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On Green's function for a three-dimensional exponentially graded elastic solid

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We dedicate this paper to Frank Rizzo on the occasion of his retirement.

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The problem of a point force acting in an unbounded, three-dimensional, isotropic elastic solid is considered. Kelvin solved this problem for homogeneous materials. Here, the material is inhomogeneous; it is 'functionally graded'. Specifically, the solid is 'exponentially graded', which means that the Lamé moduli vary exponentially in a given fixed direction. The solution for the Green's function is obtained by Fourier transforms, and consists of a singular part, given by the Kelvin solution, plus a non-singular remainder. This grading term is not obtained in simple closed form, but as the sum of single integrals over finite intervals of modified Bessel functions, and double integrals over finite regions of elementary functions. Knowledge of this new fundamental solution for graded materials permits the development of boundary-integral methods for these technologically important inhomogeneous solids.

Keywords: fundamental solutions; boundary-element methods; functionally graded materials; exponential grading

1. Introduction

Lord Kelvin obtained the Green's function $\mathbf{G}^0$ for a three-dimensional homogeneous isotropic elastic solid in 1848 (Love 1927, § 130). This gives the displacement at a point when a point force is acting at another point. $\mathbf{G}^0$ is used widely as a basic ingredient in integral-equation methods for solving elastostatic boundary-value problems.

Kelvin's solution may be generalized in two directions. First, the elastic solid could be *anisotropic*. In general, the three-dimensional Green's function cannot then be found in closed form, although it can be reduced to the evaluation of a single integral over a finite interval (see (1.2) below). For references to the extensive literature on

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anisotropic Green's functions, see Nakamura & Tanuma (1997), Ting & Lee (1997) and Pan & Yuan (2000). The incorporation of anisotropic Green's functions into a boundary-integral analysis is described in the book by Schclar (1994).

A second generalization is to consider *inhomogeneous* elastic solids. Very little can be achieved, analytically, if the material properties are allowed to vary in a smooth but arbitrary manner. For such media, it is usual to limit the analysis to the construction of a 'parametrix' (Garabedian 1964, p. 168; McLean 2000, p. 192) or 'Levi function'. These may be regarded as 'approximate Green's functions': formally, the Green's function $\mathbf{G}$ satisfies $A\mathbf{G} = \boldsymbol{\delta}$, whereas a Levi function $\mathbf{L}$ satisfies $A\mathbf{L} = \boldsymbol{\delta} + \mathbf{R}$, where $\mathbf{R}$ is a smooth 'remainder' and A is the governing differential operator. Recently, Pomp (1998) has devised a numerical algorithm for constructing Levi functions with a small remainder. Chapter 2 of his book gives a good review of the known methods for finding Green's functions when the governing partial differential equation has variable coefficients.

Rather than looking for general techniques, we concentrate here on a specific inhomogeneous material, one that has found application to functionally graded materials. Thus, we assume that the material properties vary in a simple, explicit manner. Here, we consider exponential variations, and suppose that the solid is isotropic with Lamé moduli given by

$$\lambda(\boldsymbol{x}) = \lambda_0 \exp(2\boldsymbol{\beta}\cdot\boldsymbol{x}) \quad \text{and} \quad \mu(\boldsymbol{x}) = \mu_0 \exp(2\boldsymbol{\beta}\cdot\boldsymbol{x}), \tag{1.1}$$

where λ_0 and μ_0 are constants, and $\boldsymbol{\beta}$ is a given constant vector. We say that the solid is *exponentially graded* in the direction of $\boldsymbol{\beta}$. Evidently, Poisson's ratio ν is constant for such a solid.

The assumption (1.1) is typical in the engineering literature devoted to *functionally graded materials* (FGMs). The papers by Hirai (1995) and Markworth *et al*. (1995), and the books by Suresh & Mortensen (1998) and Miyamoto *et al*. (1999), provide a good overview of current FGM research.

Inhomogeneous materials with grading in one direction have been studied extensively. For example, Booker *et al*. (1985) have considered an elastic half-space $z > 0$ with point-force loading on $z = 0$. They took ν to be constant and $\mu(z) = \mu_0 z^\alpha$. As a special case, they recovered Gibson's solution (Gibson 1967) for $\alpha = 1$ and $\nu = \frac{1}{2}$; such incompressible isotropic inhomogeneous materials are reasonable models for certain soils. Exponential grading, $\mu(z) = \mu_0 \mathrm{e}^{\beta z}$, has also been studied in this context by Giannakopoulos & Suresh (1997); see also the recent review by Suresh (2001).

Ben-Menahem (1987) has considered the elastodynamic Green's function for unbounded isotropic inhomogeneous solids. He gave a general analysis but was only able to obtain solutions when $\lambda(\boldsymbol{x})$ and $\mu(\boldsymbol{x})$ have certain specific functional forms; these do not include exponential grading (1.1). Elastodynamic half-space problems, with $\mu(z) = \mu_1 + \mu_0 \mathrm{e}^{\beta z}$, have been considered in several papers by Vrettos (1991, 1999).

In this paper, we show first that the Green's function $\mathbf{G}$ corresponding to (1.1) can be written as

$$\mathbf{G}(\boldsymbol{x};\boldsymbol{x}') = \exp\{-\boldsymbol{\beta}\cdot(\boldsymbol{x}+\boldsymbol{x}')\}\{\mathbf{G}^0(\boldsymbol{x};\boldsymbol{x}') + \mathbf{G}^{\mathrm{g}}(\boldsymbol{x};\boldsymbol{x}')\},$$

where $\mathbf{G}^0$ is the Kelvin solution, and the additional *grading term* $\mathbf{G}^{\mathrm{g}}$ is bounded. It is given as a three-dimensional Fourier integral, and the main task is to evaluate this integral. We show that it can be reduced to an explicit term, some single integrals of modified Bessel functions over a finite interval, and some double integrals of elementary functions over finite regions.

It is of interest to compare the calculation for an exponentially graded but isotropic material with that for a homogeneous but anisotropic material. In the latter, the Fourier integral over $\boldsymbol{\xi}$ involves the vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. The integral simplifies by choosing spherical polar coordinates with $\boldsymbol{r}$ along the polar axis. Moreover, the integrand contains $[\mathbf{Q}(\boldsymbol{\xi})]^{-1}$, where $Q_{i\ell}(\boldsymbol{\xi}) = C_{ijk\ell}\xi_j\xi_k$ and $C_{ijk\ell}$ are the constant stiffnesses; thus, $\mathbf{Q}$ is homogeneous ($\mathbf{Q}(t\boldsymbol{\xi}) = t^2\mathbf{Q}(\boldsymbol{\xi})$ for any $t \neq 0$), and this fact simplifies the calculation. Specifically, we have

$$\begin{aligned} \mathbf{G} &= (2\pi)^{-3} \int [\mathbf{Q}(\boldsymbol{\xi})]^{-1} \exp(-\mathrm{i}\boldsymbol{\xi} \cdot \boldsymbol{r}) \, \mathrm{d}\boldsymbol{\xi} \\ &= (2\pi)^{-3} \iint \xi^{-2} [\mathbf{Q}(\hat{\boldsymbol{\xi}})]^{-1} \cos(\xi r \cos\varphi) \xi^2 \, \mathrm{d}\xi \, \mathrm{d}\hat{\boldsymbol{\xi}}, \end{aligned}$$

where $r = |\boldsymbol{r}|$, $\xi = |\boldsymbol{\xi}|$, $\boldsymbol{\xi} = \xi\hat{\boldsymbol{\xi}}$ and we have observed that both $\mathbf{G}$ and $\mathbf{Q}$ are real. Using spherical polar coordinates (ξ, φ, χ), where $\varphi = 0$ is the polar axis, we have $\mathrm{d}\hat{\boldsymbol{\xi}} = \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\chi$, whence

$$\begin{aligned} \mathbf{G} &= (2\pi)^{-3} \lim_{X \to \infty} \int_0^\pi \mathbf{S}(\varphi) \int_0^X \cos(\xi r \cos\varphi) \, \mathrm{d}\xi \sin\varphi \, \mathrm{d}\varphi \\ &= \frac{1}{8\pi^3 r} \lim_{X \to \infty} \int_{-1}^1 \mathbf{S}(\cos^{-1}\mu) \frac{\sin(Xr\mu)}{\mu} \, \mathrm{d}\mu, \end{aligned}$$

where

$$\mathbf{S}(\varphi) = \int_0^{2\pi} [\mathbf{Q}(\hat{\boldsymbol{\xi}})]^{-1} \, \mathrm{d}\chi.$$

Note that we have evaluated the integral over ξ and then put $\mu = \cos\varphi$. The integral over μ is known as a *Dirichlet integral*; its limiting value as $X \to \infty$ is $\pi\mathbf{S}(0)$ (Knopp 1951, § 49C, p. 365), whence

$$\mathbf{G} = \frac{1}{8\pi^2 r} \oint [\mathbf{Q}(\hat{\boldsymbol{\xi}})]^{-1} \, \mathrm{d}\chi, \tag{1.2}$$

where the integral is taken around the unit circle, centred at the origin and lying in the plane perpendicular to $\boldsymbol{r}$. This derivation can be found on p. 412 of the book by Synge (1957); other derivations (involving divergent integrals and generalized functions) are available. For a generalization, see McLean (2000, Theorem 6.8).

For the exponentially graded material, we do not have homogeneity and we have *three* distinguished directions, associated with $\boldsymbol{\xi}$, $\boldsymbol{r}$ and $\boldsymbol{\beta}$. It turns out to be advantageous to use spherical polar coordinates with $\boldsymbol{\beta}$ along the polar axis.

A knowledge of $\mathbf{G}$ for an exponentially graded elastic solid permits the treatment of a variety of problems involving FGMs. For example, problems of stress analysis can be solved using boundary-integral equations. Previous work in this area includes the papers by Shaw & Gipson (1995) and by Azis & Clements (2001); the latter paper considers Lamé moduli proportional to $(\boldsymbol{\beta} \cdot \boldsymbol{x} + c)^2$, where c is a constant.

Another class of problems concerns fracture mechanics. A review of crack problems in inhomogeneous media has been given by Erdogan (1998). Most work has been devoted to two-dimensional problems with cracks aligned with the grading direction; these limitations may be removed using the Green's function derived below. Propagating cracks can also be modelled effectively using boundary-integral equations. This approach is advantageous for two main reasons. First, the crack-tip singularity can be incorporated readily, leading to very accurate stress-intensity factors. Second, the re-meshing task is much simpler as the crack propagates.

2. Governing equations

Consider an anisotropic inhomogeneous elastic solid with stiffnesses $c_{ijk\ell}(\boldsymbol{x})$, where $c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij}$ and a typical point has position vector $\boldsymbol{x} = (x_1, x_2, x_3)$ with respect to O, the origin of Cartesian coordinates. The Green's function $\mathbf{G}(\boldsymbol{x};\boldsymbol{x}')$ satisfies

$$\frac{\partial}{\partial x_j}\left\{c_{ijk\ell}(\boldsymbol{x})\frac{\partial G_{\ell m}}{\partial x_k}\right\} = -\delta_{im}\delta(\boldsymbol{x}-\boldsymbol{x}'), \quad i=1,2,3, \tag{2.1}$$

where δ_{ij} is the Kronecker delta and $\delta(\boldsymbol{x})$ is the three-dimensional Dirac delta. As usual, $G_{ij}(\boldsymbol{x};\boldsymbol{x}')$ gives the ith component of the displacement at $\boldsymbol{x}$ due to a point force acting in the jth direction at $\boldsymbol{x}'$. Also, a standard argument using the reciprocal theorem and $c_{ijk\ell} = c_{\ell kji}$ (see, for example, Bonnet 1999, p. 72) ensures that $\mathbf{G}$ is symmetric:

$$G_{ij}(\boldsymbol{x};\boldsymbol{x}') = G_{ji}(\boldsymbol{x}';\boldsymbol{x}). \tag{2.2}$$

Evaluating the left-hand side of (2.1) gives

$$c_{ijk\ell}(\boldsymbol{x})\frac{\partial^2 G_{\ell m}}{\partial x_j \partial x_k} + \left(\frac{\partial}{\partial x_j}c_{ijk\ell}(\boldsymbol{x})\right)\frac{\partial G_{\ell m}}{\partial x_k} = -\delta_{im}\delta(\boldsymbol{x}-\boldsymbol{x}'), \quad i=1,2,3. \tag{2.3}$$

We consider a particular inhomogeneous material in which the stiffnesses vary exponentially, so that

$$c_{ijk\ell}(\boldsymbol{x}) = C_{ijk\ell}\exp(2\boldsymbol{\beta}\cdot\boldsymbol{x}), \tag{2.4}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ and $C_{ijk\ell}$ and β_i are given constants; the factor of 2 in the exponent is inserted for later algebraic convenience. Hence

$$(\partial/\partial x_j)c_{ijk\ell}(\boldsymbol{x}) = 2C_{ijk\ell}\beta_j\exp(2\boldsymbol{\beta}\cdot\boldsymbol{x}) = 2\beta_j c_{ijk\ell}(\boldsymbol{x}) \tag{2.5}$$

and so (2.3) becomes

$$\begin{aligned} C_{ijk\ell}\frac{\partial^2 G_{\ell m}}{\partial x_j \partial x_k} + 2\beta_j C_{ijk\ell}\frac{\partial G_{\ell m}}{\partial x_k} &= -\delta_{im}\exp(-2\boldsymbol{\beta}\cdot\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x}') \\ &= -\delta_{im}\exp(-2\boldsymbol{\beta}\cdot\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}') \end{aligned} \tag{2.6}$$

for $i = 1, 2, 3$. Note that we can replace the right-hand side of (2.6) by

$$-\delta_{im}\exp(-\boldsymbol{\beta}\cdot[p\boldsymbol{x}+p'\boldsymbol{x}'])\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.7}$$

where p and p' are any constants that satisfy the constraint $p+p' = 2$; this flexibility will be exploited soon.

We can write (2.6) as

$$c_{ijk\ell}(\boldsymbol{x}')\frac{\partial^2 G_{\ell m}}{\partial x_j \partial x_k} + 2\beta_j c_{ijk\ell}(\boldsymbol{x}')\frac{\partial G_{\ell m}}{\partial x_k} = -\delta_{im}\delta(\boldsymbol{x}-\boldsymbol{x}'), \quad i=1,2,3,$$

which we recognize as (2.3) with the variable coefficients 'frozen' at $\boldsymbol{x} = \boldsymbol{x}'$, having used (2.5). We remark that Pomp's algorithm (1998, § 2.4) begins by freezing the coefficients of the second-order derivatives only.

Alternatively, we may work with (2.6) directly. We introduce $\mathbf{G}^0$, the Green's function for a *homogeneous* solid with constant stiffnesses $C_{ijk\ell}$, defined by

$$C_{ijk\ell}\frac{\partial^2 G^0_{\ell m}}{\partial x_j \partial x_k} = -\delta_{im}\delta(\boldsymbol{x} - \boldsymbol{x}'), \quad i = 1, 2, 3. \tag{2.8}$$

Comparing these equations with (2.6) suggests writing

$$\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}') = \exp(-2\boldsymbol{\beta} \cdot \boldsymbol{x}')\{\mathbf{G}^0(\boldsymbol{x}; \boldsymbol{x}') + \mathbf{G}^1(\boldsymbol{x}; \boldsymbol{x}')\}, \tag{2.9}$$

whence $\mathbf{G}^1$ is found to satisfy

$$C_{ijk\ell}\frac{\partial^2 G^1_{\ell m}}{\partial x_j \partial x_k} + 2\beta_j C_{ijk\ell}\frac{\partial G^1_{\ell m}}{\partial x_k} = -2\beta_j C_{ijk\ell}\frac{\partial G^0_{\ell m}}{\partial x_k} \tag{2.10}$$

for $i = 1, 2, 3$. This is a system of three coupled second-order partial differential equations, with constant coefficients. However, the decomposition (2.9) has a disadvantage: the symmetry property (2.2) is not inherited by $\mathbf{G}^1$. Thus, we change the right-hand side of (2.6), using (2.7) with $p = p' = 1$, giving

$$C_{ijk\ell}\frac{\partial^2 G_{\ell m}}{\partial x_j \partial x_k} + 2\beta_j C_{ijk\ell}\frac{\partial G_{\ell m}}{\partial x_k} = -\delta_{im}\exp\{-\boldsymbol{\beta} \cdot (\boldsymbol{x} + \boldsymbol{x}')\}\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{2.11}$$

and we replace (2.9) by

$$\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}') = \exp\{-\boldsymbol{\beta} \cdot (\boldsymbol{x} + \boldsymbol{x}')\}\{\mathbf{G}^0(\boldsymbol{x}; \boldsymbol{x}') + \mathbf{G}^{\mathrm{g}}(\boldsymbol{x}; \boldsymbol{x}')\}, \tag{2.12}$$

so that

$$G^{\mathrm{g}}_{ij}(\boldsymbol{x}; \boldsymbol{x}') = G^{\mathrm{g}}_{ji}(\boldsymbol{x}'; \boldsymbol{x}).$$

To find an equation for the *grading term* $\mathbf{G}^{\mathrm{g}}$, we simply substitute (2.12) in (2.11), making use of (2.8); the result is

$$C_{ijk\ell}\frac{\partial^2 G^{\mathrm{g}}_{\ell m}}{\partial x_j \partial x_k} + L_{i\ell}G^{\mathrm{g}}_{\ell m}(\boldsymbol{x}; \boldsymbol{x}') = -L_{i\ell}G^0_{\ell m}(\boldsymbol{x}; \boldsymbol{x}') \tag{2.13}$$

for $i = 1, 2, 3$, where the first-order differential operator $L_{i\ell}$ is defined by

$$L_{i\ell} = (C_{ijk\ell} - C_{ikj\ell})\beta_j(\partial/\partial x_k) - C_{ijk\ell}\beta_j\beta_k.$$

3. Fourier transforms

We solve the system (2.13) using three-dimensional Fourier transforms, which we define by

$$\mathcal{F}\{u\} = \hat{u}(\boldsymbol{\xi}) = \int u(\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{\xi} \cdot \boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

$$\mathcal{F}^{-1}\{\hat{u}\} = u(\boldsymbol{x}) = (2\pi)^{-3}\int \hat{u}(\boldsymbol{\xi})\exp(-\mathrm{i}\boldsymbol{\xi} \cdot \boldsymbol{x})\,\mathrm{d}\boldsymbol{\xi}.$$

Thus, we find that the Fourier transform of (2.13) is given by

$$\{Q_{i\ell}(\boldsymbol{\xi}) + B_{i\ell}(\boldsymbol{\beta}, \boldsymbol{\xi})\}\hat{G}^{\mathrm{g}}_{\ell m}(\boldsymbol{\xi}; \boldsymbol{x}') = -B_{i\ell}(\boldsymbol{\beta}, \boldsymbol{\xi})\hat{G}^{0}_{\ell m}(\boldsymbol{\xi}; \boldsymbol{x}'),$$

where

$$Q_{i\ell}(\boldsymbol{\xi}) = C_{ijk\ell}\xi_j\xi_k \quad \text{and} \quad B_{i\ell}(\boldsymbol{\beta}, \boldsymbol{\xi}) = \mathrm{i}(C_{ijk\ell} - C_{ikj\ell})\beta_j\xi_k + C_{ijk\ell}\beta_j\beta_k.$$

Note that if we define a complex vector $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma} = \boldsymbol{\xi} + \mathrm{i}\boldsymbol{\beta}$, then

$$Q_{i\ell} + B_{i\ell} = C_{ijk\ell}\gamma_j\bar{\gamma}_k, \tag{3.1}$$

where the overbar denotes complex conjugation: $\overline{\boldsymbol{\gamma}} = \boldsymbol{\xi} - \mathrm{i}\boldsymbol{\beta}$.

From (2.8), we have

$$Q_{i\ell}(\boldsymbol{\xi})\hat{G}^{0}_{\ell m}(\boldsymbol{\xi}; \boldsymbol{x}') = \delta_{im}\exp(\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}'),$$

whence

$$\hat{G}^{\mathrm{g}}_{\ell m}(\boldsymbol{\xi}; \boldsymbol{x}') = E_{\ell m}(\boldsymbol{\beta}, \boldsymbol{\xi})\exp(\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}'),$$

where

$$\mathbf{E}(\boldsymbol{\beta}, \boldsymbol{\xi}) = -\{\mathbf{Q}(\boldsymbol{\xi}) + \mathbf{B}(\boldsymbol{\beta}, \boldsymbol{\xi})\}^{-1}\mathbf{B}(\boldsymbol{\beta}, \boldsymbol{\xi})[\mathbf{Q}(\boldsymbol{\xi})]^{-1}. \tag{3.2}$$

Inverting the Fourier transform, we obtain

$$\mathbf{G}^{\mathrm{g}}(\boldsymbol{x}; \boldsymbol{x}') = (2\pi)^{-3}\int \mathbf{E}(\boldsymbol{\beta}, \boldsymbol{\xi})\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{3.3}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. It remains to evaluate this three-dimensional integral over $\boldsymbol{\xi}$. Note that $\mathbf{E}(\mathbf{0}, \boldsymbol{\xi}) = \mathbf{B}(\mathbf{0}, \boldsymbol{\xi}) = \mathbf{0}$ so that $\mathbf{G}^{\mathrm{g}} = \mathbf{0}$ when there is no grading.

So far, our specification of $\mathbf{G}$ has been incomplete: to any solution of (2.1), we can always add any solution of the corresponding homogeneous equation (with zero on the right-hand side). Thus, we *define* $\mathbf{G}$ by (2.12), wherein $\mathbf{G}^0$ is a specified solution of (2.8) (such as the Kelvin solution for isotropic materials) and $\mathbf{G}^{\mathrm{g}}$ is given by (3.3).

We conclude this section by noting a few properties of $\mathbf{E}$. First, (2.2) implies that

$$E_{ij}(\boldsymbol{\beta}, \boldsymbol{\xi}) = E_{ji}(\boldsymbol{\beta}, -\boldsymbol{\xi}). \tag{3.4}$$

Second, as $\mathbf{G}^{\mathrm{g}}$ is real, we have

$$\overline{\mathbf{E}(\boldsymbol{\beta}, \boldsymbol{\xi})} = \mathbf{E}(\boldsymbol{\beta}, -\boldsymbol{\xi}). \tag{3.5}$$

Third, as a useful check on calculations, we note that $\mathbf{B}(\boldsymbol{\xi}, \boldsymbol{\xi}) = \mathbf{Q}(\boldsymbol{\xi})$, whence $\mathbf{E}(\boldsymbol{\xi}, \boldsymbol{\xi}) = -\frac{1}{2}[\mathbf{Q}(\boldsymbol{\xi})]^{-1}$.

4. Isotropy

If the underlying homogeneous medium is isotropic, we have

$$C_{ijk\ell} = \lambda_0\delta_{ij}\delta_{k\ell} + \mu_0(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}),$$

where λ_0 and μ_0 are the (constant) Lamé moduli. Note that $\lambda_0/\mu_0 = 2\nu/(1 - 2\nu)$, where ν is Poisson's ratio. From (2.4), the corresponding graded medium has Lamé moduli given by

$$\lambda(\boldsymbol{x}) = \lambda_0 \exp(2\boldsymbol{\beta} \cdot \boldsymbol{x}) \quad \text{and} \quad \mu(\boldsymbol{x}) = \mu_0 \exp(2\boldsymbol{\beta} \cdot \boldsymbol{x}),$$

and so it has the same constant Poisson's ratio, ν.

We readily obtain

$$Q_{i\ell}(\boldsymbol{\xi}) = \mu_0\{\xi^2\delta_{i\ell} + (1-2\nu)^{-1}\xi_i\xi_\ell\}$$

and

$$Q_{i\ell}^{-1}(\boldsymbol{\xi}) = (\mu_0\xi^4)^{-1}\{\xi^2\delta_{i\ell} - [2(1-\nu)]^{-1}\xi_i\xi_\ell\},$$

where $\xi^2 = |\boldsymbol{\xi}|^2$ (see, for example, Mura 1982, eqn (3.33)). Similarly,

$$B_{i\ell}(\boldsymbol{\beta}, \boldsymbol{\xi}) = \mu_0\{\beta^2\delta_{i\ell} + (1-2\nu)^{-1}[\beta_i\beta_\ell + (4\nu - 1)\mathrm{i}(\beta_i\xi_\ell - \beta_\ell\xi_i)]\},$$

where $\beta^2 = |\boldsymbol{\beta}|^2$.

Let $C = \boldsymbol{\xi} \cdot \boldsymbol{\beta}$, $\alpha = (1-2\nu)^{-1}$, $\kappa = (1-\nu)^{-1}$ and $\sigma = 4\nu - 1$. Then we find that

$$(\mathbf{BQ}^{-1})_{\ell m} = \xi^{-4}\{\beta^2\xi^2\delta_{\ell m} + \tfrac{1}{2}\kappa\xi_\ell\xi_m(\mathrm{i}\alpha\sigma C - \beta^2) + \alpha\xi^2\beta_\ell\beta_m - \mathrm{i}\alpha\sigma\xi^2\xi_\ell\beta_m + \tfrac{1}{2}\kappa\beta_\ell\xi_m(\mathrm{i}\sigma\xi^2 - \alpha C)\}. \quad (4.1)$$

Using (3.1), we have

$$(\mathbf{Q} + \mathbf{B})_{ij} = \mu_0\{\gamma^2\delta_{ij} + 2\nu\alpha\gamma_i\bar{\gamma}_j + \bar{\gamma}_i\gamma_j\},$$

where $\gamma^2 = \gamma_j\bar{\gamma}_j$. Then, using a general result given in Appendix A, we obtain

$$(\mathbf{Q} + \mathbf{B})_{j\ell}^{-1} = (\mu_0\gamma^2\Delta)^{-1}\{\Delta\delta_{j\ell} - \tfrac{1}{2}\kappa\xi_j\xi_\ell(\gamma^2 + 8\nu\beta^2) - \tfrac{1}{2}\kappa\beta_j\beta_\ell(\gamma^2 + 8\nu\xi^2) + \tfrac{1}{2}\kappa\xi_j\beta_\ell(8\nu C + \mathrm{i}\sigma\gamma^2) + \tfrac{1}{2}\kappa\beta_j\xi_\ell(8\nu C - \mathrm{i}\sigma\gamma^2)\}, \quad (4.2)$$

where $C = \boldsymbol{\xi} \cdot \boldsymbol{\beta} = \xi\beta\cos\theta$,

$$\Delta = \xi^4 + 2\xi^2\beta^2 q + \beta^4 \quad \text{and} \quad q = 1 + 2\nu\kappa\sin^2\theta.$$

Then a lengthy calculation, using (3.2), (4.1) and (4.2), gives

$$E_{j\ell}(\boldsymbol{\beta}, \boldsymbol{\xi}) = -(\mu_0\gamma^2\xi^4\Delta)^{-1}\{\xi^2\beta^2\Delta\delta_{j\ell} + \tfrac{1}{2}\kappa\Omega_{j\ell}\}, \quad (4.3)$$

where

$$\Omega_{j\ell}(\boldsymbol{\beta}, \boldsymbol{\xi}) = \mathcal{A}\xi_j\xi_\ell + \mathcal{B}\xi_j\beta_\ell + \bar{\mathcal{B}}\beta_j\xi_\ell + \mathcal{C}\beta_j\beta_\ell, \quad (4.4)$$

$$\mathcal{A} = \xi^4(\gamma^2 + 8\nu\beta^2) - \gamma^2\Delta, \quad \mathcal{B} = -\xi^4(8\nu C + \mathrm{i}\sigma\gamma^2) \quad \text{and} \quad \mathcal{C} = \xi^4(\gamma^2 + 8\nu\xi^2).$$

One can check that these expressions satisfy (3.4) and (3.5).

5. The triple integral

There are three vectors in the triple integral (3.3), namely, $\boldsymbol{r}$, $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$. We can regard $\boldsymbol{r}$ and $\boldsymbol{\beta}$ as fixed, and integrate over $\boldsymbol{\xi}$ using spherical polar coordinates.

From (4.3), we have

$$E_{j\ell} = E\delta_{j\ell} + H_{j\ell}, \quad (5.1)$$

where

$$E = -\beta^2(\mu_0\gamma^2\xi^2)^{-1} \quad \text{and} \quad H_{j\ell} = -\tfrac{1}{2}\kappa(\mu_0\gamma^2\xi^4\Delta)^{-1}\Omega_{j\ell}.$$

Thus, there are two contributions to $\mathbf{G}^{\mathrm{g}}$, coming from the two terms on the right-hand side of (5.1); we consider them separately.

(*a*) *The first term*

We can easily evaluate the contribution due to E, using spherical polar coordinates (ξ, φ, χ), with the polar axis $(\varphi = 0)$ in the direction of $\boldsymbol{r}$. Thus, we obtain

$$\begin{aligned}\frac{1}{(2\pi)^3}\int E\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} &= \frac{-\beta^2}{(2\pi)^2\mu_0}\int_0^\infty \frac{1}{\gamma^2\xi^2}\int_0^\pi \mathrm{e}^{-\mathrm{i}\xi r\cos\varphi}\sin\varphi\,\mathrm{d}\varphi\xi^2\,\mathrm{d}\xi \\ &= \frac{-\beta^2}{2\pi^2\mu_0 r}\int_0^\infty \frac{\sin\xi r}{\xi(\xi^2+\beta^2)}\,\mathrm{d}\xi \\ &= -(4\pi\mu_0 r)^{-1}(1-\mathrm{e}^{-\beta r}),\end{aligned} \tag{5.2}$$

using a standard contour-integral method for the ξ-integral.

(*b*) *The second term*

The second term on the right-hand side of (5.1) is more complicated. In order to treat this term, it is better to take $\boldsymbol{\beta}$ as defining the polar axis of spherical coordinates (not $\boldsymbol{r}$, as is commonly done). The reason for this choice is that we have already introduced θ, the angle between $\boldsymbol{\xi}$ and $\boldsymbol{\beta}$ ($\boldsymbol{\xi}\cdot\boldsymbol{\beta} = \xi\beta\cos\theta$), and this angle appears in Δ and C.

Let $\boldsymbol{n}$ and $\boldsymbol{m}$ be any two mutually perpendicular unit vectors in the plane perpendicular to $\boldsymbol{\beta}$. Let $\boldsymbol{\beta} = \beta\hat{\boldsymbol{\beta}}$, so that $\{\boldsymbol{n}, \boldsymbol{m}, \hat{\boldsymbol{\beta}}\}$ forms an orthonormal right-handed triad. In terms of the global, fixed, Cartesian coordinates, we have $\boldsymbol{n} = (n_i)$, $\boldsymbol{m} = (m_i)$ and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_i)$. For example, we can take

$$\boldsymbol{n} = \beta_0^{-1}(\beta_3, 0, -\beta_1) \quad \text{and} \quad \boldsymbol{m} = (\beta\beta_0)^{-1}(-\beta_1\beta_2, \beta_0^2, -\beta_2\beta_3)$$

provided $\beta_0 \equiv (\beta_1^2+\beta_3^2)^{1/2} \neq 0$.

Let (ξ, θ, ϕ) be the spherical polar coordinates of the point at $\boldsymbol{\xi}$, so that

$$\boldsymbol{\xi}\cdot\boldsymbol{n} = \xi\sin\theta\cos\phi, \quad \boldsymbol{\xi}\cdot\boldsymbol{m} = \xi\sin\theta\sin\phi \quad \text{and} \quad \boldsymbol{\xi}\cdot\hat{\boldsymbol{\beta}} = \xi\cos\theta.$$

Similarly, let (r, Θ, Φ) be the spherical polar coordinates of the point at $\boldsymbol{r}$, so that

$$\boldsymbol{r}\cdot\boldsymbol{n} = r\sin\Theta\cos\Phi, \quad \boldsymbol{r}\cdot\boldsymbol{m} = r\sin\Theta\sin\Phi \quad \text{and} \quad \boldsymbol{r}\cdot\hat{\boldsymbol{\beta}} = r\cos\Theta.$$

Hence, putting $X = r\sin\Theta$ and $Z = r\cos\Theta$, we obtain

$$\boldsymbol{r}\cdot\boldsymbol{\xi} = \xi X\sin\theta\cos(\phi-\Phi) + \xi Z\cos\theta.$$

The first step is to integrate over ϕ. Extracting the dependence on ϕ, we see that $\Omega_{j\ell}$, defined by (4.4), can be written as

$$\Omega_{j\ell} = \Omega_{j\ell}^0 + \Omega_{j\ell}^1\cos\phi + \tilde{\Omega}_{j\ell}^1\sin\phi + \Omega_{j\ell}^2\cos 2\phi + \tilde{\Omega}_{j\ell}^2\sin 2\phi,$$

where

$$\begin{aligned}\Omega_{j\ell}^0 &= \tfrac{1}{2}\mathcal{A}(n_j n_\ell + m_j m_\ell)\xi^2\sin^2\theta + \{\mathcal{A}\xi^2\cos^2\theta + (\mathcal{B}+\bar{\mathcal{B}})C + \mathcal{C}\beta^2\}\hat{\beta}_j\hat{\beta}_\ell, \\ \Omega_{j\ell}^1 &= \tfrac{1}{2}\mathcal{A}(n_j\hat{\beta}_\ell + \hat{\beta}_j n_\ell)\xi^2\sin 2\theta + (\mathcal{B}n_j\hat{\beta}_\ell + \bar{\mathcal{B}}\hat{\beta}_j n_\ell)\xi\beta\sin\theta, \\ \tilde{\Omega}_{j\ell}^1 &= \tfrac{1}{2}\mathcal{A}(m_j\hat{\beta}_\ell + \hat{\beta}_j m_\ell)\xi^2\sin 2\theta + (\mathcal{B}m_j\hat{\beta}_\ell + \bar{\mathcal{B}}\hat{\beta}_j m_\ell)\xi\beta\sin\theta, \\ \Omega_{j\ell}^2 &= \tfrac{1}{2}\mathcal{A}(n_j n_\ell - m_j m_\ell)\xi^2\sin^2\theta, \\ \tilde{\Omega}_{j\ell}^2 &= \tfrac{1}{2}\mathcal{A}(n_j m_\ell + m_j n_\ell)\xi^2\sin^2\theta.\end{aligned}$$

Then, as we can integrate over any interval of length 2π, we can use

$$\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\xi X \sin\theta\cos(\phi-\Phi)} \begin{pmatrix}\cos n\phi \\ \sin n\phi\end{pmatrix} \mathrm{d}\phi = 2\pi(-\mathrm{i})^n J_n(\xi X \sin\theta) \begin{pmatrix}\cos n\Phi \\ \sin n\Phi\end{pmatrix},$$

where $J_n(w)$ is a Bessel function. Hence

$$\int_{-\pi}^{\pi} \Omega_{j\ell} \exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi})\,\mathrm{d}\phi = 2\pi\mathrm{e}^{-\mathrm{i}\xi Z\cos\theta}\{R_{j\ell}(\xi,\theta) - \mathrm{i}I_{j\ell}(\xi,\theta)\},$$

where

$$R_{j\ell}(\xi,\theta) = \Omega^0_{j\ell} J_0(\xi X\sin\theta) - [\Omega^2_{j\ell}\cos 2\Phi + \tilde{\Omega}^2_{j\ell}\sin 2\Phi] J_2(\xi X\sin\theta)$$

and

$$I_{j\ell}(\xi,\theta) = [\Omega^1_{j\ell}\cos\Phi + \tilde{\Omega}^1_{j\ell}\sin\Phi] J_1(\xi X\sin\theta).$$

Next, consider the integration over θ. As

$$R_{j\ell}(\xi,\pi-\theta) = R_{j\ell}(\xi,\theta) \quad \text{and} \quad I_{j\ell}(\xi,\pi-\theta) = -\overline{I_{j\ell}(\xi,\theta)},$$

we find that

$$\begin{aligned}&\int_0^{\pi}\int_{-\pi}^{\pi} \Omega_{j\ell}\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi})\frac{\sin\theta}{\Delta}\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &\qquad = 4\pi\int_0^{\pi/2}\{R_{j\ell}(\xi,\theta)\cos(\xi Z\cos\theta) + \mathrm{Im}\{I_{j\ell}(\xi,\theta)\mathrm{e}^{-\mathrm{i}\xi Z\cos\theta}\}\}\frac{\sin\theta}{\Delta}\,\mathrm{d}\theta,\end{aligned}$$

which is evidently real. If we put

$$s_j = n_j\cos\Phi + m_j\sin\Phi,$$

we see that

$$\begin{aligned}\mathrm{Im}\{I_{j\ell}\mathrm{e}^{-\mathrm{i}\xi Z\cos\theta}\} &= \mathrm{Im}\{I_{j\ell}\}\cos(\xi Z\cos\theta) - \mathrm{Re}\{I_{j\ell}\}\sin(\xi Z\cos\theta) \\ &= \{\tfrac{1}{2}(s_j\hat{\beta}_\ell + \hat{\beta}_j s_\ell)(2\xi^2 q + \beta^2)\beta^2\sin 2\theta\sin(\xi Z\cos\theta) \\ &\qquad - \sigma(s_j\hat{\beta}_\ell - \hat{\beta}_j s_\ell)\xi^3\beta\sin\theta\cos(\xi Z\cos\theta)\}\xi^2\gamma^2 J_1(\xi X\sin\theta),\end{aligned}$$

which is an even function of ξ; $R_{j\ell}$ is also an even function of ξ.

Finally, consider the integration over ξ. The order of integration can be interchanged (as the relevant integrals are absolutely convergent), and so we obtain

$$\frac{1}{(2\pi)^3}\int H_{j\ell}\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \frac{-\kappa\beta}{8\pi^2\mu_0}\int_0^{\pi/2} M_{j\ell}(\theta)\sin\theta\,\mathrm{d}\theta, \tag{5.3}$$

where

$$\begin{aligned}M_{j\ell}(\theta) &= \frac{1}{\beta}\int_{-\infty}^{\infty}\{R_{j\ell}\cos(\xi Z\cos\theta) + \mathrm{Im}(I_{j\ell}\mathrm{e}^{-\mathrm{i}\xi Z\cos\theta})\}\frac{\mathrm{d}\xi}{\xi^2\gamma^2\Delta} \\ &= \frac{1}{\beta}\int_{-\infty}^{\infty}(R_{j\ell} + \mathrm{i}\overline{I_{j\ell}})\mathrm{e}^{\mathrm{i}\xi Z\cos\theta}\frac{\mathrm{d}\xi}{\xi^2\gamma^2\Delta}.\end{aligned}$$

Explicitly, if we put

$$k = \beta Z \cos\theta = \beta r \cos\theta \cos\Theta, \qquad K = \beta X \sin\theta = \beta r \sin\theta \sin\Theta \geqslant 0,$$
$$\xi = \beta x \quad \text{and} \quad D(x) = x^4 + 2x^2 q + 1,$$

we can express $M_{j\ell}$ as

$$M_{j\ell} = \sum_{n=0}^{2} M_{j\ell}^{(n)} + \tilde{M}_{j\ell}^{(1)}, \tag{5.4}$$

where

$$M_{j\ell}^{(n)} = \int_{-\infty}^{\infty} \frac{f_n(x) J_n(Kx) \mathrm{e}^{\mathrm{i}kx}}{(x^2+1) D(x)} \, \mathrm{d}x, \quad n = 0, 2, \tag{5.5}$$

$$M_{j\ell}^{(1)} = \int_{-\infty}^{\infty} \frac{f_1(x)}{D(x)} J_1(Kx) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x \quad \text{and} \quad \tilde{M}_{j\ell}^{(1)} = \int_{-\infty}^{\infty} \frac{\tilde{f}_1(x)}{D(x)} J_1(Kx) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x.$$

Here, the integrands contain polynomials in x, defined by

$$\begin{aligned} f_0(x) &= \tfrac{1}{2}\{8\nu x^4 - (x^2+1)(2x^2 q + 1)\}(n_j n_\ell + m_j m_\ell) \sin^2\theta \\ &\qquad + \{8\nu x^4 \sin^2\theta + (x^2+1)[x^2 - (2x^2 q + 1)\cos^2\theta]\}\hat{\beta}_j \hat{\beta}_\ell, \\ f_1(x) &= -x^3 \sigma (s_j \hat{\beta}_\ell - \hat{\beta}_j s_\ell) \sin\theta, \\ \tilde{f}_1(x) &= -\tfrac{1}{2}\mathrm{i}(s_j \hat{\beta}_\ell + \hat{\beta}_j s_\ell)(2x^2 q + 1) \sin 2\theta, \\ f_2(x) &= -\tfrac{1}{2}[8\nu x^4 - (x^2+1)(2x^2 q + 1)]\{n_j(n_\ell \cos 2\Phi + m_\ell \sin 2\Phi) \\ &\qquad + m_j(n_\ell \sin 2\Phi - m_\ell \cos 2\Phi)\} \sin^2\theta. \end{aligned}$$

We have written $M_{j\ell}$ as (5.4) because $M_{j\ell}^{(n)}$ are all even functions of k, whereas $\tilde{M}_{j\ell}^{(1)}$ is an odd function of k. Hence, when we consider methods for evaluating $M_{j\ell}$, we can assume, without loss of generality, that

$$k \geqslant 0 \quad \text{and} \quad K \geqslant 0.$$

(*c*) *Evaluation of* $M_{j\ell}$

Sometimes, we can evaluate integrals such as $M_{j\ell}^{(n)}$ directly using the calculus of residues. For example, consider

$$\int_{\Gamma} \frac{f_n(z) J_n(Kz) \mathrm{e}^{\mathrm{i}kz}}{(z^2+1) D(z)} \, \mathrm{d}z, \quad n = 0, 2,$$

where Γ is a closed contour in the complex z-plane to be chosen later. The integrand has simple poles at

$$z = \pm \mathrm{i} y_s, \quad s = 0, 1, 2,$$

where

$$y_0 = 1, \qquad y_1 = \sqrt{q + \sqrt{q^2 - 1}}, \qquad y_2 = \sqrt{q - \sqrt{q^2 - 1}}$$

and $q = 1 + 2\nu\kappa \sin^2\theta$.

Now, as $z \to \infty$, we have

$$f_n(z)\{(z^2+1)D(z)\}^{-1} = O(z^{-2}) \quad (n=0,2),$$

and $J_n(Kz) \sim \{2/(\pi K)\}^{1/2} z^{-1/2} \cos(Kz - \frac{1}{2}n\pi - \frac{1}{4}\pi)$. This suggests that the choice of method for evaluating $M_{j\ell}$ will depend on the sign of

$$k - K = \beta r \cos(\theta + \Theta).$$

If $k \geqslant K \geqslant 0$, we can take $\Gamma = S_R \cup L_R$, where S_R is a semicircular contour in the upper half of the z-plane, of radius R, and L_R is a piece of the real axis. There is no contribution from S_R as $R \to \infty$, because the exponential growth of the Bessel functions is dominated by the exponential decay of $\mathrm{e}^{\mathrm{i}kz}$. Proceeding in the standard way, we calculate the residues at the three poles within Γ, and obtain

$$M_{j\ell}^{(n)} = 2\pi\mathrm{i} \sum_{s=0}^{2} \mathcal{M}_s^{(n)} J_n(\mathrm{i}Ky_s)\mathrm{e}^{-ky_s}, \quad n = 0, 2, \tag{5.6}$$

where

$$\mathcal{M}_0^{(n)} = \frac{f_n(\mathrm{i})}{2\mathrm{i}D(\mathrm{i})} \quad \text{and} \quad \mathcal{M}_s^{(n)} = \frac{f_n(\mathrm{i}y_s)}{(1-y_s^2)D'(\mathrm{i}y_s)} \quad \text{for } s = 1, 2;$$

we have $D(\mathrm{i}) = -4\nu\kappa \sin^2\theta$ and $D'(\mathrm{i}y_s) = 4\mathrm{i}y_s(q - y_s^2)$ for $s = 1, 2$.

The expression (5.6) is real, because $f_0(\mathrm{i}y)$ and $f_2(\mathrm{i}y)$ are real when y is real, $J_0(\mathrm{i}y) = I_0(y)$ and $J_2(\mathrm{i}y) = -I_2(y)$, where $I_n(w)$ is a modified Bessel function. Similarly,

$$M_{j\ell}^{(1)} = 2\pi\mathrm{i} \sum_{s=1}^{2} \mathcal{M}_s^{(1)} J_1(\mathrm{i}Ky_s)\mathrm{e}^{-ky_s} \quad \text{and} \quad \tilde{M}_{j\ell}^{(1)} = 2\pi\mathrm{i} \sum_{s=1}^{2} \tilde{\mathcal{M}}_s^{(1)} J_1(\mathrm{i}Ky_s)\mathrm{e}^{-ky_s}, \tag{5.7}$$

where

$$\mathcal{M}_s^{(1)} = \frac{f_1(\mathrm{i}y_s)}{D'(\mathrm{i}y_s)} \quad \text{and} \quad \tilde{\mathcal{M}}_s^{(1)} = \frac{\tilde{f}_1(\mathrm{i}y_s)}{D'(\mathrm{i}y_s)}.$$

Note that the expressions (5.7) are real because $f_1(\mathrm{i}y)$ and $\tilde{f}_1(\mathrm{i}y)$ are pure imaginary and $J_1(\mathrm{i}y) = \mathrm{i}I_1(y)$.

If $0 \leqslant k < K$, the Bessel functions are dominant. There are some standard tricks for dealing with such situations. One involves the identity

$$J_n(Kz) = \tfrac{1}{2}\{H_n^{(1)}(Kz) + H_n^{(2)}(Kz)\}, \tag{5.8}$$

where $H_n^{(1)}(w)$ and $H_n^{(2)}(w)$ are Hankel functions. (This is analogous to writing $\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})$.) However, methods based on (5.8) seem to fail here.

Instead, we proceed indirectly, and use a method suggested by Watson (1944, p. 425) in which the Bessel function is replaced by one of its integral representations; for even n, we choose

$$J_n(Kx) = \frac{2}{\pi} \int_0^{\pi/2} \cos(Kx \sin\eta) \cos(n\eta)\, \mathrm{d}\eta.$$

If we substitute this expression in (5.5) and interchange the order of integration, we obtain

$$M_{j\ell}^{(n)} = M_+ + M_-, \quad n = 0, 2,$$

where

$$M_\pm = \frac{1}{\pi}\int_0^{\pi/2} \cos(n\eta) \int_{-\infty}^{\infty} \frac{f_n(x)}{(x^2+1)D(x)} \mathrm{e}^{\mathrm{i}x(k \pm K \sin\eta)}\,\mathrm{d}x\,\mathrm{d}\eta.$$

The two inner integrals over x are easily evaluated by residues. As $k + K\sin\eta > 0$, we can always evaluate M_+ using a semicircular contour in the upper half-plane. However, $k - K\sin\eta$ will change sign if $k < K$.

As a check, suppose first that $k \geqslant K \geqslant 0$. Then we obtain

$$M_\pm = 2\mathrm{i}\sum_{s=0}^{2} \mathcal{M}_s^{(n)} \mathrm{e}^{-ky_s} \int_0^{\pi/2} \mathrm{e}^{\mp K y_s \sin\eta} \cos(n\eta)\,\mathrm{d}\eta. \tag{5.9}$$

When these are combined, using

$$2\int_0^{\pi/2} \cosh\left(K y_s \sin\eta\right) \cos(n\eta)\,\mathrm{d}\eta = \pi \mathrm{i}^n I_n(K y_s), \quad n \text{ even}, \tag{5.10}$$

we recover (5.6).

Suppose, now, that $0 \leqslant k < K$. Define η_0 by $k = K\sin\eta_0$, with $0 \leqslant \eta_0 < \frac{1}{2}\pi$. Then

$$M_- = \frac{1}{\pi}\int_0^{\eta_0} \cos(n\eta) \int_{-\infty}^{\infty} \frac{f_n(x)}{(x^2+1)D(x)} \mathrm{e}^{\mathrm{i}x(k - K\sin\eta)}\,\mathrm{d}x\,\mathrm{d}\eta + \frac{1}{\pi}\int_{\eta_0}^{\pi/2} \cos(n\eta) \int_{-\infty}^{\infty} \frac{f_n(x)}{(x^2+1)D(x)} \mathrm{e}^{-\mathrm{i}x(K\sin\eta - k)}\,\mathrm{d}x\,\mathrm{d}\eta.$$

Evaluate the first x-integral as before, and evaluate the second x-integral using a semicircular contour in the lower half-plane. As $f_n(x)$ and $D(x)$ are even functions of x, we find that

$$\begin{aligned} M_- &= 2\mathrm{i}\sum_{s=0}^{2} \mathcal{M}_s^{(n)} \left\{ \mathrm{e}^{-ky_s} \int_0^{\eta_0} \mathrm{e}^{K y_s \sin\eta} \cos(n\eta)\,\mathrm{d}\eta + \mathrm{e}^{k y_s} \int_{\eta_0}^{\pi/2} \mathrm{e}^{-K y_s \sin\eta} \cos(n\eta)\,\mathrm{d}\eta \right\} \\ &= 2\mathrm{i}\sum_{s=0}^{2} \mathcal{M}_s^{(n)} \left\{ \mathrm{e}^{-ky_s} \int_0^{\pi/2} \mathrm{e}^{K y_s \sin\eta} \cos(n\eta)\,\mathrm{d}\eta + 2\int_{\eta_0}^{\pi/2} \sinh\left(y_s[k - K\sin\eta]\right) \cos(n\eta)\,\mathrm{d}\eta \right\}. \end{aligned}$$

The first integral on the right-hand side is exactly the same as in (5.9), obtained there when $k \geqslant K$. Thus, for any non-negative choices of k and K, we have

$$M_{j\ell}^{(n)} = 2\pi \mathrm{i} \sum_{s=0}^{2} \mathcal{M}_s^{(n)} \biggl\{ \mathrm{e}^{-ky_s} \mathrm{i}^n I_n(Ky_s) + \frac{2}{\pi} H(K-k) \int_{\eta_0}^{\pi/2} \sinh\left(y_s[k - K\sin\eta]\right) \cos(n\eta)\,\mathrm{d}\eta \biggr\}, \tag{5.11}$$

where $H(x)$ is the Heaviside unit function and $n = 0, 2$.

Similar calculations succeed for $M_{j\ell}^{(1)}$ and $\tilde{M}_{j\ell}^{(1)}$. We use

$$J_1(Kx) = \frac{2}{\pi} \int_0^{\pi/2} \sin\left(Kx \sin\eta\right) \sin\eta\,\mathrm{d}\eta.$$

If $k \geqslant K \geqslant 0$, we recover (5.7), making use of

$$2 \int_0^{\pi/2} \sinh\left(Ky_s \sin\eta\right) \sin\eta\,\mathrm{d}\eta = \pi I_1(Ky_s). \tag{5.12}$$

If $0 \leqslant k < K$, we obtain

$$M_{j\ell}^{(1)} = 2\pi \sum_{s=1}^{2} \mathcal{M}_s^{(1)} \biggl\{ -\mathrm{e}^{-ky_s} I_1(Ky_s) + \frac{2}{\pi} H(K-k) \int_{\eta_0}^{\pi/2} \cosh\left(y_s[k - K\sin\eta]\right) \sin\eta\,\mathrm{d}\eta \biggr\}. \tag{5.13}$$

There is an identical formula for $\tilde{M}_{j\ell}^{(1)}$: replace $\mathcal{M}_s^{(1)}$ with $\tilde{\mathcal{M}}_s^{(1)}$.

The remaining integrals in (5.11) and (5.13) cannot be evaluated in closed form. However, we have succeeded in replacing infinite integrals of Bessel functions by finite integrals of exponentials.

We now collect up the results for $M_{j\ell}$ and substitute in (5.3), taking account of the fact that k will be negative when $\frac{1}{2}\pi < \Theta < \pi$. Let $\theta_m = |\frac{1}{2}\pi - \Theta|$. Define η_m by $|k| = K \sin\eta_m$, with $0 \leqslant \eta_m < \frac{1}{2}\pi$. Then we obtain

$$\begin{aligned}
\int H_{j\ell} \exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi}) \frac{\mathrm{d}\boldsymbol{\xi}}{(2\pi)^3} = \frac{-\kappa\beta}{4\pi\mu_0} \sum_{s=0}^{2} \sum_{n=0}^{2} \int_0^{\pi/2} \mathcal{R}_s^{(n)}(\theta) \mathrm{e}^{-|k|y_s} I_n(Ky_s) \sin\theta\,\mathrm{d}\theta \\
- \frac{\kappa\beta}{2\pi^2\mu_0} \sum_{s=0}^{2} \int_{\theta_m}^{\pi/2} \mathcal{R}_s^{(0)} \sin\theta \int_{\eta_m}^{\pi/2} \sinh\varPsi_s\,\mathrm{d}\eta\,\mathrm{d}\theta \\
+ \frac{\kappa\beta}{2\pi^2\mu_0} \sum_{s=0}^{2} \int_{\theta_m}^{\pi/2} \mathcal{R}_s^{(2)} \sin\theta \int_{\eta_m}^{\pi/2} \sinh\varPsi_s \cos 2\eta\,\mathrm{d}\eta\,\mathrm{d}\theta \\
+ \frac{\kappa\beta}{2\pi^2\mu_0} \sum_{s=1}^{2} \int_{\theta_m}^{\pi/2} \mathcal{R}_s^{(1)} \sin\theta \int_{\eta_m}^{\pi/2} \cosh\varPsi_s \sin\eta\,\mathrm{d}\eta\,\mathrm{d}\theta,
\end{aligned}$$

where

$$\Psi_s(\theta,\eta) = y_s(|k| - K\sin\eta) = K y_s(\sin\eta_m - \sin\eta);$$
$$\mathcal{R}_s^{(0)} = \mathrm{i}\mathcal{M}_s^{(0)}, \quad \mathcal{R}_s^{(2)} = -\mathrm{i}\mathcal{M}_s^{(2)}, \quad s = 0,1,2;$$
$$\mathcal{R}_0^{(1)} = 0, \quad \mathcal{R}_s^{(1)} = -(\mathcal{M}_s^{(1)} + \tilde{\mathcal{M}}_s^{(1)}\mathrm{sgn}(k)), \quad s = 1,2.$$

In these formulae, $\mathcal{R}_1^{(n)}$, $\mathcal{R}_2^{(n)}$, k, K, y_1, y_2 and η_m all depend on θ. All of the integrals are real.

Eight double integrals remain, but this number can be reduced to six, using $\cos 2\eta = 2\cos^2\eta - 1$ and an integration by parts:

$$\int_{\eta_m}^{\pi/2} \sinh\Psi_s \cos 2\eta \,\mathrm{d}\eta = \frac{2\cos\eta_m}{K y_s} - \frac{2}{K y_s}\int_{\eta_m}^{\pi/2} \cosh\Psi_s \sin\eta \,\mathrm{d}\eta - \int_{\eta_m}^{\pi/2} \sinh\Psi_s \,\mathrm{d}\eta.$$

6. Discussion and conclusion

The Green's function (or fundamental solution) for an exponentially graded elastic solid can be written as

$$\mathbf{G}(\boldsymbol{x};\boldsymbol{x}') = \exp\{-\boldsymbol{\beta}\cdot(\boldsymbol{x}+\boldsymbol{x}')\}\{\mathbf{G}^0(\boldsymbol{x};\boldsymbol{x}') + \mathbf{G}^{\mathrm{g}}(\boldsymbol{x};\boldsymbol{x}')\},$$

where $\mathbf{G}^0$ is the Kelvin solution and the vector $\boldsymbol{\beta}$ gives the grading direction and magnitude. We have shown that the triple Fourier integral defining $\mathbf{G}^{\mathrm{g}}$ can be reduced to the sum of an explicit term (given by (5.2)), finite single integrals of modified Bessel functions and finite double integrals of elementary functions. As this grading term $\mathbf{G}^{\mathrm{g}}$ is bounded as $|\boldsymbol{x}-\boldsymbol{x}'| \to 0$ (the singularity is contained within the Kelvin solution), having it available only as a computable quantity is not an impediment for a boundary-integral implementation.

Given that we have to evaluate some double integrals, it may be preferable, computationally, to replace the modified Bessel functions I_n by their integral representations, (5.10) or (5.12). Alternatively, we can express the double integrals as *infinite* series of single integrals, using

$$\cosh(K y_s \sin\eta) = I_0(K y_s) + 2\sum_{n=1}^{\infty}(-1)^n I_{2n}(K y_s)\cos 2n\eta$$

and

$$\sinh(K y_s \sin\eta) = 2\sum_{n=0}^{\infty}(-1)^n I_{2n+1}(K y_s)\sin{(2n+1)\eta},$$

as these Fourier expansions make the η-integrals trivial. However, this is unlikely to yield a good computational strategy.

Another alternative is to use the Bauer expansion for the exponential in (3.3), namely

$$\exp(-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\xi}) = 4\pi\sum_{n=0}^{\infty}(-\mathrm{i})^n j_n(\xi r)\sum_{m=-n}^{n} Y_n^m(\hat{\boldsymbol{\xi}})\overline{Y_n^m(\hat{\boldsymbol{r}})},$$

where j_n is a spherical Bessel function and Y_n^m is a spherical harmonic (Watson 1944, § 11.5). This expansion separates the dependence on ξ, θ and ϕ. Moreover, the

ϕ-integral is simple, and shows that only those values of m with $|m| \leqslant 2$ contribute. The subsequent integrals are of the form

$$\int_0^\infty \int_0^\pi g(\xi,\theta) j_n(\xi r) P_n^m(\cos\theta)\,\mathrm{d}\theta\,\mathrm{d}\xi, \quad n = 0, 1, 2, \ldots,$$

where g is known and P_n^m is an associated Legendre function. These integrals seem to be very complicated, so we did not pursue this approach further.

Further work is needed on various aspects of $\mathbf{G}(\boldsymbol{x};\boldsymbol{x}')$. First, efficient numerical algorithms should be derived for the evaluation of the remaining integrals defining $\mathbf{G}^{\mathrm{g}}$. Second, the derivatives of $\mathbf{G}^{\mathrm{g}}$ are required in boundary-integral implementations. The calculation of these derivatives is straightforward in principle but tedious in detail; symbolic software should be useful here. (Note that, in one sense, the problem is simpler than that for a homogeneous anisotropic solid: the integral in (1.2) is over a circle that moves when $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ moves.) Third, it may be useful to examine the behaviour of $\mathbf{G}(\boldsymbol{x};\boldsymbol{x}')$ in the far field, where $r = |\boldsymbol{r}|$ is large. Fourth, some special cases may lead to simplifications in the final formulae; these may arise from special material properties (such as incompressibility, for which $\nu = \frac{1}{2}$) or special grading directions (such as when $\boldsymbol{\beta}$ is aligned with one of the coordinate axes). Finally, it may be possible to extend the analysis to certain graded *anisotropic* materials. Work on these various aspects is in progress.

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Appendix A. A matrix inversion

Let $\mathbf{A}$ be an $n \times n$ matrix, with entries given by

$$A_{ij} = a\delta_{ij} + b\gamma_i\bar{\gamma}_j + c\bar{\gamma}_i\gamma_j.$$

Then, if it exists, the inverse matrix has the form

$$A_{jk}^{-1} = A\delta_{jk} + B\gamma_j\bar{\gamma}_k + C\bar{\gamma}_j\gamma_k + D\gamma_j\gamma_k + E\bar{\gamma}_j\bar{\gamma}_k.$$

Multiplication gives 15 terms, which combine to give

$$\begin{aligned} A_{ij}A_{jk}^{-1} = aA\delta_{ik} &+ \gamma_i\bar{\gamma}_k\{(a+b\gamma^2)B + bE\bar{\mathcal{G}} + bA\} + \bar{\gamma}_i\bar{\gamma}_k\{(a+c\gamma^2)E + cB\mathcal{G}\} \\ &+ \bar{\gamma}_i\gamma_k\{(a+c\gamma^2)C + cD\mathcal{G} + cA\} + \gamma_i\gamma_k\{(a+b\gamma^2)D + bC\bar{\mathcal{G}}\}, \end{aligned}$$

where $\gamma^2 = \gamma_j\bar{\gamma}_j$ and $\mathcal{G} = \gamma_j\gamma_j$. As we want $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, we obtain

$$\mathbf{Z}\begin{pmatrix} B \\ E \end{pmatrix} = \begin{pmatrix} -bA \\ 0 \end{pmatrix}, \qquad \mathbf{Z}\begin{pmatrix} D \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ -cA \end{pmatrix}$$

and $aA = 1$, where

$$\mathbf{Z} = \begin{pmatrix} a + b\gamma^2 & b\bar{\mathcal{G}} \\ c\mathcal{G} & a + c\gamma^2 \end{pmatrix}$$

is a complex 2×2 matrix with

$$Z \equiv \det \mathbf{Z} = (a + b\gamma^2)(a + c\gamma^2) - bc|\mathcal{G}|^2,$$

which is real if a, b and c are real; writing $\gamma_j = \xi_j + \mathrm{i}\beta_j$, we see that

$$Z = [a(b + c) + 4bc\beta^2 \sin^2 \theta]\xi^2 + a^2 + a(b + c)\beta^2,$$

where $\xi^2 = \xi_j\xi_j$, $\beta^2 = \beta_j\beta_j$ and $\boldsymbol{\xi} \cdot \boldsymbol{\beta} = \xi\beta\cos\theta$. So, if $Z \neq 0$, we can solve for B and E, and for D and C, whence

$$\begin{aligned} A^{-1}_{jk} &= (aZ)^{-1}\{Z\delta_{jk} - b(a + c\gamma^2)\gamma_j\bar{\gamma}_k - c(a + b\gamma^2)\bar{\gamma}_j\gamma_k + bc(\bar{\mathcal{G}}\gamma_j\gamma_k + \mathcal{G}\bar{\gamma}_j\bar{\gamma}_k)\} \\ &= (aZ)^{-1}\{Z\delta_{jk} - [a(b + c) + 4bc\beta^2]\xi_j\xi_k - [a(b + c) + 4bc\xi^2]\beta_j\beta_k \\ &\qquad + 4bc\xi\beta(\xi_j\beta_k + \xi_k\beta_j)\cos\theta + \mathrm{i}a(b - c)(\xi_j\beta_k - \xi_k\beta_j)\}. \end{aligned}$$

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