^2\,\mathrm{div}\,\tilde{\boldsymbol{v}}, \tag{9}$$

using Eqs. (5), (7), and (8).

For linear acoustics, we suppose that $\tilde{P}=P_0+P_1$, $\tilde{\boldsymbol{v}}=\boldsymbol{v}_0+\boldsymbol{v}_1$, $\tilde{\rho}=\rho_0+\rho_1$, $\tilde{S}=S_0+S_1$, $\tilde{T}=T_0+T_1$, $\tilde{c}=c_0+c_1$, and $\tilde{h}=h_0+h_1$, where the ambient flow is denoted by the subscript 0 and the small acoustic disturbance is denoted by the subscript 1. (The quantities of most interest are P_1, $\boldsymbol{v}_1$, ρ_0, S_0, and c_0.) We require that the ambient flow satisfies Eqs. (5)–(9) exactly, and then we derive a set of linear equations governing the small acoustic disturbance.

From Eq. (8), we obtain $P_0=\tilde{P}(\rho_0,S_0)$ and then

$$P_1=c_0^2\rho_1+h_0S_1, \tag{10}$$

where $c_0^2=\tilde{c}^2(\rho_0,S_0)$ and $h_0=\tilde{h}(\rho_0,S_0)$.

The leading-order equations, governing the ambient flow, follow from Eqs. (5)–(7) and (9)

$$\frac{\partial\rho_0}{\partial t}+\mathrm{div}(\rho_0\boldsymbol{v}_0)=0, \tag{11}$$

$$\rho_0\left\{\frac{\partial\boldsymbol{v}_0}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0\right\}=-\mathrm{grad}\,P_0$$
$$=-c_0^2\,\mathrm{grad}\,\rho_0-h_0\,\mathrm{grad}\,S_0, \tag{12}$$

$$\frac{\partial S_0}{\partial t}+\boldsymbol{v}_0\cdot\mathrm{grad}\,S_0=0, \tag{13}$$

$$\frac{\partial T_0}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)T_0=-\nu c_0^2T_0\,\mathrm{div}\,\boldsymbol{v}_0. \tag{14}$$

Having selected an ambient flow, the acoustic disturbance is then governed by

$$\frac{\partial\rho_1}{\partial t}+\mathrm{div}(\rho_1\boldsymbol{v}_0+\rho_0\boldsymbol{v}_1)=0, \tag{15}$$

$$\rho_0\left\{\frac{\partial \boldsymbol{v}_1}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_1+(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v}_0\right\}$$

$$+\rho_1\left\{\frac{\partial \boldsymbol{v}_0}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0\right\}=-\operatorname{grad}P_1, \tag{16}$$

$$\frac{\partial S_1}{\partial t}+\boldsymbol{v}_0\cdot\operatorname{grad}S_1+\boldsymbol{v}_1\cdot\operatorname{grad}S_0=0. \tag{17}$$

We could derive an equation for T_1, but it will not be needed below.

1. Homogeneous fluid at rest

This is the textbook case (Lighthill, 1978; Pierce, 1989; DeSanto, 1992), where the ambient flow has $\boldsymbol{v}_0=\mathbf{0}$, with P_0, ρ_0, S_0, T_0, c_0, and h_0 all constant. These choices satisfy Eqs. (11)–(14) identically. Then, Eqs. (15)–(17) reduce to

$$\frac{\partial \rho_1}{\partial t}+\rho_0\operatorname{div}\boldsymbol{v}_1=0, \tag{18}$$

$$\rho_0\frac{\partial \boldsymbol{v}_1}{\partial t}=-\operatorname{grad}P_1, \tag{19}$$

$$\frac{\partial S_1}{\partial t}=0, \tag{20}$$

together with Eq. (10). Multiplying Eq. (18) by c_0^2 and Eq. (20) by the thermodynamic coefficient h_0 and adding the results, gives

$$\frac{\partial P_1}{\partial t}+\rho_0 c_0^2\operatorname{div}\boldsymbol{v}_1=0. \tag{21}$$

Then, eliminating $\boldsymbol{v}_1$ between this equation and Eq. (19), we obtain the familiar wave equation, $\nabla^2 P_1=c_0^{-2}(\partial^2 P_1/\partial t^2)$, for the acoustic pressure. The other acoustic quantities, $\boldsymbol{v}_1$, ρ_1, and S_1, can then be calculated in terms of P_1.

In practice, the dependence on entropy is often ignored, so that Eq. (8) is replaced by $\tilde{P}=\tilde{P}(\tilde{\rho})$ when $\tilde{h}=0$ and $S_1=0$. However, we claim that entropy should be retained when the fluid is not homogeneous.

2. Constant entropy

Suppose that S_0 is constant, so that Eq. (13) is satisfied identically. Equation (12) reduces to

$$\frac{\partial \boldsymbol{v}_0}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0=-c_0^2\rho_0^{-1}\operatorname{grad}\rho_0. \tag{22}$$

If we assume that $\boldsymbol{v}_0$ is a constant vector, Eqs. (22) and (11) imply that ρ_0 is constant. Equation (12) then implies that P_0 is a function of t only. But, $P_0=\tilde{P}(\rho_0,S_0)$, so that P_0 is also a constant. Thus, constant S_0 and constant $\boldsymbol{v}_0$ imply that ρ_0 and P_0 are constant too. Also, T_0 must satisfy Eq. (14) with zero on the right-hand side.

Further remarks on the assumption of consant S_0 in the context of stratified media can be found in the book by Ostashev (1997, Sec. 2.2.4).

DeSanto (1992, Appendix 1 A) argued that the second term on the left-hand side of Eq. (22) is negligible, and then deduced that $\boldsymbol{v}_0$ cannot be constant in an inhomogeneous fluid. In fact, if the right-hand side of Eq. (22) does not depend on t, we can integrate, thus showing that $|\boldsymbol{v}_0|$ must grow linearly with t. This is an unpleasant consequence of neglecting entropy.

The derivation given by Morse and Ingard (1986, p. 408) is flawed. They begin (their first displayed equation on p. 408) with

$$0=\frac{\mathrm{D}\rho_0}{\mathrm{D}t}=\frac{\partial \rho_0}{\partial t}+\boldsymbol{v}_0\cdot\operatorname{grad}\rho_0 \tag{23}$$

(in our notation), which is incorrect; cf. Eq. (11). Next, they "add a sound wave, with its velocity **u**, its pressure p, and its additional density change δ," so that $\mathbf{u}\equiv\boldsymbol{v}_1$, $p\equiv P_1$, and $\delta\equiv\rho_1$ in our notation. The following equations are erroneous because, in their Eq. (23), they have **u** in place of $\boldsymbol{v}_0$, so that their **u** is both $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$!

3. Zero ambient velocity

Suppose, instead, that $\boldsymbol{v}_0=\mathbf{0}$. Then, Eqs. (11), (13), and (14) imply that ρ_0, S_0, T_0, c_0, and h_0 do not depend on t. Equation (12) will also be satisfied, provided that

$$c_0^2\operatorname{grad}\rho_0+h_0\operatorname{grad}S_0=\mathbf{0}. \tag{24}$$

This constraint permits us to have spatial variations in c_0^2 and ρ_0 within a stationary fluid.

For the acoustic disturbance, Eqs. (15)–(17) reduce to

$$\frac{\partial \rho_1}{\partial t}+\operatorname{div}(\rho_0\boldsymbol{v}_1)=0, \tag{25}$$

$$\rho_0\frac{\partial \boldsymbol{v}_1}{\partial t}+\operatorname{grad}P_1=\mathbf{0}, \tag{26}$$

$$\frac{\partial S_1}{\partial t}+\boldsymbol{v}_1\cdot\operatorname{grad}S_0=0. \tag{27}$$

Making use of Eq. (10), we combine Eqs. (25) and (27) to give

$$\frac{\partial P_1}{\partial t}+c_0^2\operatorname{div}(\rho_0\boldsymbol{v}_1)+h_0\boldsymbol{v}_1\cdot\operatorname{grad}S_0=0. \tag{28}$$

Using the constraint (24) so as to eliminate S_0 from Eq. (28), we obtain Eq. (21) again (except now ρ_0 and c_0^2 are not required to be constants). Finally, by eliminating $\boldsymbol{v}_1$ between Eqs. (26) and (21), we obtain Bergmann's equation (4), wherein $\rho\equiv\rho_0$, $P\equiv P_1$ and $c\equiv c_0$.

B. Reduced equations

We can reduce Bergmann's equation to an equation without first derivatives by introducing a new dependent variable (Bergmann, 1946); thus, define

$$P=\rho^{1/2}U, \tag{29}$$

whence U is found to satisfy

$$\nabla^2 U+KU=c^{-2}(\partial^2 U/\partial t^2), \tag{30}$$

where

$$K=\tfrac{1}{2}\rho^{-1}\nabla^2\rho-\tfrac{3}{4}\rho^{-2}|\mathrm{grad}\,\rho|^2 \tag{31}$$

$$=-\rho^{1/2}\nabla^2(\rho^{-1/2}). \tag{32}$$

Equations (29)–(31) [but not Eq. (32)] can be found in Brehovskikh (1960, p. 171).

Evidently, Eq. (30) can be reduced to a partial differential equation with constant coefficients if, for example, $\nabla^2(\rho^{-1/2})=\lambda\rho^{-1/2}$ and $c=c_1$, where λ and c_1 are constants.

For time-harmonic problems, we can write $U=\mathrm{Re}\{ue^{-i\omega t}\}$ and $\boldsymbol{v}=\mathrm{Re}\{\boldsymbol{w}e^{-i\omega t}\}$, whence u satisfies

$$\nabla^2u+(k^2+K)u=0, \tag{33}$$

where $k^2=\omega^2/c^2$, and then the fluid velocity is given by $\boldsymbol{w}=(i\omega\rho)^{-1}\,\mathrm{grad}(\rho^{1/2}u)$.

In a homogeneous region, we have $\rho=\rho_1$, $c=c_1$, and $k=k_1=\omega/c_1$, all constants. Then, $K\equiv 0$ and Eq. (33) reduces to the standard Helmholtz equation. If ρ is constant but c is not, we still have $K\equiv 0$ and then Eq. (33) is usually written as Eq. (2); see, for example, Colton and Kress (1992, Chap. 8).

Finally, we can write Eq. (33) as $\nabla^2u+(k_1^2-V)u=0$, where k_1^2 is a constant and $V=k_1^2-k^2-K$, which we recognize as *Schrödinger's equation* with potential V [Newton, 1982, Eq. (10.59)].

III. SPHERICAL SYMMETRY

Introduce spherical polar coordinates, r, θ, and ϕ. Assume that the inhomogeneous medium is spherically symmetric. Then, as $\nabla^2\{f(r)\}=r^{-2}[r^2f']'$ [where $f'(r)\equiv df/dr$], we find that

$$K(r)=r^{-1}(\rho'/\rho)+\tfrac{1}{2}(\rho''/\rho)-\tfrac{3}{4}(\rho'/\rho)^2.$$

Next, we seek solutions of Eq. (33) in the form

$$u(r,\theta,\phi)=u_n(r)Y_n(\theta,\phi), \tag{34}$$

where n is an integer, Y_n is a spherical harmonic, and $u_n(r)$ is to be found by substituting Eq. (34) in Eq. (33). [A typical spherical harmonic is $A_n^mP_n^m(\cos\theta)e^{im\theta}$, where P_n^m is an associated Legendre function and A_n^m is a normalization constant.]

We have

$$\nabla^2(u_nY_n)=u_n\nabla^2Y_n+2(\mathrm{grad}\,u_n)\cdot(\mathrm{grad}\,Y_n)+Y_n\nabla^2u_n. \tag{35}$$

But $(\mathrm{grad}\,u_n)\cdot(\mathrm{grad}\,Y_n)=0$ because u_n is a function of r and Y_n is a function of θ and ϕ. We also know that r^nY_n is a separated solution of Laplace's equation, so that

$$0=\nabla^2\{r^nY_n\}=r^n\nabla^2Y_n+Y_n\nabla^2\{r^n\},$$

by Eq. (35) and $\nabla^2\{r^n\}=n(n+1)r^{n-2}$ whence $\nabla^2Y_n=-n(n+1)r^{-2}Y_n$, and then Eq. (35) gives

$$\nabla^2(u_nY_n)=\{\nabla^2u_n-n(n+1)r^{-2}u_n\}Y_n.$$

Hence, Eq. (33) reduces to

$$u_n''+2r^{-1}u_n'+[k^2(r)+K(r)-n(n+1)r^{-2}]u_n=0, \tag{36}$$

which is a linear second-order differential equation for $u_n(r)$. If we have solutions of this equation, we can then use the method of separation of variables for various scattering problems involving inhomogeneous spheres. Two such problems are described next.

IV. TWO SCATTERING PROBLEMS

Acoustic scattering by spheres, with various boundary conditions, is a textbook topic (Morse and Ingard, 1986, Sec. 8.2). We shall modify the familiar method of separation of variables so as to treat inhomogeneous spheres.

Consider an inhomogeneous sphere of radius a centered at the origin. Without loss of generality, we can take the incident pressure field as

$$\frac{p_{\mathrm{inc}}}{\rho_0c_0^2}=e^{ik_0z}=\sum_{n=0}^{\infty}(2n+1)i^nj_n(k_0r)P_n(\cos\theta),$$

where $j_n(w)$ is a spherical Bessel function. Then, we can write the total pressure field outside the sphere, in $r>a$, as

$$p_0(r,\theta)=\rho_0c_0^2\sum_{n=0}^{\infty}(2n+1)i^n\{j_n(k_0r)+A_nh_n(k_0r)\}P_n(\cos\theta), \tag{37}$$

where $h_n(w)\equiv h_n^{(1)}(w)$ is a spherical Hankel function and the dimensionless coefficients A_n are to be found. This expression for p_0 satisfies the Helmholtz equation and, moreover, p_0-p_{inc} satisfies the Sommerfeld radiation condition at infinity.

A. An inhomogeneous sphere

Inside the sphere $(r<a)$, we write

$$u(r,\theta)=\rho_0^{1/2}c_0^2\sum_{n=0}^{\infty}(2n+1)i^nB_nu_n(r)P_n(\cos\theta), \tag{38}$$

where the dimensionless coefficients B_n are to be found and $u_n(r)$ is a solution of Eq. (36) that is regular at $r=0$; some explicit solutions will be given later.

We find A_n and B_n by enforcing the transmission conditions across the interface at $r=a$. Let

$$\rho_a=\lim_{r\to a-}\rho(r)\quad\text{and}\quad\kappa_a=\lim_{r\to a-}[\rho'(r)/\rho(r)],$$

so that ρ_a is the surface value of the interior density. Then, the interface conditions are

$$p_0=\rho_a^{1/2}u\quad\text{and}\quad\rho_0^{-1}\frac{\partial p_0}{\partial r}=\rho_a^{-1/2}\left(\frac{\partial u}{\partial r}+\kappa_au\right) \tag{39}$$

on $r=a$. Substituting Eqs. (37) and (38), making use of the orthogonality of the Legendre polynomials, gives

$$j_n(k_0a)+A_nh_n(k_0a)=\sigma B_nu_n(a),$$

$$k_0\{j_n'(k_0a)+A_nh_n'(k_0a)\}=\sigma^{-1}B_n\{u_n'(a)+\kappa_au_n(a)\}$$

for $n=0,1,2,\ldots$, where $\sigma=(\rho_a/\rho_0)^{1/2}$. These two equations can be solved for A_n and B_n

$$A_n\Delta=(k_0\sigma)^{-1}j_n(k_0a)\{u_n'(a)+\kappa_au_n(a)\}-\sigma j_n'(k_0a)u_n(a)$$

and $B_n\Delta=i(k_0a)^{-2}$, where

$$\Delta=\sigma h_n'(k_0a)u_n(a)-(k_0\sigma)^{-1}h_n(k_0a)\{u_n'(a)+\kappa_au_n(a)\}.$$

B. A homogeneous sphere with an inhomogeneous coating

Suppose that the sphere $r<a$ consists of a homogeneous core $r<b$ (with density ρ_c and sound speed c_c) with an inhomogeneous concentric coating, $b<r<a$.

In the coating, we can write

$$u(r,\theta)=\rho_0^{1/2}c_0^2\sum_{n=0}^{\infty}(2n+1)$$
$$\times i^n\{B_n u_n(r)+C_n v_n(r)\}P_n(\cos\theta),$$

where $u_n(r)$ and $v_n(r)$ are solutions of Eq. (36). We suppose that $u_n(r)$ is regular at $r=0$, whereas $v_n(r)$ is singular at $r=0$. In the homogeneous core, the pressure field is

$$p_c(r,\theta)=\rho_0 c_0^2\sum_{n=0}^{\infty}(2n+1)i^n D_n j_n(k_c r)P_n(\cos\theta),$$

where $k_c=\omega/c_c$.

We have to enforce two transmission conditions at $r=a$ and two at $r=b$. Let

$$\rho_b=\lim_{r\to b+}\rho(r)\ \text{ and }\ \kappa_b=\lim_{r\to b+}[\rho'(r)/\rho(r)].$$

Then, the interface conditions are Eq. (39) and

$$p_c=\rho_b^{1/2}u\ \text{ and }\ \rho_c^{-1}\frac{\partial p_c}{\partial r}=\rho_b^{-1/2}\left(\frac{\partial u}{\partial r}+\kappa_b u\right)$$

on $r=b$. These four conditions can be used to determine A_n, B_n, C_n, and D_n in a straightforward way.

V. EXPONENTIAL VARIATIONS IN ρ

Let us assume specific functional forms for $\rho(r)$ and $k(r)=\omega/[c(r)]$, namely

$$\rho(r)=\rho_1 e^{2\beta r}\ \text{ and }\ [k(r)]^2=k_1^2+2\alpha r^{-1}. \tag{40}$$

Here, ρ_1, β, k_1^2, and α are four adjustable constants. We find that Eq. (36) becomes

$$u_n''+2r^{-1}u_n'+[k_1^2-\beta^2+2(\alpha+\beta)r^{-1}$$
$$-n(n+1)r^{-2}]u_n=0. \tag{41}$$

Equation (41) has a regular singularity at $r=0$, an irregular singularity at $r=\infty$, and no others. Therefore, it can be transformed into the confluent hypergeometric equation. Make the substitution

$$u_n(r)=r^{-1}w_n(x)\quad\text{with}\quad x=\delta r$$

in Eq. (41), giving

$$w_n''(x)+\left[\frac{k_1^2-\beta^2}{\delta^2}+\frac{2}{x}\,\frac{\alpha+\beta}{\delta}-\frac{n(n+1)}{x^2}\right]w_n(x)=0, \tag{42}$$

where δ is a parameter at our disposal. There are now three cases, depending on the relative sizes of k_1^2 and β^2.

A. Case I ($k_1^2>\beta^2$)

Choose $\delta^2=k_1^2-\beta^2$ and set $\eta=-(\alpha+\beta)/\delta$. Then, Eq. (42) becomes

$$w_n''(x)+[1-2\eta x^{-1}-n(n+1)x^{-2}]w_n(x)=0,$$

which is the *Coulomb wave equation* (Abramowitz and Stegun, 1965, Chap. 14). Its general solution is

$$w_n(x)=A_nF_n(\eta,x)+B_nG_n(\eta,x),$$

where A_n and B_n are arbitrary constants, F_n is the regular Coulomb wave function (bounded at $x=0$), and G_n is the irregular Coulomb wave function. These functions arise in nuclear physics (Biedenharn and Brussard, 1965, Chap. 3, Sec. 4).

Unsurprisingly, Coulomb wave functions are wavelike, in the sense that

$$G_n(\eta,x)+iF_n(\eta,x)\sim e^{i(x-\varphi)}\quad\text{as}\quad x\to\infty,$$

where $\varphi=\eta\log 2x+\frac{1}{2}n\pi-\sigma_n$ and σ_n is known [Abramowitz and Stegun, 1965, Eq. (14.6.5)]. Moreover,

$$F_n(0,x)=xj_n(x)\quad\text{and}\quad G_n(0,x)=-xy_n(x),$$

where j_n and y_n are spherical Bessel functions, so that the known solutions for homogeneous media are recovered.

B. Case II ($k_1^2=\beta^2$)

Choose $\delta=8(\alpha+\beta)$, and then Eq. (42) becomes

$$w_n''(x)+[\tfrac{1}{4}x^{-1}-n(n+1)x^{-2}]w_n(x)=0, \tag{43}$$

which is related to Bessel's equation; the general solution of Eq. (43) is

$$w_n(x)=\sqrt{x}\{A_nJ_{2n+1}(\sqrt{x})+B_nY_{2n+1}(\sqrt{x})\}.$$

C. Case III ($k_1^2<\beta^2$)

Choose $\delta^2=4(\beta^2-k_1^2)$ and set $\kappa=2(\alpha+\beta)/\delta$ and $\mu=n+\frac{1}{2}$. Then, Eq. (42) becomes

$$w_n''(x)+[-\tfrac{1}{4}+\kappa x^{-1}+(\tfrac{1}{4}-\mu^2)x^{-2}]w_n(x)=0, \tag{44}$$

which is known as *Whittaker's equation*. Its general solution is given by

$$w_n(x)=A_nM_{\kappa,\mu}(x)+B_nW_{\kappa,\mu}(x),$$

where $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ are Whittaker functions; these are discussed by Whittaker and Watson (1927, Chap. 16), by Erdélyi *et al.* (1953, Sec. 6.9), by Abramowitz and Stegun (1965, Chap. 13), and by Buchholz (1969). The occurrence of Whittaker functions is a little surprising, because these functions do not exhibit wavelike behavior. Thus

$$M_{\kappa,\mu}(x)\sim x^{-\kappa}e^{x/2}\quad\text{and}\quad W_{\kappa,\mu}(x)\sim x^{\kappa}e^{-x/2}$$

as $x\to\infty$. Moreover [Buchholz, 1969, Sec. 2, Eqs. (11a) and (29a)]

$$M_{0,\mu}(x)=\sqrt{\pi x}I_\mu(x)\ \text{ and }\ W_{0,\mu}(x)=\sqrt{x/\pi}K_\mu(x),$$

where I_μ and K_μ are modified Bessel functions. We remark that Whittaker functions also occur when solving the steady-state heat-conduction equation, $\operatorname{div}[k(r)\operatorname{grad}u]=0$, when $k(r)$ varies exponentially with r; see Martin (2002). The solutions described above can be inserted into the formulas obtained by the method of separation of variables in Sec. IV.

VI. GAUSSIAN SPHERES

One drawback of the functional forms in Eq. (40) is that the corresponding sound speed satisfies $c(0)=0$ (unless $\alpha=0$), so that Eq. (40) may not be suitable for an inhomogeneous sphere. (This objection does not apply if the sphere has a homogeneous core, as described in Sec. IV B.)

As an alternative, we can make progress by supposing that the density is a Gaussian and that $k^2(r)$ is linear in r^2. Thus, we suppose that

$$\rho(r)=\rho_1 e^{-\beta r^2} \text{ and } [k(r)]^2=k_1^2+\gamma r^2, \tag{45}$$

where ρ_1, β, k_1^2, and γ are adjustable constants. We find that Eq. (36) becomes

$$u_n''+2r^{-1}u_n'+[(\gamma-\beta^2)r^2+(k_1^2-3\beta)-n(n+1)r^{-2}]u_n=0.$$

To simplify this equation, make the substitution $u_n(r)=r^{-3/2}w_n(x)$ with $x=\delta r^2$, where δ is a disposable parameter. This gives

$$w_n''(x)+\left[\frac{\gamma-\beta^2}{\delta^2}+\frac{k_1^2-3\beta}{4\delta x}-\left\{n(n+1)-\frac{3}{16}\right\}x^{-2}\right]w_n(x)=0. \tag{46}$$

As in Sec. V, there are now three cases, depending on the sign of $\gamma-\beta^2$. For example, we obtain the Coulomb wave equation if $\gamma>\beta^2$ and Whittaker's equation if $\gamma<\beta^2$. Explicit solutions follow readily, but are not recorded here.

VII. CONCLUSIONS

In this paper, we have done two things. First, we have given a derivation of Bergmann's equation for sound waves in inhomogeneous media, where both the ambient density and sound speed can vary with position (but not time). Second, we have studied the scattering of waves by an inhomogeneous sphere; for certain exponential variations in the density, such scattering problems can be solved by the method of separation of variables, where the radial dependence involves some less-familiar but well-studied special functions. Perhaps the main value of these solutions is to provide nontrivial benchmarks against which numerical schemes (based, for example, on volume integral equations) can be tested.

ACKNOWLEDGMENTS

Thanks are due to C. J. Chapman and J. A. DeSanto for their constructive comments on the derivation in Sec. II A.

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