Scattering relations for point sources: Acoustic and electromagnetic waves

C. Athanasiadis
Department of Mathematics, University of Athens, Panepistimiopolis, GR 157 84 Athens, Greece

P. A. Martin[a)]
Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, Colorado 80401-1887

A. Spyropoulos and I. G. Stratis
Department of Mathematics, University of Athens, Panepistimiopolis, GR 157 84 Athens, Greece

(Received 25 March 2002; accepted 20 May 2002)

The problem of scattering of spherical waves by a bounded obstacle is considered. General scattering theorems are proved. These relate the far-field patterns due to scattering of waves from a point source put in any two different locations. The scatterer can have any of the usual properties, penetrable or impenetrable. The optical theorem is recovered as a corollary. Mixed scattering relations are also established, relating the scattered fields due to a point source and a plane wave.
 [DOI: 10.1063/1.1509089]

I. INTRODUCTION

In classical scattering theory, there are some general results that connect the solutions of two related problems. The most familiar of these is *reciprocity*: the scattered field at A due to a source at B is simply related to the scattered field at B due to a source at A.

There are also internal relations within a single problem. A well-known example is the *optical theorem* for scattering of plane waves: it relates the far-field pattern in the forward direction to a certain integral of the far-field pattern over all directions.

In this article, we derive some general relations for scattering of waves emanating from point sources. Thus, we relate one problem with a point source at A to a similar problem with a point source at B. By setting $A=B$ and then letting A recede to infinity, we recover the optical theorem. If we keep A fixed and let B recede to infinity, we obtain so-called *mixed scattering theorems*, relating plane-wave incidence to point-source incidence. An example of these is the *mixed reciprocity theorem*, which has found much use recently in methods for solving inverse scattering problems.[1]

As Logan[2] points out, Clebsch considered the scattering of elastic waves from a point source by a rigid sphere 140 years ago, a decade before Lord Rayleigh published his solution for the scattering of a plane sound wave by a sphere. Collected results for scattering of point-source fields by simple shapes are given in Ref. 3. More recently, Dassios and his co-workers have studied incident waves generated by a point source in the vicinity of a scatterer; see, for example, Refs. 4–6. There is also some recent work on near-field inverse problems: in addition to the previous papers, we note the work by Coyle[7] and Potthast,[1] as well as recent work by three of the present authors.[8,9] The revival of interest in problems related to point-generated wave fields has several reasons. One is due to the variety of applications coming from the theory of composite materials and of acoustic emission, from the theoretical analysis of biological studies at the cell level, from nondestructive testing and evaluation, from geophysics, from modeling in medicine and health

[a)]Author to whom correspondence should be addressed. Electronic mail: pamartin@mines.edu

sciences, and from scattering problems connected to environmental data analysis. Another reason is due to the fact that a point-source field is more easily realizable in a laboratory.

We give results for both acoustic and electromagnetic waves, and we permit all the usual kinds of scattering obstacles, penetrable and impenetrable. Extensions to elastic-wave problems are expected.

II. FORMULATION: ACOUSTICS

Let Ω^- be a bounded three-dimensional obstacle with a smooth closed boundary S (the scatterer). The exterior $\mathbb{R}^3 \setminus \Omega^- = \Omega$ of the scatterer is an infinite homogeneous isotropic lossless acoustic medium, that is the compressional viscosity δ is zero, and with mass density ρ, phase velocity c, mean compressibility γ and real wave number $k = \omega\sqrt{\gamma\rho}$, ω being the angular frequency. The interior of the scatterer Ω^- is filled with a lossy medium, in general, with corresponding physical parameters δ^-, ρ^-, c^- and γ^-. We consider an incident spherical acoustic wave due to a source located at a point with position vector $\mathbf{a}$. Suppressing the harmonic time dependence $\exp\{-\mathrm{i}\omega t\}$, and following the normalization introduced by Dassios and Kamvyssas,[4] we assume the following form for the incident field:

$$u_a^{\mathrm{i}}(\mathbf{r}) = a\,\mathrm{e}^{-\mathrm{i}ka}\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}-\mathbf{a}|}}{|\mathbf{r}-\mathbf{a}|}, \quad \mathbf{r} \neq \mathbf{a}, \tag{1}$$

where $a = |\mathbf{a}|$. We note that when $a \to \infty$, the spherical wave reduces to a plane wave with direction of propagation $-\hat{\mathbf{a}}$, where $\mathbf{a} = a\hat{\mathbf{a}}$.[4] The total field u_a^{t} in the exterior of the scatterer is given by

$$u_a^{\mathrm{t}}(\mathbf{r}) = u_a^{\mathrm{i}}(\mathbf{r}) + u_a^{\mathrm{s}}(\mathbf{r}), \quad \mathbf{r} \in \Omega \setminus \{\mathbf{a}\}, \tag{2}$$

where u_a^{s} is the scattered acoustic field, and solves the Helmholtz equation

$$\nabla^2 u_a^{\mathrm{t}}(\mathbf{r}) + k^2 u_a^{\mathrm{t}}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega. \tag{3}$$

The scattered as well as the incident fields are solutions of Helmholtz's equation that satisfy the Sommerfeld radiation condition

$$\hat{\mathbf{r}} \cdot \nabla u(\mathbf{r}) - \mathrm{i}ku(\mathbf{r}) = o(r^{-1}), \quad r \to \infty, \tag{4}$$

uniformly in all directions $\hat{\mathbf{r}} \in S^2$, where S^2 is the unit sphere. We note that u_a^{t} also satisfies the Sommerfeld radiation condition.

In our analysis, we can permit impenetrable or penetrable scatterers. In the former case, we could have Dirichlet $(u_a^{\mathrm{t}} = 0)$, Neumann $(\partial u_a^{\mathrm{t}}/\partial n = 0)$ or Robin conditions on S; the Robin (or impedance) condition is

$$\left(\frac{\partial}{\partial n} + \mathrm{i}k\lambda\right) u_a^{\mathrm{t}}(\mathbf{r}) = 0, \quad \mathbf{r} \in S, \tag{5}$$

where λ is a dimensionless real parameter. In the penetrable case, the incident wave is transmitted into the scatterer; let u_a^- be the total acoustic field in Ω^-. Then the following *transmission conditions* must hold on the scatterer's surface

$$u_a^{\mathrm{t}}(\mathbf{r}) = u_a^-(\mathbf{r}) \ \text{ and } \ \frac{\partial u_a^{\mathrm{t}}(\mathbf{r})}{\partial n} = \beta \frac{\partial u_a^-(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S, \tag{6}$$

where the constant $\beta = (\rho/\rho^-)(1 - \mathrm{i}kc\,\delta^-\gamma^-)$ is complex for a lossy scatterer and real for a lossless scatterer. More details on the physical parameters of the above problems can be found in Ref. 6. The field u_a^- solves the Helmholtz equation in Ω^-,

$$\nabla^2 u_a^-(\mathbf{r}) + \eta^2 k^2 u_a^-(\mathbf{r}), \quad \mathbf{r}\in\Omega^-, \tag{7}$$

where $\eta=(c/c^-)(1-\mathrm{i}kc\gamma^-\delta^-)^{-1/2}$ is the (complex) index of refraction between the two media in Ω and Ω^-. The choice of the branch of the square root is such that $\mathrm{Im}(\eta)\geqslant 0$ and hence $\mathrm{Im}(\eta k)\geqslant 0$.

In addition, in order to cover various cases that arise in applications, we can easily extend our analysis to include an impenetrable core in the interior of the scatterer.

As it is well known, all the above problems are well posed.

The behavior of the scattered wave in the far field is given by

$$u_a^{\mathrm{s}}(\mathbf{r}) = g_a(\hat{\mathbf{r}})h_0(kr) + O(r^{-2}), \quad r\to\infty, \tag{8}$$

where $h_0(x)=\mathrm{e}^{\mathrm{i}x}/(\mathrm{i}x)$ is the spherical Hankel function of the first kind and order zero. Moreover,

$$g_a(\hat{\mathbf{r}}) = -\frac{\mathrm{i}k}{4\pi}\int_S\left[\frac{\partial u_a^{\mathrm{s}}(\mathbf{r}')}{\partial n} + \mathrm{i}k(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}})u_a^{\mathrm{s}}(\mathbf{r}')\right]\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{r}'}ds(\mathbf{r}') \tag{9}$$

is the far-field pattern.[5]

III. GENERAL SCATTERING THEOREM

In what follows, we consider two locations for the point source, **a** and **b**, from which the time-harmonic incident spherical waves emanate. Each source generates a corresponding scattered field, u_a^{s} and u_b^{s}, respectively. We are interested in relations between these fields.

Let S_r denote a large sphere of radius r, surrounding the points **a** and **b**, and let

$$S_{a,\varepsilon} = \{\mathbf{r}\in\mathbb{R}^3 : |\mathbf{a}-\mathbf{r}| = \varepsilon\}, \tag{10}$$

a small sphere of radius ε, surrounding the point **a**. Then, we introduce the following notation,

$$[u,v]_{\mathcal{S}} = \int_{\mathcal{S}}\left(\overline{u}\frac{\partial v}{\partial n} - v\frac{\partial \overline{u}}{\partial n}\right)ds,$$

where the overbar denotes complex conjugation; in particular, we write $[u,v]\equiv[u,v]_S$.

Lemma 1: Let u_a^{i} *be a point source at* **a**. *Let* u_b^{i} *be a point source at* **b**, *with corresponding scattered field* u_b^{s} *and far-field pattern* g_b. *Then*

$$\lim_{r\to\infty}[u_a^{\mathrm{i}}, u_b^{\mathrm{s}}]_{S_r} = 2a\mathrm{e}^{\mathrm{i}ka}\int_{S^2} g_b(\hat{\mathbf{r}})\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}) \tag{11}$$

and

$$\lim_{\varepsilon\to 0}[u_a^{\mathrm{i}}, u_b^{\mathrm{s}}]_{S_{a,\varepsilon}} = 4\pi a\mathrm{e}^{\mathrm{i}ka}u_b^{\mathrm{s}}(\mathbf{a}), \tag{12}$$

where S_r *is a large sphere of radius* r *surrounding* **a** *and* **b**, *and* $S_{a,\varepsilon}$ *is the small sphere defined by Eq. (10).*

Proof: For Eq. (11), we use the asymptotic relations

$$|\mathbf{r}-\mathbf{a}| = r - \hat{\mathbf{r}}\cdot\mathbf{a} + O(r^{-1}) \quad \text{and} \quad |\mathbf{r}-\mathbf{a}|^{-1} = r^{-1} + O(r^{-2}), \tag{13}$$

as $r\to\infty$, and obtain

$$u_a^{\mathrm{i}}(\mathbf{r}) = h_0(kr)g_a^{\mathrm{i}}(\hat{\mathbf{r}}) + O(r^{-2}), \quad r\to\infty, \tag{14}$$

where $g_a^{\mathrm{i}}(\hat{\mathbf{r}}) = \mathrm{i}ka\exp\{-\mathrm{i}ka(1+\hat{\mathbf{a}}\cdot\hat{\mathbf{r}})\}$ is the far-field pattern of the point-source incident wave. Using Eqs. (8) and (14) gives Eq. (11).

For Eq. (12), we evaluate $\overline{u_a^{\mathrm{i}}}$ and its normal derivative on $S_{a,\varepsilon}$. There, we have $\overline{u_a^{\mathrm{i}}} = (a/\varepsilon)\mathrm{e}^{\mathrm{i}k(a-\varepsilon)}$ and $(\partial/\partial n)\overline{u_a^{\mathrm{i}}} = -(\mathrm{i}k+\varepsilon^{-1})\overline{u_a^{\mathrm{i}}}(\mathbf{r})$, where $\partial/\partial n$ denotes normal differentiation in the outward direction. Using the mean value theorem and letting $\varepsilon \to 0$, Eq. (12) is proved. □

Now, for two point sources, with position vectors **a** and **b**, we define a *spherical far-field pattern generator* for spherical acoustic waves by

$$G_b(\mathbf{a}) = \mathrm{i}ka\mathrm{e}^{\mathrm{i}ka}\left[u_b^{\mathrm{s}}(\mathbf{a}) - \frac{1}{2\pi}\int_{S^2} g_b(\hat{\mathbf{r}})\mathrm{e}^{\mathrm{i}k\mathbf{a}\cdot\hat{\mathbf{r}}}ds(\hat{\mathbf{r}})\right]. \tag{15}$$

This terminology and definition are appropriate for the following reason [see Eq. (50) later in this work]. When both the point source and the observation point recede to infinity, $G_b(\mathbf{a})$ reduces to the far-field pattern for an incident plane wave propagating in the direction $-\hat{\mathbf{b}}$. Using this notation, the general scattering theorem for spherical waves is formulated as follows.

Theorem 2: *For any two point-source locations in* Ω, **a** *and* **b**, *we have*

$$G_b(\mathbf{a}) + \overline{G_a(\mathbf{b})} + \frac{1}{2\pi}\int_{S^2} g_b(\hat{\mathbf{r}})\overline{g_a(\hat{\mathbf{r}})}ds(\hat{\mathbf{r}}) = \mathcal{E}_{a,b}, \tag{16}$$

where

$$\mathcal{E}_{a,b} = -\frac{\mathrm{i}k}{4\pi}[u_a^{\mathrm{t}}, u_b^{\mathrm{t}}]. \tag{17}$$

The value of $\mathcal{E}_{a,b}$ *depends on the scatterer:*

$$\mathcal{E}_{a,b} = 0 \quad \textit{for Dirichlet or Neumann conditions on } S; \tag{18}$$

$$\mathcal{E}_{a,b} = -\frac{k^2\lambda}{2\pi}\int_S u_b^{\mathrm{t}}(\mathbf{r})\overline{u_a^{\mathrm{t}}(\mathbf{r})}ds \quad \textit{for the Robin condition (5) on } S; \tag{19}$$

or

$$\mathcal{E}_{a,b} = -\frac{k}{2\pi}\mathrm{Im}(\beta)\int_{\Omega^-}\nabla\overline{u_a^-(\mathbf{r})}\cdot\nabla u_b^-(\mathbf{r})dv(\mathbf{r}) \quad \textit{for a penetrable scatterer,} \tag{20}$$

where β *is the constant in the transmission conditions (6).*

Proof: Let us first evaluate $\mathcal{E}_{a,b}$ directly. For Dirichlet ($u^{\mathrm{t}}=0$) or Neumann ($\partial u^{\mathrm{t}}/\partial n = 0$) conditions, we immediately obtain Eq. (18). Similarly, the Robin condition (5) gives Eq. (19). For a penetrable scatterer, we use the transmission conditions (6), apply Green's first theorem and take into account that $\mathrm{Im}(\beta\eta^2)=0$ (see p. 9 of Ref. 6); this gives Eq. (20).

Next, we give an alternative evaluation of $\mathcal{E}_{a,b}$. Thus, by the relations $u_\alpha^{\mathrm{t}} = u_\alpha^{\mathrm{i}} + u_\alpha^{\mathrm{s}}$ for $\alpha = a,b$ we have

$$[u_a^{\mathrm{t}}, u_b^{\mathrm{t}}] = [u_a^{\mathrm{i}}, u_b^{\mathrm{i}}] + [u_a^{\mathrm{s}}, u_b^{\mathrm{i}}] + [u_a^{\mathrm{i}}, u_b^{\mathrm{s}}] + [u_a^{\mathrm{s}}, u_b^{\mathrm{s}}]. \tag{21}$$

Since $\overline{u_a^{\mathrm{i}}}$ and u_b^{i} are regular solutions of the Helmholtz equation in Ω^-, Green's second theorem gives

$$[u_a^{\mathrm{i}}, u_b^{\mathrm{i}}] = 0. \tag{22}$$

For the other terms in Eq. (21), we consider two small spheres, S_{a,ε_1} and S_{b,ε_2}, centered at **a** and **b** with radii ε_1 and ε_2, respectively, with $S_{a,\varepsilon_1}\cap S_{b,\varepsilon_2} = \emptyset$, as well as a large sphere S_R centered at the origin surrounding the whole system of the scatterer and the two small spheres. Since $\overline{u_a^{\mathrm{i}}}(\mathbf{r})$ and $u_b^{\mathrm{s}}(\mathbf{r})$ are solutions of the Helmholtz equation, for $\mathbf{r}\neq\mathbf{a},\mathbf{b}$, Green's second theorem gives

$$[u_a^{\mathrm{i}},u_b^{\mathrm{s}}]=[u_a^{\mathrm{i}},u_b^{\mathrm{s}}]_{S_R}-[u_a^{\mathrm{i}},u_b^{\mathrm{s}}]_{S_{a,\varepsilon_1}}-[u_a^{\mathrm{i}},u_b^{\mathrm{s}}]_{S_{b,\varepsilon_2}}.$$

The third term is zero because $\overline{u_a^{\mathrm{i}}}$ and u_b^{s} are regular solutions of the Helmholtz equation in the interior of S_{b,ε_2}. Then, letting $R\to\infty$ and $\varepsilon_1\to 0$, using Lemma 1, we obtain

$$[u_a^{\mathrm{i}},u_b^{\mathrm{s}}]=-4\pi a\mathrm{e}^{\mathrm{i}ka}u_b^{\mathrm{s}}(\mathbf{a})+2a\mathrm{e}^{\mathrm{i}ka}\int_{S^2}g_b(\hat{\mathbf{r}})\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}). \tag{23}$$

As $[u_a^{\mathrm{s}},u_b^{\mathrm{i}}]=-\overline{[u_b^{\mathrm{i}},u_a^{\mathrm{s}}]}$, we easily deduce that

$$[u_a^{\mathrm{s}},u_b^{\mathrm{i}}]=4\pi b\mathrm{e}^{-\mathrm{i}kb}\overline{u_a^{\mathrm{s}}(\mathbf{b})}-2b\mathrm{e}^{-\mathrm{i}kb}\int_{S^2}\overline{g_a(\hat{\mathbf{r}})}\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{b}}ds(\hat{\mathbf{r}}). \tag{24}$$

Finally, in view of the regularity of $\overline{u_a^{\mathrm{s}}}$ and u_b^{s} in the region exterior to S, we have

$$[u_a^{\mathrm{s}},u_b^{\mathrm{s}}]=[u_a^{\mathrm{s}},u_b^{\mathrm{s}}]_{S_R}.$$

Then, letting $R\to\infty$, we pass to the radiation zone and thus we can use the asymptotic form (8), giving

$$[u_a^{\mathrm{s}},u_b^{\mathrm{s}}]=\frac{2\mathrm{i}}{k}\int_{S^2}\overline{g_a(\hat{\mathbf{r}})}g_b(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}). \tag{25}$$

Substituting Eqs. (22)–(25) in Eq. (21), making use of Eq. (17) and its evaluation, gives Eq. (16), and the theorem is proved. □

Let us make two remarks. First, if the scatterer is a lossless penetrable obstacle, that is, the physical parameter β is real, then we obtain $\mathcal{E}_{a,b}=0$ in Eq. (16), just as for soft or hard scatterers.

Second, suppose that we have a penetrable scatterer with a core S^- on which $u^-=0$ or $\partial u^-/\partial n=0$. Then, the relation (20) still holds, where, now, Ω^- denotes the part of the scatterer between the surfaces S and S^-.

IV. RECIPROCITY RELATIONS

The proof of Theorem 2 uses two evaluations of $[u_a^{\mathrm{t}},u_b^{\mathrm{t}}]$. If, instead, we start from $[\overline{u_a^{\mathrm{t}}},u_b^{\mathrm{t}}]$, we obtain a reciprocity theorem. This is not surprising, because u_a^{t} can be regarded as the exact Green's function for the scattering problem. The reciprocity theorem can be found on p. 48 of Ref. 6, for example. We quote it here, using our normalizations.

Theorem 3: *For any two point-source locations in* Ω, $\mathbf{a}$ *and* $\mathbf{b}$, *and for any scatterer, we have*

$$h_0(ka)u_a^{\mathrm{s}}(\mathbf{b})=h_0(kb)u_b^{\mathrm{s}}(\mathbf{a}), \tag{26}$$

where $h_0(x)=\mathrm{e}^{\mathrm{i}x}/(\mathrm{i}x)$.

From Eq. (1), we see that the same reciprocity relation holds for the incident fields, as well: $h_0(ka)u_a^{\mathrm{i}}(\mathbf{b})=h_0(kb)u_b^{\mathrm{i}}(\mathbf{a})$. Hence, by Eq. (2) we conclude that the total exterior fields satisfy $h_0(ka)u_a^{\mathrm{t}}(\mathbf{b})=h_0(kb)u_b^{\mathrm{t}}(\mathbf{a})$.

We note that the presence of the multiplicative constant in the above reciprocity relation is due to the form of the modified spherical wave, Eq. (1). If we consider the point sources lying on the same sphere, that is, $a=b$, then we obtain the following results:

$$u_a^{\mathrm{s}}(\mathbf{b})=u_b^{\mathrm{s}}(\mathbf{a}),\quad u_a^{\mathrm{i}}(\mathbf{b})=u_b^{\mathrm{i}}(\mathbf{a}),\quad u_a^{\mathrm{t}}(\mathbf{b})=u_b^{\mathrm{t}}(\mathbf{a}). \tag{27}$$

These relations express the fact that interchanging the excitation with the observation point, the scattered, incident and total fields remain unchanged. From Eq. (26) we cannot have a reciprocity relation for the corresponding spherical far-field patterns, because when the point sources go to infinity the spherical waves reduce to plane waves.

V. THE OPTICAL THEOREM

In Ref. 8, an optical theorem for spherical waves incident upon a soft scatterer has been proved. Now, this theorem as well as optical theorems for scatterers of other types can be derived as corollaries of the general scattering theorem. First we define the scattering cross-section due to a point source at **a** (Ref. 6) as

$$\sigma_a^{\mathrm{s}} = \frac{1}{k^2} \int_{S^2} |g_a(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}), \tag{28}$$

the absorption cross-section as

$$\sigma_a^{\mathrm{a}} = \frac{1}{k} \operatorname{Im} \int_S u_a^{\mathrm{t}} \frac{\partial \overline{u_a^{\mathrm{t}}}}{\partial n} ds, \tag{29}$$

and the extinction cross-section as

$$\sigma_a^{\mathrm{e}} = \sigma_a^{\mathrm{s}} + \sigma_a^{\mathrm{a}}. \tag{30}$$

If we put $\mathbf{a}=\mathbf{b}$ in Theorem 2, we obtain

$$2 \operatorname{Re}\{G_a(\mathbf{a})\} + \frac{1}{2\pi} \int_{S^2} |g_a(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}) = \mathcal{E}_{a,a}.$$

We can rewrite this equation using Eq. (28) to give

$$\sigma_a^{\mathrm{s}} = -4\pi k^{-2} \operatorname{Re}\{G_a(\mathbf{a})\} + 2\pi k^{-2} \mathcal{E}_{a,a}. \tag{31}$$

From the definitions (17) and (29), we have

$$\sigma_a^{\mathrm{a}} = \tfrac{1}{2} ik^{-1} [u_a^{\mathrm{t}}, u_a^{\mathrm{t}}] = -2\pi k^{-2} \mathcal{E}_{a,a}. \tag{32}$$

Hence, adding Eqs. (31) and (32), Eq. (30) gives

$$\sigma_a^{\mathrm{e}} = -4\pi k^{-2} \operatorname{Re}\{G_a(\mathbf{a})\}. \tag{33}$$

The value of $\mathcal{E}_{a,a}$ is given in Theorem 2. In particular, for a penetrable scatterer, we obtain

$$\sigma_a^{\mathrm{a}} = -\frac{1}{k} \operatorname{Im}(\beta) \int_{\Omega^-} |\nabla u_a^-(\mathbf{r})|^2 dv(\mathbf{r}), \tag{34}$$

whereas for an impedance surface, we have

$$\sigma_a^{\mathrm{a}} = \lambda k \int_S |u_a^{\mathrm{t}}|^2 ds.$$

We remark that the absorption cross-section σ_a^{a} provides a measure of the total energy taken from the incident spherical wave and absorbed by the surface of the scatterer in the impedance boundary case, or by the lossy medium occupying Ω^- in the penetrable case. It is clear that $\sigma_a^{\mathrm{a}} = 0$ for the soft and hard scatterers and $\sigma_a^{\mathrm{a}} \geqslant 0$ for the other cases.

VI. MIXED SCATTERING RELATIONS

For inverse problems, one effective reconstruction method is the point-source method.[1] One of the main steps of this method is the derivation of mixed reciprocity relations. These relations were introduced in Ref. 10 for sound-soft scatterers, and in Ref. 1 for sound-hard scatterers.

In this section, we allow one of the two point sources considered previously to recede to infinity, so that we have one spherical incident wave and one plane incident wave. We let both sources recede to infinity at the end of this section, and recover known results for plane-wave incidence.

An incident plane wave propagating in the direction of the unit vector $\hat{\mathbf{d}}$ is given by

$$u^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{d}})=\exp\{\mathrm{i}k\hat{\mathbf{d}}\cdot\mathbf{r}\}. \tag{35}$$

We have already noted that $u_a^{\mathrm{i}}(\mathbf{r})\to u^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{a}})$ as $a\to\infty$.

For an incident plane wave, Eq. (35), we denote the total field in Ω, the scattered field and the far-field pattern by $u^{\mathrm{t}}(\mathbf{r};\hat{\mathbf{d}})$, $u^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{d}})$ and $g(\hat{\mathbf{r}};\hat{\mathbf{d}})$, respectively, indicating the dependence on the incident direction $\hat{\mathbf{d}}$. We have[4]

$$u_a^{\mathrm{s}}(\mathbf{r})\to u^{\mathrm{s}}(\mathbf{r};-\hat{\mathbf{a}}),\quad \text{as}\quad a\to\infty \tag{36a}$$

and

$$g_a(\hat{\mathbf{r}})\to g(\hat{\mathbf{r}};-\hat{\mathbf{a}}),\quad \text{as}\quad a\to\infty. \tag{36b}$$

Now, consider our previous results, involving $\mathbf{a}$ and $\mathbf{b}$, and let $b\to\infty$. Lemma 1 gives the following results.

Lemma 4: Let $u_a^{\mathrm{i}}(\mathbf{r})$ *be an incident spherical wave and let* $u^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}})$ *be an incident plane wave. Then*

$$\lim_{r\to\infty}[u_a^{\mathrm{i}},u^{\mathrm{s}}(\cdot,-\hat{\mathbf{b}})]_{S_r}=2a\mathrm{e}^{\mathrm{i}ka}\int_{S^2}g(\hat{\mathbf{r}};-\hat{\mathbf{b}})\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}) \tag{37}$$

and

$$\lim_{\varepsilon\to 0}[u_a^{\mathrm{i}},u^{\mathrm{s}}(\cdot,-\hat{\mathbf{b}})]_{S_{a,\varepsilon}}=4\pi a\mathrm{e}^{\mathrm{i}ka}u^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}}). \tag{38}$$

Next, we define a *plane far-field pattern generator* by

$$G(\mathbf{a};-\hat{\mathbf{b}})=\lim_{b\to\infty}G_b(\mathbf{a}) \tag{39}$$

$$=\mathrm{i}ka\mathrm{e}^{\mathrm{i}ka}\left[u^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}})-\frac{1}{2\pi}\int_{S^2}g(\hat{\mathbf{r}};-\hat{\mathbf{b}})\mathrm{e}^{\mathrm{i}k\mathbf{a}\cdot\hat{\mathbf{r}}}ds(\hat{\mathbf{r}})\right], \tag{40}$$

where the spherical far-field pattern generator $G_b(\mathbf{a})$ is defined by Eq. (15). We will also require $\lim_{a\to\infty}G_b(\mathbf{a})$; this limit is contained in the following theorem.

Theorem 5: *For two incident spherical waves,* u_a^{i} *and* u_b^{i}, *we have*

$$\lim_{a\to\infty}G_b(\mathbf{a})=g_b(-\hat{\mathbf{a}}) \tag{41}$$

and

$$\lim_{a\to\infty} G(\mathbf{a};-\hat{\mathbf{b}}) = g(-\hat{\mathbf{a}};-\hat{\mathbf{b}}). \tag{42}$$

Proof: We prove Eq. (41); a very similar argument gives Eq. (42).

We choose coordinates so that the point source at $\mathbf{a}$ is on the z-axis. Then, take spherical polar coordinates (θ, φ) on S^2, so that $\hat{\mathbf{a}}\cdot\hat{\mathbf{r}} = \cos\theta$. Hence for $\theta=0$ we have $\hat{\mathbf{r}}=\hat{\mathbf{a}}$, while for $\theta=\pi$ we have $\hat{\mathbf{r}}=-\hat{\mathbf{a}}$. If we define

$$F_b(\theta) = \int_0^{2\pi} g_b(\hat{\mathbf{r}})\,d\varphi, \tag{43}$$

then we have $F_b(0)=2\pi g_b(\hat{\mathbf{a}})$ and $F_b(\pi)=2\pi g_b(-\hat{\mathbf{a}})$. Hence

$$\begin{aligned}\int_{S^2} g_b(\hat{\mathbf{r}})\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}) &= \int_0^{\pi} F_b(\theta)\mathrm{e}^{\mathrm{i}ka\cos\theta}\sin\theta d\theta \\ &= \frac{\mathrm{i}}{ka}\int_0^{\pi} F_b(\theta)\frac{d}{d\theta}(\mathrm{e}^{\mathrm{i}ka\cos\theta})d\theta \\ &= \frac{2\pi\mathrm{i}}{ka}[g_b(-\hat{\mathbf{a}})\mathrm{e}^{-\mathrm{i}ka} - g_b(\hat{\mathbf{a}})\mathrm{e}^{\mathrm{i}ka}] - \frac{\mathrm{i}}{ka}\int_0^{\pi}\mathrm{e}^{\mathrm{i}ka\cos\theta}\frac{dF_b(\theta)}{d\theta}d\theta.\end{aligned}$$

From this equation and Eq. (15), we find that

$$G_b(\mathbf{a}) = \mathrm{i}ka\mathrm{e}^{\mathrm{i}ka}u_b^s(\mathbf{a}) + \mathrm{e}^{\mathrm{i}ka}[g_b(-\hat{\mathbf{a}})\mathrm{e}^{-\mathrm{i}ka} - g_b(\hat{\mathbf{a}})\mathrm{e}^{\mathrm{i}ka}] + \frac{\mathrm{e}^{\mathrm{i}ka}}{2\pi}\int_0^{\pi}\mathrm{e}^{\mathrm{i}ka\cos\theta}\frac{dF_b(gv)}{d\theta}d\theta. \tag{44}$$

In view of Eq. (8) and taking into account that the integral in Eq. (44) tends to zero as $a\to\infty$, by the Riemann-Lebesgue lemma, we get Eq. (41).

We can now let $b\to\infty$ in the general scattering theorem, Theorem 2; this gives the following results.

Theorem 6: *Let* $u_a^{\mathrm{i}}(\mathbf{r})$ *be an incident spherical wave and let* $u^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}})$ *be an incident plane wave. Then*

$$G(\mathbf{a};\hat{\mathbf{b}}) + \overline{g_a(-\hat{\mathbf{b}})} + \frac{1}{2\pi}\int_{S^2} g(\hat{\mathbf{r}};-\hat{\mathbf{b}})\overline{g_a(\hat{\mathbf{r}})}ds(\hat{\mathbf{r}}) = \mathcal{M}_a(-\hat{\mathbf{b}}), \tag{45}$$

where $\mathcal{M}_a(-\hat{\mathbf{b}}) = \lim_{b\to\infty}\mathcal{E}_{a,b}$:

$$\mathcal{M}_a(-\hat{\mathbf{b}}) = 0 \quad \textit{for Dirichlet or Neumann conditions on } S; \tag{46}$$

$$\mathcal{M}_a(-\hat{\mathbf{b}}) = -\frac{k^2\lambda}{2\pi}\int_S u^t(\mathbf{r};-\hat{\mathbf{b}})\overline{u_a^{\mathrm{t}}(\mathbf{r})}ds(\hat{\mathbf{r}}) \quad \textit{for the Robin condition (5) on } S; \tag{47}$$

or

$$\mathcal{M}_a(-\hat{\mathbf{b}}) = -\frac{k}{2\pi}\mathrm{Im}(\beta)\int_{\Omega^-}\nabla\overline{u_a^-(\mathbf{r})}\cdot\nabla u^-(\mathbf{r};-\hat{\mathbf{b}})dv(\mathbf{r}) \quad \textit{for a penetrable scatterer.} \tag{48}$$

The mixed reciprocity principle is contained in the next theorem.

Theorem 7: *Let* $u_a^{\mathrm{i}}(\mathbf{r})$ *be an incident spherical wave and let* $u^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}})$ *be an incident plane wave. Then, we have the following reciprocity relation:*

$$g_a(\hat{\mathbf{b}}) = \mathrm{i}ka\,\mathrm{e}^{-\mathrm{i}ka}u^s(\mathbf{a};-\hat{\mathbf{b}}). \tag{49}$$

Proof: Let $b\to\infty$ in Theorem 3, using Eqs. (8) and (36a). □

This result means that the spherical far-field pattern of the point source at $\mathbf{a}$ in the direction $\hat{\mathbf{b}}$ is proportional to the scattered field at $\mathbf{a}$ due to the incident plane wave with direction of propagation $-\hat{\mathbf{b}}$.

Finally, if we combine Eqs. (36b), (39), (41) and (42), we obtain

$$\lim_{a\to\infty}\lim_{b\to\infty} G_b(\mathbf{a}) = \lim_{b\to\infty}\lim_{a\to\infty} G_b(\mathbf{a}) = g(-\hat{\mathbf{a}};-\hat{\mathbf{b}}). \tag{50}$$

We can then check that letting both point sources recede to infinity, $a\to\infty$ and $b\to\infty$, yields the known scattering and optical theorems for plane-wave scattering; see Refs. 11–13, 6, and 8.

VII. FORMULATION: ELECTROMAGNETICS

In the remainder of the article, we consider electromagnetic problems. The exterior Ω is an infinite homogeneous medium with electric permittivity ε, magnetic permeability μ, phase velocity c and conductivity $\sigma=0$. The scatterer Ω^- is filled with a homogeneous medium with corresponding physical parameters ε^-, μ^-, c^- and $\sigma^-\neq 0$.

We consider an incident spherical electromagnetic wave due to a source located at a point with position vector $\mathbf{a}$, with respect to the origin $\mathcal{O}$. This incident wave $(\mathbf{E}_a^{\mathrm{i}},\mathbf{H}_a^{\mathrm{i}})$ has the form[9]

$$\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}) = \frac{a\,\mathrm{e}^{-\mathrm{i}ka}}{\mathrm{i}k}\nabla\times\left(\frac{\mathrm{e}^{\mathrm{i}k|r-a|}}{|\mathbf{r}-\mathbf{a}|}\hat{\mathbf{a}}\times\hat{\mathbf{p}}\right), \tag{51}$$

$$\mathbf{H}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}) = (\mathrm{i}k)^{-1}(\varepsilon/\mu)^{1/2}\nabla\times\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}), \tag{52}$$

where $\hat{\mathbf{p}}$ is a constant unit vector with $\hat{\mathbf{p}}\cdot\hat{\mathbf{a}}=0$, $k=\omega\sqrt{\varepsilon\mu}>0$ is the free-space wave number, and $a=|\mathbf{a}|$. Physically, $(\mathbf{E}_a^{\mathrm{i}},\mathbf{H}_a^{\mathrm{i}})$ represents the field generated by a magnetic dipole with dipole moment $\hat{\mathbf{a}}\times\hat{\mathbf{p}}$; see p. 163 of Ref. 14 or p. 23 of Ref. 6. The coefficient $a\mathrm{e}^{-\mathrm{i}ka}/(\mathrm{i}k)$ in Eq. (51) assures that when the point source tends to infinity the spherical wave reduces to a plane electric wave with direction of propagation $-\hat{\mathbf{a}}$ and polarization $\hat{\mathbf{p}}$. The total electric exterior field $\mathbf{E}_a^{\mathrm{t}}$ is given by

$$\mathbf{E}_a^{\mathrm{t}}(\mathbf{r};\hat{\mathbf{p}}) = \mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}) + \mathbf{E}_a^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}}), \quad \mathbf{r}\in\Omega\setminus\{\mathbf{a}\}, \tag{53}$$

where $\mathbf{E}_a^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}})$ is the scattered electric field, which satisfies the Silver-Müller radiation condition

$$\lim_{r\to\infty}(\mathbf{r}\times\nabla\times\mathbf{E}_a^{\mathrm{s}} + \mathrm{i}kr\mathbf{E}_a^{\mathrm{s}}) = \mathbf{0} \tag{54}$$

uniformly in all directions $\hat{\mathbf{r}}\in S^2$, where S^2 is the unit sphere. $\mathbf{E}_a^{\mathrm{t}}$ solves the equation

$$\nabla\times\nabla\times\mathbf{E}_a^{\mathrm{t}} = k^2\mathbf{E}_a^{\mathrm{t}} \text{ in } \Omega. \tag{55}$$

We note that the incident electric field satisfies the radiation condition (54), and hence the total electric field also satisfies Eq. (54).

The surface of the scatterer may be perfectly conducting, in which case

$$\hat{\mathbf{n}}\times\mathbf{E}_a^{\mathrm{t}} = \mathbf{0} \text{ on } S, \tag{56}$$

or it may be an impedance surface, in which case

$$\hat{\mathbf{n}}\times\nabla\times\mathbf{E}_a^{\mathrm{t}} = -(\mathrm{i}k/Z_s)\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{E}_a^{\mathrm{t}}) \text{ on } S, \tag{57}$$

where the dimensionless parameter Z_s denotes the surface impedance relative to the characteristic impedance of the medium and may vary on S.

If the scatterer is a dielectric, the incident electromagnetic waves are transmitted into the scatterer. Let $\mathbf{E}_a^-$ be the total electric field in the interior. Then $\mathbf{E}_a^-$ satisfies

$$\nabla\times\nabla\times\mathbf{E}_a^- = \eta^2 k^2 \mathbf{E}_a^- \quad \text{in} \quad \Omega^-, \tag{58}$$

where the complex constant η is the relative index of refraction; on the surface of the scatterer we have the following transmission conditions:

$$\hat{\mathbf{n}}\times\mathbf{E}_a^{\mathrm{t}} = \hat{\mathbf{n}}\times\mathbf{E}_a^- \quad \text{and} \quad \hat{\mathbf{n}}\times\nabla\times\mathbf{E}_a^{\mathrm{t}} = (\mu/\mu^-)\hat{\mathbf{n}}\times\nabla\times\mathbf{E}_a^- \quad \text{on} \quad S.$$

The behavior of the scattered electric field in the radiation zone is given by

$$\mathbf{E}_a^{\mathrm{s}}(\mathbf{r}) = h_0(kr)\mathbf{g}_a(\hat{\mathbf{r}}) + O(r^{-2}), \quad r\to\infty, \tag{59}$$

where $\mathbf{g}_a(\hat{\mathbf{r}})$ is the far-field pattern.

More details on the physical parameters of the above problems can be found in Ref. 6. Each problem has a unique and stable classical solution.[14,15]

VIII. GENERAL SCATTERING THEOREM: ELECTROMAGNETICS

In the sequel, for an incident time-harmonic spherical wave $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}})$ due to a point source located at $\mathbf{a}$, we will denote the total field in Ω, the scattered field and the far-field pattern by writing $\mathbf{E}_a^{\mathrm{t}}(\mathbf{r};\hat{\mathbf{p}})$, $\mathbf{E}_a^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}})$ and $\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}})$, respectively, indicating the dependence on the position $\mathbf{a}$ of the point source and the polarization $\hat{\mathbf{p}}$. Also, the total electric field in Ω^- will be denoted by $\mathbf{E}_a^-(\mathbf{r};\hat{\mathbf{p}})$. We consider a point source at $\mathbf{a}$ with polarization $\hat{\mathbf{p}}_1$ and another point source at $\mathbf{b}$ with polarization $\hat{\mathbf{p}}_2$.

For a shorthand notation, we use

$$\{\mathbf{E},\mathbf{E}'\}_S = \int_S [(\hat{\mathbf{n}}\times\overline{\mathbf{E}})\cdot(\nabla\times\mathbf{E}') - (\hat{\mathbf{n}}\times\mathbf{E}')\cdot(\nabla\times\overline{\mathbf{E}})]ds;$$

in particular, we write $\{\mathbf{E},\mathbf{E}'\}\equiv\{\mathbf{E},\mathbf{E}'\}_S$.

Lemma 8: Let $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ *be a point source at* $\mathbf{a}$. *Let* $\mathbf{E}_b^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_2)$ *be a point source at* $\mathbf{b}$, *with corresponding scattered field* $\mathbf{E}_b^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}}_2)$ *and far-field pattern* $\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)$. *Then*

$$\lim_{r\to\infty}\{\mathbf{E}_a^{\mathrm{i}}(\cdot;\hat{\mathbf{p}}_1),\mathbf{E}_b^{\mathrm{s}}(\cdot;\hat{\mathbf{p}}_2)\}_{S_r} = 2a\mathrm{e}^{\mathrm{i}ka}\int_{S^2}\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)\cdot(\hat{\mathbf{r}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1))\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}) \tag{60}$$

and

$$\lim_{\varepsilon\to 0}\{\mathbf{E}_a^{\mathrm{i}}(\cdot;\hat{\mathbf{p}}_1),\mathbf{E}_b^{\mathrm{s}}(\cdot,\hat{\mathbf{p}}_2)\}_{S_{a,\varepsilon}} = 4\pi\mathrm{i}(a/k)\mathrm{e}^{\mathrm{i}ka}(\nabla\times\mathbf{E}_b^{\mathrm{s}}(\mathbf{a};\hat{\mathbf{p}}_2))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1), \tag{61}$$

where S_r *is a large sphere of radius* r *enclosing* $\mathbf{a}$ *and* $\mathbf{b}$, *and* $S_{a,\varepsilon}$ *is a small sphere surrounding* $\mathbf{a}$, *defined by Eq. (10).*

Proof: For Eq. (60), we use the asymptotic forms (13). These show that the incident electric wave takes the form

$$\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1) = h_0(kr)\mathbf{g}_a^{\mathrm{i}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_1) + O(r^{-2}), \quad r\to\infty, \tag{62}$$

where $\mathbf{g}_a^{\mathrm{i}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_1) = \mathrm{i}ka\exp\{-\mathrm{i}ka(1+\hat{\mathbf{r}}\cdot\hat{\mathbf{a}})\}\,(\hat{\mathbf{r}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1))$ is the far-field pattern of the point source incident wave. Using Eqs. (59) and (62) we establish Eq. (60). Note that $\hat{\mathbf{r}}\cdot\mathbf{g}_a^{\mathrm{i}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)=0$.

For Eq. (61), some calculations show that

$$\{\mathbf{E}_a^{\mathrm{i}}(\cdot,\hat{\mathbf{p}}_1),\mathbf{E}_b^{\mathrm{s}}(\cdot,\hat{\mathbf{p}}_2)\}_{S_{a,\varepsilon}}=a\mathrm{e}^{\mathrm{i}ka}\Bigg\{\int_{S_{a,\varepsilon}}\hat{\mathbf{n}}\cdot\nabla\times[((\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1)\cdot\overline{\nabla h_0(k|\mathbf{r}-\mathbf{a}|)})\mathbf{E}_b^{\mathrm{s}}]ds$$
$$+\int_{S_{a,\varepsilon}}(\mathrm{i}k+|\mathbf{r}-\mathbf{a}|^{-1})\overline{h_0(k|\mathbf{r}-\mathbf{a}|)}(\nabla\times\mathbf{E}_b^{\mathrm{s}})\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1)ds$$
$$-k^2\int_{S_{a,\varepsilon}}\hat{\mathbf{n}}\cdot[\mathbf{E}_b^{\mathrm{s}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1)]\overline{h_0(k|\mathbf{r}-\mathbf{a}|)}ds\Bigg\}.$$

The first integral on the right-hand side vanishes by Stokes's theorem. Applying the mean value theorem on the remaining integrals and letting $\varepsilon\to 0$, we obtain Eq. (61). □

For two incident spherical electric waves, $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ and $\mathbf{E}_b^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_2)$, we define a *spherical far-field pattern generator*, as follows:

$$\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)=\mathrm{e}^{\mathrm{i}ka}\mathbf{a}\times\left[\nabla\times\mathbf{E}_b^{\mathrm{s}}(\mathbf{a};\hat{\mathbf{p}}_2)-\frac{\mathrm{i}k}{2\pi}\int_{S^2}\hat{\mathbf{r}}\times\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}})\right]. \tag{63}$$

As we shall see later, when the point sources recede to infinity, $\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)$ is reduced to the far-field pattern for an incident plane electric wave propagating in the direction $-\hat{\mathbf{a}}$ and of polarization $\hat{\mathbf{p}}_2$. Using this notation, the general scattering theorem for spherical electric waves is formulated as follows.

Theorem 9: *For any two point-source locations in* Ω, $\mathbf{a}$ *and* $\mathbf{b}$, *and for any polarizations,* $\hat{\mathbf{p}}_1$ *and* $\hat{\mathbf{p}}_2$, *we have*

$$\hat{\mathbf{p}}_1\cdot\mathbf{G}_1(\mathbf{a};\hat{\mathbf{p}}_2)+\hat{\mathbf{p}}_2\cdot\overline{\mathbf{G}_a(\mathbf{b};\hat{\mathbf{p}}_1)}+\frac{1}{2\pi}\int_{S^2}\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)\cdot\overline{\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)}ds(\hat{\mathbf{r}})=\mathcal{E}_{a,b}(\hat{\mathbf{p}}_1;\hat{\mathbf{p}}_2), \tag{64}$$

where

$$\mathcal{E}_{a,b}(\hat{\mathbf{p}}_1;\hat{\mathbf{p}}_2)=-\frac{\mathrm{i}k}{4\pi}\{\mathbf{E}_a^{\mathrm{t}}(\cdot,\hat{\mathbf{p}}_1),\mathbf{E}_b^{\mathrm{t}}(\cdot,\hat{\mathbf{p}}_2)\}. \tag{65}$$

The value of $\mathcal{E}_{a,b}$ *depends on the scatterer:*

$$\mathcal{E}_{a,b}=0 \quad \textit{for a perfectly conducting surface;} \tag{66}$$

$$\mathcal{E}_{a,b}(\hat{\mathbf{p}}_1;\hat{\mathbf{p}}_2)=-\frac{k^2}{2\pi}\int_S\frac{\mathrm{Re}(Z_S)}{|Z_S|^2}(\hat{\mathbf{n}}\times\overline{\mathbf{E}_a^{\mathrm{t}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)})\cdot(\hat{\mathbf{n}}\times\mathbf{E}_b^{\mathrm{t}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_2))ds(\mathbf{r}) \tag{67}$$

for the impedance boundary condition (57); or

$$\mathcal{E}_{a,b}(\hat{\mathbf{p}}_1;\hat{\mathbf{p}}_2)=-\frac{k^3\mu}{2\pi\mu^-}\mathrm{Im}(\eta^2)\int_{\Omega^-}\overline{\mathbf{E}_a^-(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)}\cdot\mathbf{E}_b^-(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)dv \tag{68}$$

for a dielectric scatterer.

Proof: We proceed exactly as in the proof of Theorem 2. First, we evaluate $\mathcal{E}_{a,b}$ directly, using the boundary or transmission conditions on S; for the dielectric scatterer, we have to apply the divergence theorem in Ω^-. This gives the stated expressions for $\mathcal{E}_{a,b}$.

Next, we give an alternative evaluation of $\mathcal{E}_{a,b}$, using the relations $\mathbf{E}_\alpha^{\mathrm{t}}=\mathbf{E}_\alpha^{\mathrm{i}}+\mathbf{E}_\alpha^{\mathrm{s}}$, for $\alpha=a,\ b$. Formally, the calculations proceed as before, with $\{\cdot,\cdot\}$ in place of $[\cdot,\cdot]$. We also use the vector version of Green's second theorem, which gives

$$\{\mathbf{E}_a^{\mathrm{i}},\mathbf{E}_b^{\mathrm{i}}\}=0, \tag{69}$$

$$\{\mathbf{E}_a^{\mathrm{i}},\mathbf{E}_b^{\mathrm{s}}\} = -4\pi \mathrm{i}(a/k)\mathrm{e}^{ika}(\nabla\times\mathbf{E}_b^{\mathrm{s}}(\mathbf{a};\hat{\mathbf{p}}_2))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1) + 2a\mathrm{e}^{ika}\int_{S^2}\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)\cdot(\hat{\mathbf{r}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1))\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}}) \tag{70}$$

and

$$\{\mathbf{E}_a^{\mathrm{s}},\mathbf{E}_b^{\mathrm{s}}\} = \frac{2\mathrm{i}}{k}\int_{S^2}\overline{\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)}\cdot\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)ds(\hat{\mathbf{r}}). \tag{71}$$

The remaining details are omitted. □

There is also a reciprocity theorem, as $\mathbf{E}_a^{\mathrm{t}}$ is an exact Green's function. It can be found on p. 63 of Ref. 6, for example. With our normalizations, it takes the following form.

Theorem 10: *For any two point-source locations in* Ω, $\mathbf{a}$ *and* $\mathbf{b}$, *for any polarizations,* $\hat{\mathbf{p}}_1$ *and* $\hat{\mathbf{p}}_2$, *and for any scatterer, we have*

$$h_0(ka)(\hat{\mathbf{b}}\times\hat{\mathbf{p}}_2)\cdot(\nabla\times\mathbf{E}_a^{\mathrm{s}}(\hat{\mathbf{b}};\hat{\mathbf{p}}_1)) = h_0(kb)(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1)\cdot(\nabla\times\mathbf{E}_b^{\mathrm{s}}(\hat{\mathbf{a}};\hat{\mathbf{p}}_2)). \tag{72}$$

IX. OPTICAL THEOREM: ELECTROMAGNETICS

In Ref. 9, an optical theorem for spherical waves incident upon a perfect conductor has been proved. Here, we generalize this result to other scatterers, using the general scattering theorem.

First we define the scattering cross-section due to a point source at $\mathbf{a}$ (Ref. 6) as

$$\sigma_a^{\mathrm{s}} = \frac{1}{k^2}\int_{S^2}|\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}})|^2 ds(\hat{\mathbf{r}}), \tag{73}$$

the absorption cross-section as

$$\sigma_a^{\mathrm{a}} = \frac{1}{k}\,\mathrm{Im}\int_S \hat{n}\cdot(\mathbf{E}_a^{\mathrm{t}}\times\nabla\times\overline{\mathbf{E}_a^{\mathrm{t}}})ds \tag{74}$$

and the extinction cross-section, σ_a^{e}, by Eq. (30).

If we put $\mathbf{a}=\mathbf{b}$ and $\hat{\mathbf{p}}_1=\hat{\mathbf{p}}_2=\hat{\mathbf{p}}$ in Theorem 9, we obtain

$$2\,\mathrm{Re}[\hat{\mathbf{p}}\cdot\mathbf{G}_a(a;\hat{\mathbf{p}})] + \frac{1}{2\pi}\int_{S^2}|\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}})|^2 ds(\hat{\mathbf{r}}) = \mathcal{E}_{a,a}(\hat{\mathbf{p}};\hat{\mathbf{p}}),$$

which we can rewrite as

$$\sigma_a^{\mathrm{s}} = -4\pi k^{-2}\,\mathrm{Re}[\hat{\mathbf{p}}\cdot\mathbf{G}_a(\mathbf{a};\hat{\mathbf{p}})] + 2\pi k^{-2}\mathcal{E}_{a,a}(\hat{\mathbf{p}};\hat{\mathbf{p}}). \tag{75}$$

From the definitions (65) and (74), we have

$$\sigma_a^{\mathrm{a}} = \tfrac{1}{2}\mathrm{i}k^{-1}\{\mathbf{E}_a^{\mathrm{t}}(\cdot,\hat{\mathbf{p}}),\mathbf{E}_a^{\mathrm{t}}(\cdot,\hat{\mathbf{p}})\} = -2\pi k^{-2}\mathcal{E}_{a,a}(\hat{\mathbf{p}};\hat{\mathbf{p}}). \tag{76}$$

Hence, adding Eqs. (75) and (76), the definition (30) gives

$$\sigma_a^{\mathrm{e}} = -4\pi k^{-2}\,\mathrm{Re}[\hat{\mathbf{p}}\cdot\mathbf{G}_a(\mathbf{a};\hat{\mathbf{p}})]. \tag{77}$$

The value of $\mathcal{E}_{a,a}(\hat{\mathbf{p}};\hat{\mathbf{p}})$ is given in Theorem 9; it depends on the scatterer's properties.

X. MIXED SCATTERING RELATIONS

Let

$$\mathbf{E}^{\mathrm{i}}(r;\hat{\mathbf{d}},\hat{\mathbf{p}}) = \hat{\mathbf{p}}\exp\{\mathrm{i}k\hat{\mathbf{d}}\cdot\mathbf{r}\}. \tag{78}$$

be an incident time-harmonic plane electric wave, where the unit vector $\hat{\mathbf{d}}$ describes the direction of propagation and the unit vector $\hat{\mathbf{p}}$ gives the polarization. We will indicate the dependence of the total field in Ω, the total field in Ω^-, the scattered field and the electric far-field pattern on the incident direction $\hat{\mathbf{d}}$ and the polarization $\hat{\mathbf{p}}$ by writing $\mathbf{E}^{\mathrm{t}}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})$, $\mathbf{E}^-(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})$, $\mathbf{E}^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})$ and $\mathbf{g}(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}})$, respectively.

Here, we consider mixed situations, and relate fields due to one spherical electric wave $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ and one plane electric wave $\mathbf{E}^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)$; we do this by letting $b\to\infty$ in our previous results.

Using the asymptotic forms (13), we can easily show that for the spherical electric wave (51) we have

$$\lim_{b\to\infty}\mathbf{E}_b^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}})=\mathbf{E}^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}}), \tag{79}$$

that is the spherical electric wave, when the point source goes to infinity, reduces to a plane electric wave with direction of propagation $-\hat{\mathbf{b}}$ and polarization $\hat{\mathbf{p}}$. Similarly, we have $\mathbf{E}_b^{\mathrm{t}}(\mathbf{r};\hat{\mathbf{p}})\to\mathbf{E}^{\mathrm{t}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}})$, $\mathbf{E}_b^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}})\to\mathbf{E}^{\mathrm{s}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}})$ and $\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}})\to\mathbf{g}(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}})$ as $b\to\infty$.

Next, let $b\to\infty$ in Lemma 8 to give the following result.

Lemma 11: Let $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ *be an incident spherical electric wave and let* $\mathbf{E}^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)$ *be an incident plane electric wave. Then*

$$\lim_{r\to\infty}\{\mathbf{E}_a^{\mathrm{i}}(\cdot;\hat{\mathbf{p}}_1),\mathbf{E}^{\mathrm{s}}(\cdot;-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\}_{S_r}=2a\mathrm{e}^{\mathrm{i}ka}\int_{S^2}\mathbf{g}(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\cdot(\hat{\mathbf{r}}\times(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1))\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}})$$

and

$$\lim_{\varepsilon\to0}\{\mathbf{E}_a^{\mathrm{i}}(\cdot;\hat{\mathbf{p}}_1),\mathbf{E}^{\mathrm{s}}(\cdot;-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\}_{S_{a,\varepsilon}}=4\pi\mathrm{i}(a/k)\mathrm{e}^{\mathrm{i}ka}(\nabla\times\mathbf{E}^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1).$$

We define a *plane far-field pattern generator* by the formula

$$\mathbf{G}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)=\lim_{b\to\infty}\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)=\mathrm{e}^{\mathrm{i}ka}\mathbf{a}\times\left[\nabla\times\mathbf{E}^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)-\frac{\mathrm{i}k}{2\pi}\int_{S^2}\hat{\mathbf{r}}\times\mathbf{g}(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}})\right],$$

where $\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)$ is defined by Eq. (63). Other limiting values are given in the next theorem.

Theorem 12: *For two incident point source electric waves,* $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ *and* $\mathbf{E}_b^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_2)$, *we have*

$$\lim_{a\to\infty}\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}})=\mathbf{g}_b(-\hat{\mathbf{a}};\hat{\mathbf{p}}_2) \tag{80}$$

and

$$\lim_{a\to\infty}\mathbf{G}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)=\mathbf{g}(-\hat{\mathbf{a}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2). \tag{81}$$

Proof: For Eq. (80), we use spherical polar coordinates (θ, φ) as in the proof of Theorem 5, and define

$$\mathbf{F}_b(\theta)=\int_0^{2\pi}\hat{\mathbf{r}}\times\mathbf{g}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)d\varphi.$$

In particular, we have $\mathbf{F}_b(0)=2\pi\hat{\mathbf{a}}\times\mathbf{g}_b(\hat{\mathbf{a}};\hat{\mathbf{p}}_2)$ and $\mathbf{F}_b(\pi)=-2\pi\hat{\mathbf{a}}\times\mathbf{g}_b(-\hat{\mathbf{a}};\hat{\mathbf{p}}_2)$. Hence

$$\int_{S^2}\hat{\mathbf{r}}\times\hat{\mathbf{g}}_b(\hat{\mathbf{r}};\hat{\mathbf{p}}_2)\mathrm{e}^{\mathrm{i}k\hat{\mathbf{r}}\cdot\mathbf{a}}ds(\hat{\mathbf{r}})=\int_0^\pi \mathbf{F}_b(\theta)\mathrm{e}^{\mathrm{i}ka\cos\theta}\sin\theta d\theta$$
$$=\frac{\mathrm{i}}{ka}[\mathbf{F}_b(\pi)\mathrm{e}^{-\mathrm{i}ka}-\mathbf{F}_b(0)\mathrm{e}^{\mathrm{i}ka}]-\frac{\mathrm{i}}{ka}\int_0^\pi \mathrm{e}^{\mathrm{i}ka\cos\theta}\frac{d\mathbf{F}_b(\theta)}{d\theta}d\theta$$
$$\sim-\frac{2\pi\mathrm{i}}{ka}\hat{\mathbf{a}}\times[\mathbf{g}_b(-\hat{\mathbf{a}};\hat{\mathbf{p}}_2)\mathrm{e}^{-\mathrm{i}ka}+\mathbf{g}_b(\hat{\mathbf{a}};\hat{\mathbf{p}}_2)\mathrm{e}^{\mathrm{i}ka}]$$

for large a. From this equation and Eq. (63), we arrive at Eq. (80). The proof of Eq. (81) is similar. □

We can now let $b\to\infty$ in the general scattering theorem, Theorem 9.

Theorem 13: *Let $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ be an incident spherical electric wave and let $\mathbf{E}^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)$ be an incident plane electric wave. Then*

$$\hat{\mathbf{p}}_1\cdot\mathbf{G}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)+\hat{\mathbf{p}}_2\cdot\overline{\mathbf{g}_a(\hat{\mathbf{b}};\hat{\mathbf{p}}_1)}+\frac{1}{2\pi}\int_{S^2}\mathbf{g}(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\cdot\overline{\mathbf{g}_a(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)}ds(\hat{\mathbf{r}})=\mathcal{M}_a(-\hat{\mathbf{b}};\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2),$$

where $\mathcal{M}_a(-\hat{\mathbf{b}};\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)=\lim_{b\to\infty}\mathcal{E}_{a,b}(\hat{\mathbf{p}}_1;\hat{\mathbf{p}}_2)$:

$$\mathcal{M}_a(-\hat{\mathbf{b}};\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)=0 \tag{82}$$

for a perfectly conducting surface;

$$\mathcal{M}_a(-\hat{\mathbf{b}};\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)=-\frac{k^2}{2\pi}\int_S\frac{\mathrm{Re}(Z_S)}{|Z_S|^2}(\hat{\mathbf{n}}\times\mathbf{E}^{\mathrm{t}}(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2))\cdot(\hat{\mathbf{n}}\times\overline{\mathbf{E}_a^{\mathrm{t}}(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)})ds(\hat{\mathbf{r}}) \tag{83}$$

for the impedance boundary condition; or

$$\mathcal{M}_a(-\hat{\mathbf{b}};\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)=-\frac{k^3\mu}{2\pi\mu_-}\mathrm{Im}(\eta^2)\int_{\Omega^-}\overline{\mathbf{E}_a^-(\hat{\mathbf{r}};\hat{\mathbf{p}}_1)}\cdot\mathbf{E}^-(\hat{\mathbf{r}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)dv \tag{84}$$

for a dielectric scatterer.

Proof: The proof is similar to that of Theorem 9. The only substantial difference appears in the formula for $\{\mathbf{E}_a^{\mathrm{s}},\mathbf{E}^{\mathrm{i}}\}$; cf. Eq. (70). Now, using the plane electric wave (78), we find that (see p. 59 of Ref. 6)

$$\{\mathbf{E}_a^{\mathrm{s}}(\cdot;\hat{\mathbf{p}}_1),\mathbf{E}^{i}(\cdot;-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\}=\hat{\mathbf{p}}_2\cdot\int_S[\hat{\mathbf{n}}\times\nabla\times\overline{\mathbf{E}_a^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}}_1)}+\mathrm{i}k\hat{\mathbf{b}}\times(\hat{\mathbf{n}}\times\overline{\mathbf{E}_a^{\mathrm{s}}(\mathbf{r};\hat{\mathbf{p}}_1)})]\mathrm{e}^{-\mathrm{i}k\hat{\mathbf{b}}\cdot\mathbf{r}}ds(\mathbf{r})$$
$$=4\pi\mathrm{i}k^{-1}\hat{\mathbf{p}}_2\cdot\overline{\mathbf{g}_a(-\hat{b},\hat{\mathbf{p}}_1)}.$$

□

The mixed reciprocity relation for perfect conductors is Theorem 2.3.4 in Ref. 1. It is valid more generally, as follows.

Theorem 14: *Let $\mathbf{E}_a^{\mathrm{i}}(\mathbf{r};\hat{\mathbf{p}}_1)$ be an incident spherical electric wave and let $\mathbf{E}^{\mathrm{i}}(\mathbf{r};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)$ be an incident plane electric wave. Then*

$$\hat{\mathbf{p}}_2\cdot\mathbf{g}_a(\hat{\mathbf{b}},\hat{\mathbf{p}}_1)=\mathrm{e}^{-\mathrm{i}ka}\hat{\mathbf{p}}_1\cdot[(\nabla\times\mathbf{E}^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2))\times\hat{\mathbf{a}}]. \tag{85}$$

Proof: Working as in the proof of Theorem 10 and taking into account that

$$\{\overline{\mathbf{E}_a^{\mathrm{i}}(\cdot;\hat{\mathbf{p}}_1)},\mathbf{E}^{\mathrm{s}}(\cdot;-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\}=4\pi\mathrm{i}(a/k)\mathrm{e}^{-\mathrm{i}ka}(\nabla\times\mathbf{E}^{\mathrm{s}}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_1))\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{p}}_1)$$

and

$$\{\overline{\mathbf{E}_a^{\mathrm{s}}(\cdot;\hat{\mathbf{p}}_1)},\mathbf{E}^{\mathrm{i}}(\cdot;-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)\}=-4\pi\mathrm{i}k^{-1}\hat{\mathbf{p}}_2\cdot\mathbf{g}_a(\hat{\mathbf{b}},\hat{\mathbf{p}}_1),$$

Theorem 14 is proved. □

To conclude, we note that we also have

$$\lim_{a\to\infty}\lim_{b\to\infty}\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)=\lim_{b\to\infty}\lim_{a\to\infty}\mathbf{G}_b(\mathbf{a};\hat{\mathbf{p}}_2)=\mathbf{g}(-\hat{\mathbf{a}};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2).$$

This can be used to verify that the known scattering relations for plane-wave incidence[14,6] are recovered when $a\to\infty$ and $b\to\infty$. Furthermore, Eq. (81) and the reciprocity principle for plane waves[6] gives the following limiting property:

$$\lim_{a\to\infty}\hat{\mathbf{p}}_1\cdot\mathbf{G}(\mathbf{a};-\hat{\mathbf{b}},\hat{\mathbf{p}}_2)=\lim_{b\to\infty}\hat{\mathbf{p}}_2\cdot\mathbf{G}(-\mathbf{b};\hat{\mathbf{a}},\hat{\mathbf{p}}_1).$$

[1] R. Potthast, *Point-Sources and Multipoles in Inverse-Scattering Theory* (Chapman & Hall/CRC, London, 2001).
[2] N. A. Logan, Proc. IEEE **53**, 773 (1965).
[3] *Electromagnetic and Acoustic Scattering by Simple Shapes*, edited by J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi (North Holland, New York, 1969).
[4] G. Dassios and G. Kamvyssas, IMA J. Appl. Math. **55**, 67 (1995).
[5] G. Dassios and G. Kamvyssas, Q. J. Mech. Appl. Math. **50**, 321 (1997).
[6] G. Dassios and R. Kleinman, *Low Frequency Scattering* (Clarendon, Oxford, 2000).
[7] J. Coyle, Inverse Probl. **16**, 275 (2000).
[8] C. Athanasiadis, P. A. Martin, and I. G. Stratis, IMA J. Appl. Math. **66**, 539 (2001).
[9] C. Athanasiadis, P. A. Martin, and I. G. Stratis, "On the scattering of point-generated electromagnetic waves by a perfectly conducting sphere, and related near-field inverse problems," Z. Angew. Math. Mech., to appear.
[10] R. Kress, "Integral equation methods in inverse acoustic and electromagnetic scattering," in *Boundary Integral Formulations for Inverse Analysis*, edited by D. B. Ingham and L. C. Wrobel (Computational Mechanics Publications, Southampton, 1997), pp. 67–92.
[11] V. Twersky, J. Appl. Phys. **25**, 859 (1954).
[12] A. T. de Hoop, *Handbook of Radiation and Scattering of Waves* (Academic, London, 1995).
[13] C. Athanasiadis, J. Austral. Math. Soc., Ser. B **39**, 431 (1998).
[14] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer, Berlin, 1992).
[15] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory* (Wiley, New York, 1983).