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The incompressible limit in linear anisotropic elasticity, with applications to surface waves and elastostatics

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Abstract

Incompressibility is established for three- and two-dimensional deformations of an anisotropic linearly elastic material, as conditions to be satisfied by the elastic compliances. These conditions make it straightforward to derive results for incompressible materials from those established for compressible materials. As an illustration, the explicit secular equation is obtained for surface waves in incompressible monoclinic materials with the symmetry plane at $x_3 = 0$. This equation also covers the case of incompressible orthotropic materials. The displacements and stresses for surface waves are often expressed in terms of the elastic stiffnesses, which can be unbounded in the incompressible limit. An alternative formalism in terms of the elastic compliances presented recently by Ting (Proc. R. Soc. London (2002), in press) is employed so that surface wave solutions in the incompressible limit can be obtained. A different formalism, also by Ting (Proc. R. Soc. London A 455 (1999) 69), is employed to study the solutions to two-dimensional elastostatic problems. In the special case of incompressible monoclinic materials with the symmetry plane at $x_3 = 0$, one of the three Barnett–Lothe tensors $\mathbf{S}$ vanishes while the other two tensors $\mathbf{H}$ and $\mathbf{L}$ are the inverse of each other. Moreover, $\mathbf{H}$ and $\mathbf{L}$ are diagonal with the first two diagonal elements being identical. Many interesting physical phenomena can be deduced using this property. For instance, there is no interpenetration of the interface crack surfaces in an incompressible bimaterial. When only the inplane deformation is considered, the image force due to a line dislocation in a half-space or in a bimaterial depends on the magnitude, not on the direction, of the Burgers vector.

Keywords: Dislocations; Dynamics; Anisotropic material; Elastic material; Stress waves

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1. Introduction

Linear isotropic elasticity is characterized by two material constants, which can be taken as the shear modulus μ and Poisson's ratio v. These constants satisfy $\mu > 0$ and $-1 < v < 1/2$. The incompressible limit is $v \to 1/2$. To see why this is so, we write down Hooke's law, relating the stress components σ_{ij} to the strain components ε_{ks} as

$$\sigma_{ij} = \mu \left(2\varepsilon_{ij} + \frac{2v}{1-2v}\delta_{ij}\varepsilon_{kk} \right). \tag{1.1}$$

In the above, δ_{ij} is the Kronecker delta and repeated indices imply summation. Contracting, we obtain

$$\varepsilon_{ii} = \frac{1-2v}{2\mu(1+v)}\sigma_{ii} = \frac{v}{\lambda(1+v)}\sigma_{ii}, \tag{1.2}$$

where λ is a Lamé constant. If the material is incompressible, $\varepsilon_{ii} = 0$ for any stresses, whence $(1.2)_1$ gives $v = 1/2$.

Let us now turn to linear *anisotropic* elasticity, and consider the corresponding incompressible limit. For such materials, we have $\sigma_{ij} = C_{ijks}\varepsilon_{ks}$ where the C's are the elastic stiffnesses. In the special case of isotropy, the non-trivial stiffnesses are $C_{1111} = C_{2222} = C_{3333} = \lambda + 2\mu$, $C_{1122} = C_{1133} = C_{2233} = \lambda$ and $C_{1212} = C_{1313} = C_{2323} = \mu$. From $(1.2)_2$ the incompressible limit corresponds to $\lambda \to \infty$. This suggests that, in general, some of the stiffnesses will become unbounded in the incompressible limit, and therefore it will be safer to work with the elastic *compliance* matrix **s** rather than with the elastic stiffness matrix **C**. This is so because **s** is the inverse of **C**, and possible infinite components of **C** will simply correspond to some components (or combination of components) of **s** being equal to zero.

In order to consider incompressible linearly elastic anisotropic materials directly, some authors have modified the stress–strain law by introducing a hydrostatic pressure P, as $\sigma_{ij} = -P\delta_{ij} + C_{ijks}\varepsilon_{ks}$. Incompressibility is then imposed by supplementing the condition $\varepsilon_{ii} = 0$. Although formally acceptable, and supported by similar considerations in finite elasticity, this approach is risky as it may lead to potentially meaningless results, when the stiffnesses appear in the final expressions.

For example, consider some recent developments in the theory of surface waves in linear anisotropic elastic materials. For compressible materials the secular equation was obtained explicitly for monoclinic materials with the symmetry plane at $x_3 = 0$ (Destrade, 2001a; Ting, 2002). At the same time, some attention has been given to the consideration of interface waves in anisotropic materials which are *incompressible* (see, for instance, Nair and Sotiropoulos (1999) or Destrade (2001b) and the references therein). In this paper we show that results obtained in the general (compressible) case can be easily specialized to the incompressible case, simply by imposing the conditions for incompressibility on the elastic compliances, without having to introduce an arbitrary pressure.

We adopt the following plan for the paper. In Section 2, we recall the three-dimensional stress–strain laws of linear anisotropic elasticity, and establish that the constraint of incompressibility yields simple mathematical conditions, which are written for the elastic compliances $s_{\alpha\beta}$. Unlike the case of isotropic elastic materials, the conditions of incompressibility are different for two-dimensional deformations. These conditions are established in Section 3 and written for the *reduced elastic compliances* $s'_{\alpha\beta}$. In both sections, a necessary and sufficient condition for the strain energy density to be positive semidefinite is presented. We show in Section 4 how simple it is to deduce an explicit secular equation for surface waves in a monoclinic material with the symmetry plane at $x_3 = 0$ for the incompressible case from that for the compressible case. The secular equation is only a part of the surface wave solution. In the literature, the stresses and displacements for surface waves in an anisotropic elastic material are expressed in terms of the elastic stiffnesses, as briefly summarized in Section 5. These expressions have to be converted to ones for the reduced elastic compliances. This has been done by Ting (2002) and is outlined in Section 6. The conversion presented in Section 6 does not apply to elastostatics. A different formulation, again by Ting (1999), is reviewed in Section 7. In Section 8 we consider the special case of incompressible monoclinic materials with the symmetry plane at $x_3 = 0$ under a static loading. Interesting physical phenomena are discovered due to the incompressibility of the material.

2. Incompressibility for three-dimensional deformations

When the displacement $\mathbf{u}$ in an anisotropic linear elastic material depends on the three material coordinates x_1, x_2, x_3, the deformation is three-dimensional. The relation between the strains ε_α and the stresses σ_α in the contracted notation (Voigt, 1910) is

$$\varepsilon_\alpha = s_{\alpha\beta}\sigma_\beta, \tag{2.1}$$

where $s_{\alpha\beta}$ are the elastic compliances. In an incompressible material the vanishing of the volume change is given by

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = (s_{1\beta} + s_{2\beta} + s_{3\beta})\sigma_\beta = 0. \tag{2.2}$$

If this is to hold for any stresses we must have

$$s_{1\beta} + s_{2\beta} + s_{3\beta} = 0 \quad \text{for } \beta = 1, 2, 3, 4, 5, 6. \tag{2.3}$$

There are six conditions for incompressibility. When the material is isotropic (see, for example, Ting, 1996, p. 52), (2.3) is trivially satisfied for $\beta = 4, 5, 6$ while for $\beta = 1, 2, 3$ it recovers the single condition that $v = 1/2$. The number of conditions for incompressibility is also one for cubic materials. It is two for transversely isotropic, tetragonal, and trigonal materials; three for orthotropic materials; and four for monoclinic materials.

If the right-hand side of the equation in (2.3) is replaced by a non-zero constant for $\beta = 1, 2, 3$, (2.3) gives the condition for the material to have a uniform contraction under a uniform pressure (Ting, 2001). Isotropic and cubic materials are the only ones that can have a uniform contraction under a uniform pressure without any conditions on the elastic constants.

We show that (2.3) is *structurally invariant* (Ting, 2000). If (2.3) holds for a coordinate system x_j, it holds for any other coordinate system x_i^* obtained from x_j by an orthogonal transformation Ω_{ij}, say. Let

$$x_i^* = \Omega_{ij} x_j \qquad \Omega_{ik}\Omega_{jk} = \delta_{ij} = \Omega_{ki}\Omega_{kj}. \tag{2.4}$$

In the four-index tensor notation, the elastic compliances s_{ijks} referred to the rotated coordinate system x_i^* become

$$s^*_{ijks} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{st}s_{pqrt}. \tag{2.5}$$

By contracting $i = j$ and using $(2.4)_3$, this yields

$$s^*_{iiks} = \Omega_{kr}\Omega_{st}s_{pprt}. \tag{2.6}$$

However, (2.3) in the four-index tensor notation is $s_{pprt} = 0$. Eq. (2.6) then gives $s^*_{iiks} = 0$. This completes the proof.

The constraint (2.3) says that the first three rows of the 6×6 matrix $\mathbf{s}$ are linearly dependent. This means that $\mathbf{s}$ is singular, and that the rank of $\mathbf{s}$ is at most five. We assume that the rank is five, because that is the case for isotropic materials. The strain energy density cannot be negative for an incompressible material. Hence $\mathbf{s}$ must be positive semidefinite. The rank of $\mathbf{s}$ being five implies that there exists a 5×5 submatrix that is non-singular. According to Theorems 9.17.1 and 9.17.2 presented by Hohn (1965), a necessary and sufficient condition for the matrix $\mathbf{s}$ of rank five to be positive semidefinite is that the five leading principal minors of a non-singular submatrix be positive. It means that this non-singular submatrix must be positive definite.

To apply the theorem we write the matrix $\mathbf{s}$ satisfying the constraint (2.3) in the form

$$\mathbf{s} = \begin{bmatrix} s_{22} + 2s_{23} + s_{33} & & & & & \\ -(s_{22} + s_{23}) & s_{22} & & & & \\ -(s_{23} + s_{33}) & s_{23} & s_{33} & & & \\ -(s_{24} + s_{34}) & s_{24} & s_{34} & s_{44} & & \\ -(s_{25} + s_{35}) & s_{25} & s_{35} & s_{45} & s_{55} & \\ -(s_{26} + s_{36}) & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix}. \tag{2.7}$$

Only the lower triangle of the matrix is shown since it is symmetric. The 5×5 submatrix on the lower right corner of $\mathbf{s}$ can be prescribed arbitrarily. The elements in the first column (and hence the first row) of $\mathbf{s}$ are then determined. We will therefore take the 5×5 submatrix on the lower right corner of s to be positive definite. Before we write down the leading principal minors of this submatrix, we introduce the following notation for the minors of $\mathbf{s}$. Let $s(n_1, \ldots, n_k \,|\, m_1, \ldots, m_k)$ be the $k \times k$ minor of the matrix $s_{\alpha\beta}$, the elements of which belong to the rows of $s_{\alpha\beta}$ numbered $n_1, \ldots, n_k$ and columns numbered $m_1, \ldots, m_k$, $1 \leqslant k \leqslant 6$. A principal minor is $s(n_1, \ldots, n_k \,|\, n_1, \ldots, n_k)$, which is written as $s(n_1, \ldots, n_k)$ for simplicity. If the leading principal minors are taken from the lower right corner of the submatrix, a necessary and sufficient condition for the matrix $\mathbf{s}$ to be positive semidefinite is

$$s_{66} > 0,\ s(5,6) > 0,\ s(4,5,6) > 0,\ s(3,4,5,6) > 0,\ s(2,3,4,5,6) > 0. \tag{2.8}$$

If they are taken from the top left corner of the submatrix, we have

$$s_{22} > 0,\ s(2,3) > 0,\ s(2,3,4) > 0,\ s(2,3,4,5) > 0,\ s(2,3,4,5,6) > 0. \tag{2.9}$$

Eq. (2.8) or (2.9) is the necessary and sufficient condition for the matrix $\mathbf{s}$ to be positive semidefinite.

The first two inequalities in (2.9) are the necessary and sufficient conditions for the 3×3 submatrix on the top left corner of the matrix $\mathbf{s}$ to be positive semidefinite. When the three equations for $\beta = 1,2,3$ in (2.3) are solved for s_{12}, s_{23}, s_{31}, we have

$$\begin{aligned} s_{12} &= \tfrac{1}{2}(s_{33} - s_{11} - s_{22}),\\ s_{23} &= \tfrac{1}{2}(s_{11} - s_{22} - s_{33}),\\ s_{31} &= \tfrac{1}{2}(s_{22} - s_{33} - s_{11}). \end{aligned} \tag{2.10}$$

Hence s_{11}, s_{22}, s_{33} are all we need to prescribe the 3×3 submatrix. The s_{11}, s_{22}, s_{33} are, respectively, $1/E_1$, $1/E_2$, $1/E_3$, where E_i are the Young's moduli. With the s_{23} given in (2.10), the second inequality in (2.9) is

$$s(2,3) = s_{22}s_{33} - \tfrac{1}{4}(s_{11} - s_{22} - s_{33})^2 > 0. \tag{2.11}$$

Since $s_{22} > 0$, (2.11) tells us that $s_{33} > 0$. Eq. (2.11) can then be written as

$$\left[\left(\sqrt{s_{22}} + \sqrt{s_{33}}\right)^2 - s_{11}\right]\left[s_{11} - \left(\sqrt{s_{22}} - \sqrt{s_{33}}\right)^2\right] > 0. \tag{2.12}$$

It tells us that $s_{11} > 0$. This is rewritten in a form symmetric with respect to s_{11}, s_{22}, s_{33} as

$$(U + V + W)(U + V - W)(V + W - U)(W + U - V) > 0, \tag{2.13}$$

where $U = \sqrt{s_{11}}$, $V = \sqrt{s_{22}}$, $W = \sqrt{s_{33}}$. Scott (2000) obtained the same inequality involving the area modulus of elasticity. From Hero's formula, the left-hand side of (2.13) is, after taking the square root and dividing the result by 4, the area of a triangle whose three sides are U, V, W. Thus $\sqrt{s_{11}}$, $\sqrt{s_{22}}$, $\sqrt{s_{33}}$ must form a triangle with a non-zero area for the 3×3 submatrix to be positive semidefinite.

Another geometrical interpretation of the constraint on s_{11}, s_{22}, s_{33} can be made by noticing that (2.13) is equivalent to

$$V + W > U > |V - W|. \tag{2.14}$$

In a rectangular coordinate system U, V, W, the point (U, V, W) is inside a triangular cone (or pyramid) in the space $U > 0$, $V > 0$, $W > 0$. The three edges of the cone lie on the three coordinate planes making an equal angle $(\pi/4)$ with the coordinate axes.

Eq. (2.11) can be written in a symmetric form as

$$s(2,3) = \tfrac{1}{2}(s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11}) - \tfrac{1}{4}(s_{11} + s_{22} + s_{33})^2 > 0. \tag{2.15}$$

Hence the three 2×2 minors $s(2,3)$, $s(3,1)$ and $s(1,2)$ are identical.

3. Incompressibility for two-dimensional deformations

When the displacement $\mathbf{u}$ depends on x_1, x_2 but not on x_3, the deformation is two-dimensional. In this case $\varepsilon_3 = u_{3,3} = 0$ and (2.1) is replaced by

$$\varepsilon_\alpha = s'_{\alpha\beta}\sigma_\beta, \tag{3.1}$$

where

$$s'_{\alpha\beta} = s_{\alpha\beta} - \frac{s_{\alpha 3}s_{3\beta}}{s_{33}} \tag{3.2}$$

are the reduced elastic compliances (Lekhnitskii, 1963). It should be noted that $s'_{\alpha 3} = s'_{3a} = 0$. With (3.1) the incompressibility condition $\varepsilon_1 + \varepsilon_2 = 0$ yields

$$s'_{1\beta} + s'_{2\beta} = 0, \quad \text{for } \beta = 1, 2, 4, 5, 6. \tag{3.3}$$

When the material is isotropic (see, for example, Ting, 1996, p. 53), (3.3) is trivially satisfied for $\beta = 4, 5, 6$ while for $\beta = 1, 2$ it recovers the condition that $v = 1/2$.

Under a rotation of the coordinate system about the x_3-axis, Ting (2000) has shown that the following relations for the elastic stiffnesses $C_{\alpha\beta}$ in the contracted notation are *structurally invariant*:

$$C_{16} + C_{26} = C_{11} - C_{22} = 0, \qquad C_{14} + C_{24} = C_{15} + C_{25} = 0. \tag{3.4}$$

They are called Types 1A and 4A, respectively. He pointed out that (3.4) applies also to $s'_{\alpha\beta}$. Following his derivation it can be shown that

$$C_{11} + C_{12} = C_{12} + C_{22} = C_{16} + C_{26} = 0 \tag{3.5}$$

is structurally invariant, and that it applies to $s'_{\alpha\beta}$. Thus the incompressibility condition (3.3) is *structurally invariant* under a rotation of the coordinate system about the x_3-axis.

The reduced elastic compliance matrix that satisfies (3.3) has the structure

$$\mathbf{s}' = \begin{bmatrix} s'_{22} & & & & \\ -s'_{22} & s'_{22} & & & \\ -s'_{24} & s'_{24} & s'_{44} & & \\ -s'_{25} & s'_{25} & s'_{45} & s'_{55} & \\ -s'_{26} & s'_{26} & s'_{46} & s'_{56} & s'_{66} \end{bmatrix}. \tag{3.6}$$

The matrix $\mathbf{s}'$ must be positive semidefinite. A necessary and sufficient condition for the matrix $\mathbf{s}'$ to be positive semidefinite is that the four leading principal minors of the 4×4 submatrix on the lower right corner of $\mathbf{s}'$ be positive. If the leading principal minors are taken from the lower right corner of the submatrix, a necessary and sufficient condition for $\mathbf{s}'$ to be positive semidefinite is

$$s'_{66} > 0,\ s'(5,6) > 0,\ s'(4,5,6) > 0,\ s'(2,4,5,6) > 0. \tag{3.7}$$

If they are taken from the top left corner of the submatrix, we have

$$s'_{22} > 0,\ s'(2,4) > 0,\ s'(2,4,5) > 0,\ s'(2,4,5,6) > 0. \tag{3.8}$$

Using (3.2), (3.3) can be rewritten as

$$s_{1\beta} + s_{2\beta} + ws_{3\beta} = 0, \qquad w = -(s_{13} + s_{23})/s_{33}. \tag{3.9}$$

This is an identity when $\beta = 3$. An elastic compliance matrix that satisfies (3.9) has the structure

$$\mathbf{s} = \begin{bmatrix} s_{22} + 2ws_{23} + w^2 s_{33} & & & & & \\ -(s_{22} + ws_{23}) & s_{22} & & & & \\ -(s_{23} + ws_{33}) & s_{23} & s_{33} & & & \\ -(s_{24} + ws_{34}) & s_{24} & s_{34} & s_{44} & & \\ -(s_{25} + ws_{35}) & s_{25} & s_{35} & s_{45} & s_{55} & \\ -(s_{26} + ws_{36}) & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix}, \tag{3.10}$$

where w is arbitrary. It reduces to (2.7) when $w = 1$. Thus incompressibility in three-dimensional deformations implies incompressibility in two-dimensional deformations, but the converse need not hold. A necessary and sufficient condition for the matrix $\mathbf{s}$ in (3.10) to be positive semidefinite is identical to the one in (2.8) or (2.9). It should be noted that (2.8) or (2.9) does not involve w. If the matrix $\mathbf{s}$ in (3.10) is positive semidefinite for any w, then s_{11} and $s(1,2)$, which can be computed easily, should be non-negative for any w. It can be shown that

$$s_{11} = \left(w\sqrt{s_{33}} + \frac{s_{23}}{\sqrt{s_{33}}} \right)^2 + \frac{s(2,3)}{s_{33}}, \quad s(1,2) = w^2 s(2,3), \tag{3.11}$$

so that s_{11} and $s(1,2)$ are indeed non-negative for any w. When $w = 0$, $s(1,2) = 0$ but the rank of the 3×3 submatrix on the top left corner of the matrix $\mathbf{s}$ is two for any w.

4. Secular equation for surface waves in incompressible monoclinic materials

The interest for considering incompressibility for surface waves in linear anisotropic elasticity is threefold. From a *historical* perspective, it must be remembered that

Lord Rayleigh (1885), the initiator of the theoretical study of elastic surface waves, did treat the case of incompressible linearly isotropic elastic half-space. Although some literature can be found on the subject of surface waves in incompressible, finitely elastic, stress-induced anisotropic half-spaces (Flavin, 1963; Wilson, 1973; Dowaikh and Ogden, 1990; Chadwick, 1997), very few papers are placed within the counterpart context of linearly elastic, anisotropic half-spaces, subject to the internal constraint of incompressibility. Second, from an *experimental* point of view, it is accepted that certain elastic materials may be modeled as incompressible, linearly elastic, anisotropic materials (Nair and Sotiropoulos, 1997, 1999; Sotiropoulos and Nair, 1999; Sutcu, 1992; Guz and Guz, 1999). According to Nair and Sotiropoulos (1997), such is the case for "polymer Kratons, thermo-plastic elastomers, rubber composites when low frequency waves are considered to justify the assumption of material inhomogeneity, etc". Third, the *theoretical* aspect of incompressibility in linear anisotropic elasticity has not been addressed in this context, and it is important to derive the secular equation in terms of the compliances rather than in terms of the stiffnesses.

Here attention is turned to surface waves propagating with speed v in the direction of the x_1-axis in the half-space $x_2 \geqslant 0$. The material is monoclinic with the symmetry plane at $x_3 = 0$. In the general (compressible) case the secular equation for the surface wave has been obtained explicitly by Destrade (2001a) using the method of first integrals introduced by Mozhaev (1995), and by Ting (2002) using a modified Stroh (1962) formalism. Letting $X = \rho v^2$ where ρ is the mass density, the secular equation is

$$[\eta - (1 + r_2)X]((\eta - X)[(\eta - X)(n_{66}X - 1) + r_6^2 X] + X^2[(\eta - X)n_{22} + r_2^2]) \\ + 2r_6 X^2(\eta - X)[(\eta - X)n_{26} + r_2 r_6] = 0. \tag{4.1}$$

It is a quartic in X. In (4.1), (Ting, 2002)

$$\eta = \frac{1}{s'_{11}}, \qquad r_2 = \frac{-s'_{12}}{s'_{11}}, \qquad r_6 = \frac{-s'_{16}}{s'_{11}},$$
$$n_{66} = \frac{s'(1,6)}{s'_{11}}, \qquad n_{26} = \frac{s'(1,2|1,6)}{s'_{11}}, \qquad n_{22} = \frac{s'(1,2)}{s'_{11}}. \tag{4.2}$$

The incompressible case was first studied by Nair and Sotiropoulos (1999), but they did not establish the secular equation explicitly. The secular equation for incompressible materials can be deduced directly from (4.1) by imposing the incompressibility conditions $s'_{2\beta} = -s'_{1\beta}$. The quantities r_2, n_{26}, n_{22} in (4.2) simplify to

$$r_2 = 1, \qquad n_{26} = 0, \qquad n_{22} = 0, \tag{4.3}$$

and the secular equation (4.1) reduces to

$$(\eta - 2X)((\eta - X)^2(n_{66}X - 1) + X^2) + r_6^2 \eta X(\eta - X) = 0. \tag{4.4}$$

It can be written in a non-dimensional form as

$$(1-2\xi)((1-\xi)^2(\kappa\xi-1)+\xi^2)+r_6^2\xi(1-\xi)=0, \tag{4.5}$$

$$\xi = X/\eta = \rho v^2 s'_{11}, \qquad \kappa = n_{66}/s'_{11}. \tag{4.6}$$

For incompressible orthotropic materials for which $s'_{16}=0$, the secular equation further simplifies to

$$(1-\xi)^2(1-\kappa\xi)=\xi^2, \qquad \kappa = s'_{66}/s'_{11}, \tag{4.7}$$

since $(1-2\xi)\neq 0$. This cubic in ξ has a more compact and satisfying form than that obtained by Destrade (2001b) in terms of the stiffnesses which, as stressed in Section 1, are not easily defined for incompressible anisotropic materials.

The secular equation is only a part of the surface wave solution. A complete solution requires the computation of the displacements and stresses. This is discussed next.

5. The Stroh formalism for steady state motion

In a fixed rectangular coordinate system x_i $(i=1,2,3)$ the stress–strain law and the equation of motion are

$$\sigma_{ij} = C_{ijks}u_{k,s}, \tag{5.1}$$

$$C_{ijks}u_{k,sj} = \rho\ddot{u}_i, \tag{5.2}$$

in which the dot stands for differentiation with time t. Consider a steady state motion with the steady wave speed v propagating in the direction of the x_1-axis. A solution for the displacement vector $\mathbf{u}$ of (5.2) can be written as (Stroh, 1962)

$$\mathbf{u} = \mathbf{a}f(z), \qquad z = x_1 - vt + px_2, \tag{5.3}$$

in which f is an arbitrary function of z, and p and $\mathbf{a}$ satisfy the eigenrelation

$$\boldsymbol{\Gamma}\mathbf{a} = \mathbf{0}, \tag{5.4}$$

$$\boldsymbol{\Gamma} = \mathbf{Q} - X\mathbf{I} + p(\mathbf{R}+\mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}, \tag{5.5}$$

$$X = \rho v^2. \tag{5.6}$$

In the above the superscript T stands for the transpose, $\mathbf{I}$ is the unit matrix, and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$ are 3×3 matrices whose elements are

$$Q_{ik} = C_{i1k1}, \qquad R_{ik} = C_{i1k2}, \qquad T_{ik} = C_{i2k2}. \tag{5.7}$$

The matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and so is the matrix $\boldsymbol{\Gamma}$. Introducing the new vector $\mathbf{b}$ defined by

$$\mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = -[p^{-1}(\mathbf{Q}-X\mathbf{I})+\mathbf{R}]\mathbf{a}, \tag{5.8}$$

in which the second equality follows from (5.4), the stress determined from (5.1) can be written as

$$\sigma_{i1} = -\phi_{i,2} - \rho v \dot{u}_i, \qquad \sigma_{i2} = \phi_{i,1}. \tag{5.9}$$

The ϕ_i $(i = 1, 2, 3)$ are the components of the stress function vector

$$\boldsymbol{\phi} = \mathbf{b} f(z). \tag{5.10}$$

There are six eigenvalues p_α and six Stroh eigenvectors $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ $(\alpha = 1, 2, \ldots, 6)$. When p_α are complex, they consist of three pairs of complex conjugates. If p_1, p_2, p_3 are the eigenvalues with a positive imaginary part, the remaining three eigenvalues are the complex conjugates of p_1, p_2, p_3. Assuming that p_1, p_2, p_3 are distinct, the general solution obtained from superposing three solutions of (5.3) and (5.10) associated with p_1, p_2, p_3 can be written in matrix notation as

$$\mathbf{u} = \mathbf{A}\langle f(z_*)\rangle \mathbf{q}, \qquad \boldsymbol{\phi} = \mathbf{B}\langle f(z_*)\rangle \mathbf{q}, \tag{5.11}$$

where $\mathbf{q}$ is an arbitrary constant vector and

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3], \tag{5.12a}$$

$$\langle f(z_*)\rangle = \mathrm{diag}[f(z_1), f(z_2), f(z_3)], \tag{5.12b}$$

$$z_\alpha = x_1 - vt + p_\alpha x_2. \tag{5.12c}$$

For surface waves in the half-space $x_2 \geqslant 0$, the function $f(z)$ is chosen as

$$f(z) = \mathrm{e}^{\mathrm{i}kz}, \tag{5.13}$$

where k is the real wave number. Since the imaginary parts of p_1, p_2, p_3 are positive, $(5.11)_1$ assures us that $\mathbf{u} \to \mathbf{0}$ as $x_2 \to \infty$. The surface traction at $x_2 = 0$ vanishes if $\boldsymbol{\phi} = \mathbf{0}$ at $x_2 = 0$, i.e.,

$$\mathbf{Bq} = \mathbf{0}. \tag{5.14}$$

This has a non-trivial solution for $\mathbf{q}$ when the determinant of $\mathbf{B}$ vanishes, i.e.,

$$|\mathbf{B}| = 0. \tag{5.15}$$

This is the secular equation for v. For a monoclinic material with the symmetry plane at $x_3 = 0$, (5.15) leads to (4.1).

The displacement $\mathbf{u}$ and the stress function vector $\boldsymbol{\phi}$ given in (5.11) require the computation of the eigenvalues p_α and the eigenvectors $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ $(\alpha = 1, 2, 3)$. They are provided by (5.4) and (5.8) which are in terms of the elastic stiffnesses. They are not suitable for taking the incompressible limit. A different expression in terms of the reduced elastic compliances is needed. This is presented next.

6. Steady state motion for incompressible materials

The two equations in (5.8) can be written in a standard eigenrelation as (Ingebrigtsen and Tonning, 1969; Barnett and Lothe, 1973; Chadwick and

Smith, 1977)

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{6.1}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 + X\mathbf{I} & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{6.2}$$

$$\mathbf{N}_1 = -\,\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \qquad \mathbf{N}_2 = \mathbf{T}^{-1}, \qquad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}. \tag{6.3}$$

It was shown by Ting (1988) that $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ have the structure

$$-\mathbf{N}_1 = \begin{bmatrix} r_6 & 1 & s_6 \\ r_2 & 0 & s_2 \\ r_4 & 0 & s_4 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} n_{66} & n_{26} & n_{46} \\ n_{26} & n_{22} & n_{24} \\ n_{46} & n_{24} & n_{44} \end{bmatrix}, \quad -\mathbf{N}_3 = \begin{bmatrix} m_{55} & 0 & -m_{15} \\ 0 & 0 & 0 \\ -m_{15} & 0 & m_{11} \end{bmatrix}. \tag{6.4}$$

An explicit expression of the elements of $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ was given in Ting (1988) in terms of the reduced elastic compliances $s'_{\alpha\beta}$ and in Barnett and Chadwick (1990) in terms of the elastic stiffnesses $C_{\alpha\beta}$. The expressions in term of the reduced elastic compliances are (Ting, 1996, p. 167)

$$\begin{aligned} r_\alpha &= \frac{1}{\varDelta} s'(1,\,5|5,\,\alpha), \qquad s_\alpha = \frac{1}{\varDelta} s'(1,\,5|\alpha,\,1), \\ n_{\alpha\beta} &= \frac{1}{\varDelta} s'(\alpha, 1,\, 5|\beta, 1,\, 5), \qquad m_{\alpha\beta} = \frac{1}{\varDelta} a\beta, \quad \varDelta = s'(1,5). \end{aligned} \tag{6.5}$$

Since $s'_{2\beta} = -\,s'_{1\beta}$ for incompressible materials, it can be shown that

$$r_2 = 1,\; s_2 = 0,\; n_{26} = n_{22} = n_{24} = 0. \tag{6.6}$$

Thus, for incompressible materials, the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ have the simpler expressions (see also Chadwick, 1997)

$$-\mathbf{N}_1 = \begin{bmatrix} r_6 & 1 & s_6 \\ 1 & 0 & 0 \\ r_4 & 0 & s_4 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} n_{66} & 0 & n_{46} \\ 0 & 0 & 0 \\ n_{46} & 0 & n_{44} \end{bmatrix}. \tag{6.7}$$

Eq. (6.1) consists of six scalar equations. The second and the fifth equations provide the identities

$$a_1 + pa_2 = 0, \qquad b_1 + pb_2 = Xa_2. \tag{6.8}$$

The first identity could have been deduced by inserting the solution (5.3) into the condition of incompressibility

$$\varepsilon_1 + \varepsilon_2 = u_{1,1} + u_{2,2} = 0. \tag{6.9}$$

With $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ expressed in terms of $s'_{\alpha\beta}$, (6.1) can be employed to compute the eigenvalues p and the eigenvectors $\mathbf{a}$ and $\mathbf{b}$. Eq. (6.1) consists of two equations,

$$(\mathbf{N}_1 - p\mathbf{I})\mathbf{a} + \mathbf{N}_2\mathbf{b} = \mathbf{0}, \qquad (\mathbf{N}_3 + X\mathbf{I})\mathbf{a} + (\mathbf{N}_1^{\mathrm{T}} - p\mathbf{I})\mathbf{b} = \mathbf{0}. \tag{6.10}$$

Assuming that $(\mathbf{N}_3 + X\mathbf{I})$ is not singular, $(6.10)_2$ can be solved for $\mathbf{a}$ and $(6.10)_1$ can be written as

$$\hat{\mathbf{\Gamma}}\mathbf{b} = \mathbf{0}, \tag{6.11}$$

$$\hat{\mathbf{\Gamma}} = \hat{\mathbf{Q}} + p(\hat{\mathbf{R}} + \hat{\mathbf{R}}^{\mathrm{T}}) + p^2\hat{\mathrm{T}}. \tag{6.12}$$

In the above,

$$\hat{\mathbf{T}} = (-\mathbf{N}_3 - X\mathbf{I})^{-1}, \quad \hat{\mathbf{R}} = -\mathbf{N}_1\hat{\mathbf{T}}, \quad \hat{\mathbf{Q}} = \mathbf{N}_1\hat{\mathbf{T}}\mathbf{N}_1^{\mathrm{T}} + \mathbf{N}_2. \tag{6.13}$$

An explicit expression of the elements of $\hat{\mathbf{T}}, \hat{\mathbf{R}}, \hat{\mathbf{Q}}$ is given in Ting (2002). Eq. (6.11) provides the eigenvalue p and the eigenvector $\mathbf{b}$. The eigenvector $\mathbf{a}$ obtained from $(6.10)_2$ is, using (6.11) and (6.13),

$$\mathbf{a} = -(\hat{\mathbf{R}}^{\mathrm{T}} + p\hat{\mathbf{T}})\mathbf{b} = (p^{-1}\hat{\mathbf{Q}} + \hat{\mathbf{R}})\mathbf{b}. \tag{6.14}$$

We have thus presented equations for computing the eigenvalues p and the eigenvectors $\mathbf{a}$ and $\mathbf{b}$ needed for the surface wave solution in terms of $s'_{\alpha\beta}$. The surface wave solution for an incompressible material is then complete.

7. Elastostatics for incompressible materials

The solutions (5.11) and (5.12) remain valid for elastostatics if we set $v = 0$. The derivation in (6.1)–(6.9) also holds for elastostatics if we let $X = 0$. However, the derivation from (6.11) to (6.14) is not valid for elastostatics because $(\mathbf{N}_3 + X\mathbf{I})$ is singular when $X = 0$. A different approach is needed to find $\mathbf{a}$ and $\mathbf{b}$ in terms of $s'_{\alpha\beta}$.

A modified Lekhnitskii formalism in the style of Stroh was proposed by Ting (1999) in which the vector $\mathbf{b}$ satisfies the eigenrelation (see also Barnett and Kirchner, 1997; Yin, 1997)

$$\begin{bmatrix} 1 & -p & 0 \\ 0 & \ell_4 & -\ell_3 \\ 0 & -\ell_3 & \ell_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{0}. \tag{7.1}$$

In the above

$$\begin{aligned} \ell_2 &= s'_{55}p^2 - 2s'_{45}p + s'_{44}, \\ \ell_3 &= s'_{15}p^3 - (s'_{14} + s'_{56})p^2 + (s'_{25} + s'_{46})p - s'_{24}, \\ \ell_4 &= s'_{11}p^4 - 2s'_{16}p^3 + (s'_{66} + 2s'_{12})p^2 - 2s'_{26}p + s'_{22}. \end{aligned} \tag{7.2}$$

The first of the three scalar equations in (7.1) recovers the identity $(6.8)_2$ when $X=0$. From (7.1) the eigenvalues p are computed from the sextic equation

$$\ell_2\ell_4 - \ell_3\ell_3 = 0, \tag{7.3}$$

originally given by Lekhnitskii (1963). The vector **a** is (Ting, 1999)

$$\mathbf{a} = \begin{bmatrix} g_1 & -h_1 \\ p^{-1}g_2 & -p^{-1}h_2 \\ g_5 & h_5 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}, \tag{7.4}$$

in which

$$g_\alpha = s'_{\alpha 1}p^2 - s'_{\alpha 6}p + s'_{\alpha 2}, \qquad h_\alpha = s'_{\alpha 5}p - s'_{\alpha 4}. \tag{7.5}$$

We have thus the eigenvalues p and the eigenvectors **a** and **b** all in terms of $s'_{\alpha\beta}$.

When the material is incompressible, $s'_{2\beta} = -s'_{1\beta}$ and the ℓ_3, ℓ_4 in (7.2) simplify to

$$\begin{aligned} \ell_3 &= (s'_{15}p - s'_{14})(p^2-1) - s'_{56}p^2 + s'_{46}p, \\ \ell_4 &= s'_{11}(p^2-1)^2 - 2s'_{16}p(p^2-1) + s'_{66}p^2. \end{aligned} \tag{7.6}$$

Also, (7.5) gives

$$g_2 = -g_1, \quad h_2 = -h_1, \tag{7.7}$$

and (7.4) can be written as

$$\mathbf{a} = \begin{bmatrix} g_1 & -h_1 \\ -p^{-1}g_1 & p^{-1}h_1 \\ g_5 & h_5 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}. \tag{7.8}$$

The a_1, a_2 components of the vector **a** computed from (7.8) indeed satisfy the identity $(6.8)_1$.

In the next section we study the special case of incompressible monoclinic materials with the symmetry plane at $x_3 = 0$.

8. Monoclinic materials with the symmetry plane at $x_3 = 0$

When the material is monoclinic with the symmetry plane at $x_3 = 0$, ℓ_3 vanishes identically so that the sextic equation (7.3) leads to $\ell_2 = 0$ or $\ell_4 = 0$. If the material is incompressible, ℓ_4 is given by (7.6) and we have

$$(p - p^{-1})^2 - 2\alpha(p - p^{-1}) + \beta = 0, \tag{8.1}$$

where

$$\alpha = s'_{16}/s'_{11}, \quad \beta = s'_{66}/s'_{11}. \tag{8.2}$$

Since p_1, p_2 are the roots of (8.1) with a positive imaginary part, (8.1) gives

$$p - p^{-1} = \alpha + i\gamma, \tag{8.3}$$

in which

$$\gamma = \sqrt{\beta - \alpha^2} = \sqrt{s'(1,6)}/s'_{11}. \tag{8.4}$$

Eq. (8.3) tells us that

$$p_1 + p_2 = \alpha + i\gamma, \quad p_1 p_2 = -1. \tag{8.5}$$

We also obtain explicit expression for p_1, p_2 as

$$p_1, p_2 = \frac{\alpha + i\gamma}{2} \pm \sqrt{\left(\frac{\alpha + i\gamma}{2}\right)^2 + 1}. \tag{8.6}$$

The three Barnett–Lothe (1973) tensors $\mathbf{S}, \mathbf{H}, \mathbf{L}$ appear often in the solutions to anisotropic elasticity problems. They are real. Explicit expressions of $\mathbf{S}, \mathbf{H}, \mathbf{L}$ for monoclinic materials with the symmetry plane at $x_3 = 0$ have been presented in (Ting, 1996, p. 174). Specialized to incompressible materials using (8.5) leads to

$$\mathbf{S} = \mathbf{0}, \quad \mathbf{H} = \mathbf{L}^{-1} = \mathrm{diag}[\gamma s'_{11},\ \gamma s'_{11},\ 1/\mu], \tag{8.7}$$

$$\mu = [s'(4,5)]^{-1/2}. \tag{8.8}$$

The quantity μ is the shear modulus when the material is isotropic. The structure of $\mathbf{S}, \mathbf{H}, \mathbf{L}$ in (8.7) provides the following interesting results in elastostatics for incompressible materials.

The order of the stress singularity at an interfacial crack tip in a bimaterial consisting of two dissimilar materials bonded together is not a complex number when $\mathbf{SL}^{-1}$ in the two materials are identical. In this case, the physically unrealistic interpenetration of the crack surface displacement does not occur (see, for example, Ting, 1996, p. 144). For a bimaterial for which both materials are incompressible, $\mathbf{S} = \mathbf{0}$ according to (8.7). Hence $\mathbf{SL}^{-1}$ vanishes in both materials. Therefore, there is no interpenetration of the crack surfaces when the material is incompressible and monoclinic with the symmetry plane at $x_3 = 0$.

The inplane displacement and the antiplane displacement for a monoclinic material with the symmetry plane at $x_3 = 0$ are uncoupled. We can therefore consider the inplane and antiplane deformations separately. Consider the inplane deformation. The Barnett–Lothe tensors now require only the 2×2 submatrix located at the top left corner of $\mathbf{S}, \mathbf{H}, \mathbf{L}$. From (8.7) we have

$$\mathbf{S} = \mathbf{0}, \quad \mathbf{H} = \mathbf{L}^{-1} = \gamma s'_{11}\mathbf{I}, \tag{8.9}$$

where $\mathbf{I}$ is the 2×2 identity matrix. Consider now an infinite monoclinic material subject to a line of concentrated force $\mathbf{f}$ and a line of dislocation with Burgers vector $\hat{\mathbf{b}}$ applied along the x_3-axis. The strain energy in the annular region bounded by the two radii

$r_2 > r_1$ can be shown to be (Eshelby, 1956; Ting, 1996, p. 249)

$$\frac{1}{4\pi}\ln\left(\frac{r_2}{r_1}\right)(\mathbf{f}^{\mathrm{T}}\mathbf{H}\mathbf{f} + \hat{\mathbf{b}}^{\mathrm{T}}\mathbf{L}\hat{\mathbf{b}}), \tag{8.10}$$

for a compressible material. When the material is incompressible and when the vectors $\mathbf{f}$ and $\hat{\mathbf{b}}$ lie on the $x_3 = 0$ plane, use of (8.9) in (8.10) yields

$$\frac{1}{4\pi}\ln\left(\frac{r_2}{r_1}\right)(\gamma s_{11}'|\mathbf{f}|^2 + (\gamma s_{11}')^{-1}|\hat{\mathbf{b}}|^2). \tag{8.11}$$

The strain energy depends on the magnitudes, not on the direction, of the vectors $\mathbf{f}$ and $\hat{\mathbf{b}}$.

Consider next a half-space with a traction-free boundary surface subject to a line dislocation with Burgers vector $\hat{\mathbf{b}}$ in the half-space. The image force F_{A} that is *attracted* to the free-surface is (Barnett and Lothe, 1974; Ting, 1996, pp. 264–265)

$$F_{\mathrm{A}} = \frac{1}{4\pi d}\hat{\mathbf{b}}^{\mathrm{T}}\mathbf{L}\hat{\mathbf{b}}, \tag{8.12}$$

where d is the distance between the line dislocation and the half-space boundary. When the material is incompressible, by virtue of (8.9), F_{A} depends on the magnitude, not on the direction, of the Burgers vector $\hat{\mathbf{b}}$. Likewise, if the boundary surface is a rigid surface, the image force F_{R} that is *repelled* by the rigid surface is (Ting and Barnett, 1993)

$$F_{\mathrm{R}} = \frac{1}{4\pi d}\hat{\mathbf{b}}^{\mathrm{T}}(2\mathbf{H}^{-1} - \mathbf{L})\hat{\mathbf{b}} \geqslant F_{\mathrm{A}}. \tag{8.13}$$

When the material is incompressible, F_{R} depends on the magnitude, not on the direction, of the Burgers vector $\hat{\mathbf{b}}$. Moreover, $F_{\mathrm{R}} = F_{\mathrm{A}}$ due to (8.9).

The same result applies to a line dislocation in a bimaterial that consists of two dissimilar materials bonded together (Barnett and Lothe, 1974; Ting, 1996, p. 286). When the material is incompressible, the image force that is attracted to, or repelled by, the interface depends on the magnitude, not on the direction, of the Burgers vector.

Clearly, other interesting physical phenomena can be cited when the material is incompressible and monoclinic with the symmetry plane at $x_3 = 0$.

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