Short Communication

On the scattering of point-generated electromagnetic waves by a perfectly conducting sphere, and related near-field inverse problems

C. Athanasiadis[1], **P. A. Martin**[2,*], **and I. G. Stratis**[3,**]

[1] Department of Mathematics, University of Athens, Panepistimiopolis, GR 157 84 Athens, Greece
[2] Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA
[3] Department of Mathematics, University of Athens, Panepistimiopolis, GR 157 84 Athens, Greece

Received 10 October 2001, revised 8 February 2002, accepted 5 March 2002
Published online 27 January 2003

Key words Point sources, inverse problems, Maxwell's equations
MSC (2000) 78A45, 78A46

A spherical electromagnetic wave is scattered by a bounded perfectly conducting obstacle. A generalization of the plane-wave optical theorem is established. For a spherical scatterer, low frequency results are obtained by approximating the known exact solution (separation of variables). In particular, a closed-form approximation of the scattered wavefield at the source of the incident spherical wave is obtained. This leads to the solution of a near-field inverse problem, where both the source and coincident receiver are located at several points in the vicinity of a small sphere. The same inverse problem is also treated from the knowledge of the leading order term in the low-frequency asymptotic expansion of the scattering cross-section.

1 Introduction

A basic electromagnetic inverse problem is the following: determine the shape of a perfectly conducting object from a knowledge of the scattered field for several incident fields. The standard version of this problem uses incident plane waves and measurements in the far field; for an excellent survey of what is known about this problem, see the book by Colton and Kress [2]. Several methods for solving this problem make essential use of point sources; for a review, see the recent book by Potthast [8].

However, in practice, one cannot realise an actual plane wave and one may not be able to take measurements in the far field. For these reasons, there has been some interest in the use of point-generated incident spherical waves, and in near-field measurements.

In the acoustics case, Dassios and his co-workers (see, for example, [3] and [4]) and the present authors [1] have studied incident waves generated by a point source in the vicinity of the scatterer. Such incident wavefields introduce an extra parameter (the distance of the source from the scatterer) which may be and is exploited in all the above works for the study of inverse problems. Dassios et al. developed a low-frequency theory for arbitrary smooth scatterers (for a number of different boundary conditions), which they then specialized to small spherical scatterers. In [1], we noted that if spheres are of primary interest, then these low-frequency results can be extracted more easily by first solving the boundary-value problem for the Helmholtz equation exactly. In the present paper, we generalise these results to electromagnetic scattering by a perfectly conducting sphere; the classical results for a sphere can be found in [9].

Two kinds of inverse problem, with point-source generated incident fields, can be considered. The first involves far-field measurements, and is discussed in Section 6.1. The second involves near-field measurements: specifically, one can measure the scattered field *at the location of the point source*. This problem was first studied in [1], where it was shown how to recover the location and radius of a small spherical scatterer, using acoustic waves. We consider the analogous problem for electromagnetic waves in Section 6.2.

We begin with the direct problem. For the scattering of a time-harmonic spherical electromagnetic wave by a bounded three-dimensional perfectly conducting body, located in the vicinity of the point source generating the incident wavefield, we derive a new general result. Thus, we establish an optical theorem for point-source excitation, relating the scattering cross-section due to a point source at a given point to the scattered field at this point, and a Herglotz wave function with Herglotz

* Corresponding author, e-mail: pamartin@mines.edu
** e-mail: istratis@math.uoa.gr

kernel depending on the electric far-field pattern. We recover the standard optical theorem for plane-wave incidence when we let the point source recede to infinity.

Next, we study the case in which the scatterer is spherical. After expanding the spherical incident field in terms of spherical wave functions, we obtain an exact solution of the boundary-value problem under consideration, as well as an expansion for the electric far-field pattern. This point-source solution can be regarded as the exact Green's function for the problem. Under the low-frequency assumption ($ka \ll 1$), we calculate the electric far-field pattern with an error of fourth order in ka (where k is the free-space wave number and a is the radius of the spherical scatterer), and the scattering cross-section with an error of sixth order in ka. Again, the classical results for plane incident waves are recovered as the point source recedes to infinity.

Finally, we consider the two inverse problems mentioned above. For the far-field experiments, we measure the scattering cross-section for various point-source locations. We use this data to recover the location and radius of the small perfectly conducting spherical scatterer. We then obtain similar results using near-field experiments, in which the scattered field is measured at the source, for various point-source locations.

2 Formulation

Consider a bounded three-dimensional perfectly conducting body B with a smooth closed boundary S, surrounded by an infinite dielectric medium. We consider a time harmonic spherical electromagnetic wave due to a point source at P_0 with position vector $\boldsymbol{r}_0$ with respect to an origin O in the vicinity of B. This incident wave $\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}$, $\boldsymbol{H}^{\mathrm{inc}}_{\boldsymbol{r}_0}$ has the form

$$\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) = A^{\mathrm{inc}}\,\nabla\times\left(\frac{\mathrm{e}^{i\,k\,|\boldsymbol{r}-\boldsymbol{r}_0|}}{|\boldsymbol{r}-\boldsymbol{r}_0|}\,\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}}\right), \quad \boldsymbol{H}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) = -ik^{-1}\,\nabla\times\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}), \tag{1}$$

where $\widehat{\boldsymbol{b}}$ is a constant unit vector with $\widehat{\boldsymbol{r}}_0\cdot\widehat{\boldsymbol{b}}=0$, and $k>0$ is the free-space wave number. Physically, (1) represents the electromagnetic field generated by a magnetic dipole with dipole moment $\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}}$; see, for example, [2, p. 163] or [5, p. 23]. The constant A^{inc} is evaluated so that as the location of the point source goes to infinity along the ray in the direction $\widehat{\boldsymbol{r}}_0$, the point source field degenerates into a plane electromagnetic wave propagating in the direction from P_0 towards O. Furthermore, the normal electric energy flux at the origin due to the spherical electric incident field must be equal to the normal electric energy flux of the plane electric wave that the point source field assumes as $r_0=|\boldsymbol{r}_0|\to\infty$. The spherical electric wave that satisfies these demands is given by (1) with $A^{\mathrm{inc}}=-i(r_0/k)\,\mathrm{e}^{-ikr_0}$. It is convenient to write the incident spherical field as

$$\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) = \frac{1}{ik}\,\nabla\times\left(\frac{h(kR)}{h(kr_0)}\,\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}}\right), \tag{2}$$

where $R=|\boldsymbol{r}-\boldsymbol{r}_0|$ and $h(x)\equiv h_0^{(1)}(x)=\mathrm{e}^{ix}/(ix)$ is the zeroth-order spherical Hankel function of the first kind. When $r_0\to\infty$, $\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}})$ reduces to the plane electric wave

$$\boldsymbol{E}^{\mathrm{inc}}(\boldsymbol{r};-\widehat{\boldsymbol{r}}_0,\widehat{\boldsymbol{b}}) = \widehat{\boldsymbol{b}}\,\mathrm{e}^{-ik\widehat{\boldsymbol{r}}_0\cdot\boldsymbol{r}},$$

with direction of propagation $-\widehat{\boldsymbol{r}}_0$ and polarization $\widehat{\boldsymbol{b}}$.

We want to calculate the scattered electric field $\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}$, where $\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}$ satisfies

$$\nabla\times\nabla\times\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0} = k^2\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}, \tag{3}$$

everywhere in the exterior of B, the Silver–Müller radiation condition

$$\lim_{r\to\infty}(\boldsymbol{r}\times\nabla\times\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0} + ikr\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}) = \boldsymbol{0}, \tag{4}$$

where $r=|\boldsymbol{r}|$, and the boundary condition

$$\widehat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0} = -\widehat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0} \quad \text{on } S, \tag{5}$$

where $\widehat{\boldsymbol{n}}$ is the outward unit normal vector to S.

The behaviour of the scattered wave in the far field is given by

$$\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) \sim \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})\,h(kr) \quad \text{as } r\to\infty, \tag{6}$$

where $\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}$ is the far-field pattern.

The total exterior electric field $\boldsymbol{E}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}})$ is given by

$$\boldsymbol{E}_{\boldsymbol{r}_0} = \boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0} + \boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}, \tag{7}$$

and satisfies (3) for $\boldsymbol{r} \neq \boldsymbol{r}_0$ and the boundary condition

$$\widehat{\boldsymbol{n}} \times \boldsymbol{E}_{\boldsymbol{r}_0} = \boldsymbol{0} \quad \text{on } S. \tag{8}$$

Let us note that $\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}$, $\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}$, and $\boldsymbol{E}_{\boldsymbol{r}_0}$ are required to be divergence-free.

In concluding this section let us make some comments on plane versus spherical excitation. For plane incident waves the energy is infinite, the energy density is uniformly distributed, the considered radiation condition is not satisfied, the number of parameters on which the incident wave depends is 5 (the wave number, the two spherical angles of the direction of propagation, and the two spherical angles of the direction of polarization), while the number of parameters on which the far–field pattern depends is 7 (the wave number, the two spherical angles of the direction of propagation, the two spherical angles of the direction of polarization, and the two spherical angles of the direction of observation). On the other hand, for spherical incident waves the energy is finite, there is geometrical attenuation of the energy density, the considered radiation condition is satisfied, the number of parameters on which the incident wave depends is 6 (the wave number, the two spherical angles of the direction of polarization of the corresponding plane wave, and the three components of the source point), while the number of parameters on which the far–field pattern depends is 8 (the wave number, the two spherical angles of the direction of polarization of the corresponding plane wave, the three components of the source point, and the two spherical angles of the direction of observation).

3 An optical theorem for point-source excitation

Consider the incident field (2) and fix P_0. Then

$$\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) \sim \boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})\, h(kr), \quad \text{as } r \to \infty,$$

where

$$\boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}}) = \frac{\mathrm{e}^{-ik\widehat{\boldsymbol{r}}\cdot\boldsymbol{r}_0}}{h(kr_0)}\, \widehat{\boldsymbol{r}} \times (\widehat{\boldsymbol{r}}_0 \times \widehat{\boldsymbol{b}})$$

is the far-field pattern of the point-source incident field. Thus $\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}$ satisfies the radiation condition (4) at infinity (with respect to $\boldsymbol{r}$). Let us note that

$$\widehat{\boldsymbol{r}} \cdot \boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}}) = 0. \tag{9}$$

From $\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}$ given by (6) we can calculate the scattering cross-section $\sigma_{\boldsymbol{r}_0}$, defined by

$$\sigma_{\boldsymbol{r}_0} = \frac{1}{k^2} \int_{S^2} |\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})|^2 \, ds(\widehat{\boldsymbol{r}}), \tag{10}$$

where S^2 is the unit-sphere.

Consider a volume B_r bounded internally by S and externally by a large sphere S_r centred at the origin with radius r large enough to include the scatterer B in its interior. We also exclude a small ball centred on the source point P_0; the boundary of this ball is a sphere S_ε of radius ε. Let

$$\{\boldsymbol{U},\boldsymbol{V}\}_S := \int_S \left[(\widehat{\boldsymbol{n}} \times \boldsymbol{U}) \cdot (\nabla \times \boldsymbol{V}) - (\widehat{\boldsymbol{n}} \times \boldsymbol{V}) \cdot (\nabla \times \boldsymbol{U})\right] ds.$$

Apply the vector Green's theorem in B_r to $\boldsymbol{E}_{\boldsymbol{r}_0}$ and $\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}$, where the overbar denotes complex conjugation. As $\boldsymbol{E}_{\boldsymbol{r}_0}$ and $\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}$ both satisfy (3) in B_r, and they both satisfy (8) on S, we obtain

$$\{\boldsymbol{E}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}\}_{S_r} + \{\boldsymbol{E}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}\}_{S_\varepsilon} = 0.$$

As

$$\boldsymbol{E}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) \sim \boldsymbol{F}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})\, h(kr), \quad r \to \infty,$$

where $\boldsymbol{F}_{\boldsymbol{r}_0} := \boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0} + \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}$, and since in view of (9) and the fact that $\widehat{\boldsymbol{r}} \cdot \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0} = 0$ [2, Theorem 6.8], $\boldsymbol{F}_{\boldsymbol{r}_0}$ satisfies $\widehat{\boldsymbol{r}} \cdot \boldsymbol{F}_{\boldsymbol{r}_0} = 0$, we find that

$$\begin{aligned}
\{\boldsymbol{E}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}\}_{S_\infty} &= \frac{2}{ik} \int_{S^2} |\widehat{\boldsymbol{r}} \times \boldsymbol{F}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})|^2 \, ds(\widehat{\boldsymbol{r}}) \\
&= \frac{2}{ik} \left[\int_{S^2} |\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}|^2 \, ds + \int_{S^2} |\boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0}|^2 \, ds + 2\,\Re \int_{S^2} \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0} \cdot \boldsymbol{F}^{\mathrm{inc}}_{\boldsymbol{r}_0} \, ds\right] \\
&= \frac{2}{ik} \left[k^2\, \sigma_{\boldsymbol{r}_0} + 4\pi k^2 r_0^2 + 2\, k r_0\, \Im \left(\mathrm{e}^{i\,kr_0} \int_{S^2} \mathrm{e}^{i\,k\widehat{\boldsymbol{r}}\cdot\boldsymbol{r}_0} \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}}) \cdot \left(\widehat{\boldsymbol{r}} \times (\widehat{\boldsymbol{r}}_0 \times \widehat{\boldsymbol{b}})\right) ds(\widehat{\boldsymbol{r}})\right)\right],
\end{aligned}$$

where $\{\boldsymbol{U},\boldsymbol{V}\}_{S_\infty}$ denotes $\lim_{r\to\infty}\{\boldsymbol{U},\boldsymbol{V}\}_{S_r}$.

Next, consider $\{\boldsymbol{E}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}$. In view of the bilinearity of $\{\cdot,\cdot\}_S$ and using (7), we have

$$\{\boldsymbol{E}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}=\{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}+\{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}+\{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}+\{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}.$$

Now, due to the regularity of $\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}$, $\overline{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}}$ inside S_ε, we have $\{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}=0$. Using the mean value theorem, we obtain $\{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}\sim 8i\pi k\, r_0^2$ as $\varepsilon\to 0$. Implementing in addition Stokes' theorem we get

$$\{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}+\{\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0},\overline{\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}}\}_{S_\varepsilon}\sim 8\pi(r_0/k)\,\Im\left[\mathrm{e}^{-i\,kr_0}(\nabla\times\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{b}}))\cdot(\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}})\right],$$

as $\varepsilon\to 0$. Combining the above we finally obtain

$$\begin{aligned}\sigma_{\boldsymbol{r}_0} \;=\; \frac{2r_0}{k}\,\Im\Bigg[&\frac{2\pi}{k}\,\mathrm{e}^{-i\,kr_0}(\nabla\times\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{b}}))\cdot(\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}})\\ &-\mathrm{e}^{i\,kr_0}\int_{S^2}\mathrm{e}^{i\,k\widehat{\boldsymbol{r}}\cdot\boldsymbol{r}_0}\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})\cdot(\widehat{\boldsymbol{r}}\times(\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}}))\,ds(\widehat{\boldsymbol{r}})\Bigg].\end{aligned}\tag{11}$$

This is the analogue of the optical theorem for a point-source incident field. It shows that the scattering cross-section due to a point source at $\boldsymbol{r}_0$ is related to the scattered field at $\boldsymbol{r}_0$ and a Herglotz vector wavefunction, [2], with Herglotz kernel $\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{b}})\cdot(\widehat{\boldsymbol{r}}\times(\widehat{\boldsymbol{r}}_0\times\widehat{\boldsymbol{b}}))$.

The behaviour of $\sigma_{\boldsymbol{r}_0}$ as the point source recedes to infinity can be examined as in [1], and it turns out that (11) reduces to

$$\sigma=-4\pi k^{-2}\,\Re\widehat{\boldsymbol{b}}\cdot\boldsymbol{F}(-\widehat{\boldsymbol{r}}_0;-\widehat{\boldsymbol{r}}_0,\widehat{\boldsymbol{b}})$$

which is the standard optical theorem for plane-wave incidence; here

$$\sigma=\frac{1}{k^2}\int_{S^2}|\boldsymbol{F}(\widehat{\boldsymbol{r}};-\widehat{\boldsymbol{r}}_0,\widehat{\boldsymbol{b}})|^2\,ds(\widehat{\boldsymbol{r}})$$

is the scattering cross-section for plane-wave incidence, where $\boldsymbol{F}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}},\widehat{\boldsymbol{b}})$ is the far-field pattern in the direction $\widehat{\boldsymbol{r}}$ due to a plane wave propagating in the direction $\widehat{\boldsymbol{p}}$ with polarization $\widehat{\boldsymbol{b}}$.

We conclude this section by noting that the solution for a point source in the presence of a scatterer is an exact Green's function, and so we should expect a reciprocity theorem. Indeed, such a result is known: see equation (2.126) in [5].

4 Exact Green's function for a perfectly conducting sphere

Consider a spherical scatterer of radius a. Take spherical polar coordinates (r,θ,ϕ) with the origin at the centre of the sphere, so that the point source is at $r=r_0$, $\theta=0$, and so that the polarization vector $\widehat{\boldsymbol{b}}$ is in the x-direction. Thus, $\boldsymbol{r}_0=r_0\widehat{\boldsymbol{z}}$ and $\widehat{\boldsymbol{b}}=\widehat{\boldsymbol{x}}$, where $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{z}}$ are unit vectors in the x and z directions, respectively.

Using spherical vector wave functions, and in particular (13.3.68), (13.3.69), (13.3.70) of [7] we obtain the following expansion for the incident field:

$$\boldsymbol{E}^{\mathrm{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{x}})=\frac{i}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{h_n(kr_0)\,\boldsymbol{N}^1_{e1n}(\boldsymbol{r})-\widetilde{h}_n(kr_0)\,\boldsymbol{M}^1_{o1n}(\boldsymbol{r})\right\}\tag{12}$$

for $r<r_0$, where $h_n\equiv h_n^{(1)}$ is a spherical Hankel function, $\widetilde{h}(x)=x^{-1}h_n(x)+h_n'(x)=x^{-1}[xh_n(x)]'$, and $\boldsymbol{M}^\rho_{\sigma 1n}$ and $\boldsymbol{N}^\rho_{\sigma 1n}$ are defined in the Appendix. The scattered field has a similar expression; taking the radiation condition into account we have

$$\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{x}})=\frac{i}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{\alpha_n h_n(kr_0)\,\boldsymbol{N}^3_{e1n}(\boldsymbol{r})-\beta_n\widetilde{h}_n(kr_0)\,\boldsymbol{M}^3_{o1n}(\boldsymbol{r})\right\},\tag{13}$$

where the dimensionless coefficients α_n and β_n are to be determined. Indeed, using the boundary condition (5), on $r=a$, we obtain

$$\alpha_n=-\frac{j_n(ka)+kaj_n'(ka)}{h_n(ka)+kah_n'(ka)}\quad\text{and}\quad\beta_n=-\frac{j_n(ka)}{h_n(ka)}.\tag{14}$$

Let us calculate the electric far-field pattern. Since

$$h_n(x)\sim(-i)^n h_0(x)\quad\text{and}\quad h_n'(x)\sim(-i)^{n-1}h_0(x)\quad x\to\infty,$$

and using (13.3.68) and (13.3.69) of [7] we find that

$$\begin{aligned} \boldsymbol{M}^3_{o1n}(\boldsymbol{r}) &= \sqrt{n(n+1)}\, h_n(kr)\, \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}}) \\ &\sim \sqrt{n(n+1)}\, (-i)^n\, h_0(kr)\, \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}}) \end{aligned} \tag{15}$$

and

$$\boldsymbol{N}^3_{e1n}(\boldsymbol{r}) = n(n+1)\,(kr)^{-1}\, h_n(kr)\, \boldsymbol{P}_{e1n}(\widehat{\boldsymbol{r}}) + \sqrt{n(n+1)}\,\widetilde{h}_n(kr)\, \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}}) \tag{16}$$

$$\sim \sqrt{n(n+1)}\,(-i)^{n-1}\, h_0(kr)\, \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}}) \tag{17}$$

as $kr \to \infty$, where $\boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}})$, $\boldsymbol{P}_{e1n}(\widehat{\boldsymbol{r}})$, and $\boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}})$ are defined in the Appendix. Therefore for the electric far-field pattern, we have

$$\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{x}}) = -\sum_{n=1}^{\infty} \frac{(2n+1)(-i)^n}{\sqrt{n(n+1)}} \left\{ \alpha_n\, \frac{h_n(kr_0)}{h_0(kr_0)}\, \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}}) - i\beta_n\, \frac{\widetilde{h}_n(kr_0)}{h_0(kr_0)} \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}}) \right\}. \tag{18}$$

5 Far-field results for a small perfectly conducting sphere

So far, all of our formulae are exact. In the asymptotic results to follow, there are two parameters,

$$\kappa = ika \quad \text{and} \quad \tau = a/r_0.$$

We assume that $|\kappa| = ka \ll 1$; that is we make the so-called low-frequency assumption. We also note that the geometrical parameter τ must satisfy $0 < \tau < 1$ because the point source is outside the sphere.

From (14), we obtain

$$\alpha_n \sim \frac{i(n+1)\,(ka)^{2n+1}}{n(2n+1)\,c_n^2} \quad \text{and} \quad \beta_n \sim -\frac{i(ka)^{2n+1}}{(2n+1)\,c_n^2} \tag{19}$$

as $ka \to 0$, where $c_n := 1\cdot 3\cdot 5\cdots(2n-1) = (2n)!/(2^n n!)$. In particular,

$$\begin{aligned} &\alpha_1 = -\tfrac{2}{3}\kappa^3 + O(\kappa^5), \quad \beta_1 = \tfrac{1}{3}\kappa^3 + O(\kappa^5), \\ &\alpha_2 = \tfrac{1}{30}\kappa^5 + O(\kappa^7), \quad \beta_2 = -\tfrac{1}{45}\kappa^5 + O(\kappa^7), \\ &\alpha_3 = -\tfrac{4}{4725}\kappa^7 + O(\kappa^9), \quad \beta_3 = \tfrac{1}{1575}\kappa^7 + O(\kappa^9), \end{aligned}$$

as $ka \to 0$. Moreover we have

$$\frac{h_n(kr_0)}{h_0(kr_0)} \sim \frac{c_n}{(kr_0)^n} \quad \text{and} \quad \frac{\widetilde{h}_n(kr_0)}{h_0(kr_0)} \sim \frac{-n\, c_n}{(kr_0)^{n+1}} \quad \text{as } kr_0 \to 0. \tag{20}$$

With the use of the 'angular differential operator'

$$D_a := \widehat{\boldsymbol{\theta}}\, \frac{\partial}{\partial\theta} + \frac{\widehat{\boldsymbol{\phi}}}{\sin\theta}\, \frac{\partial}{\partial\phi}$$

we have

$$\begin{aligned} \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}}) &= -\{n(n+1)\}^{-1/2}\, \widehat{\boldsymbol{r}} \times D_a \left\{ P_n^1(\cos\theta)\, \sin\phi \right\}, \\ \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}}) &= \{n(n+1)\}^{-1/2}\, D_a \left\{ P_n^1(\cos\theta)\, \cos\phi \right\}, \end{aligned}$$

where P_n^1 is an associated Legendre function; see the Appendix.

In order to calculate $\boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}$ with an error of $O((ka)^4)$ we only need the following:

$$\begin{aligned} \boldsymbol{C}_{o11}(\widehat{\boldsymbol{r}}) &= 2^{-1/2} \left\{ \widehat{\boldsymbol{\theta}}\, \cos\phi - \widehat{\boldsymbol{\phi}}\, \cos\theta\, \sin\phi \right\}, \\ \boldsymbol{B}_{e11}(\widehat{\boldsymbol{r}}) &= 2^{-1/2} \left\{ \widehat{\boldsymbol{\theta}}\, \cos\theta\, \cos\phi - \widehat{\boldsymbol{\phi}}\, \sin\phi \right\}, \\ \boldsymbol{C}_{o12}(\widehat{\boldsymbol{r}}) &= (3/2)^{1/2} \left\{ \widehat{\boldsymbol{\theta}}\, \cos\theta\, \cos\phi - \widehat{\boldsymbol{\phi}}\, \cos 2\theta\, \sin\phi \right\}, \\ \boldsymbol{B}_{e12}(\widehat{\boldsymbol{r}}) &= (3/2)^{1/2} \left\{ \widehat{\boldsymbol{\theta}}\, \cos 2\theta\, \cos\phi - \widehat{\boldsymbol{\phi}}\, \cos\theta\, \sin\phi \right\}, \\ \boldsymbol{C}_{o13}(\widehat{\boldsymbol{r}}) &= \tfrac{1}{4}\sqrt{3} \left\{ \widehat{\boldsymbol{\theta}}\, (4 - 5\sin^2\theta)\, \cos\phi + \widehat{\boldsymbol{\phi}}\, (4\cos\theta - 15\sin^2\theta\cos\theta)\, \sin\phi \right\}. \end{aligned}$$

From (18) we finally obtain

$$\begin{aligned} \boldsymbol{F}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{b}}) \;=\;& -\kappa\frac{\tau^2}{\sqrt{2}}\boldsymbol{C}_{o11}(\widehat{\boldsymbol{r}}) + \kappa^2\left[\frac{\tau}{\sqrt{2}}\left(\boldsymbol{C}_{o11}(\widehat{\boldsymbol{r}}) - 2\boldsymbol{B}_{e11}(\widehat{\boldsymbol{r}})\right) + \frac{2\tau^3}{3\sqrt{6}}\boldsymbol{C}_{o12}(\widehat{\boldsymbol{r}})\right] \\ &+ \kappa^3\left[\frac{1}{\sqrt{2}}\left(2\boldsymbol{B}_{e11}(\widehat{\boldsymbol{r}}) + \boldsymbol{C}_{o11}(\widehat{\boldsymbol{r}})\right) - \frac{\tau^2}{\sqrt{6}}\left(\frac{1}{2}\boldsymbol{B}_{e12}(\widehat{\boldsymbol{r}}) + \frac{2}{3}\boldsymbol{C}_{o12}(\widehat{\boldsymbol{r}})\right)\right. \\ &\left. + \frac{\tau^4}{5\sqrt{12}}\boldsymbol{C}_{o13}(\widehat{\boldsymbol{r}})\right] + O(\kappa^4), \quad \text{as } \kappa \to 0, \end{aligned} \tag{21}$$

where $\tau = a/r_0$. In particular, when $r_0 \to \infty$, (21) yields

$$\boldsymbol{F}^{\mathrm{sc}}(\boldsymbol{r};\widehat{\boldsymbol{b}}) = \tfrac{1}{2}i(ka)^3\{-\widehat{\boldsymbol{\theta}}(2\cos\theta + 1)\cos\phi + \widehat{\boldsymbol{\phi}}(2+\cos\theta)\sin\phi\} + O((ka)^4)$$

as $ka \to 0$, recovering (10.198) of [9, p. 406].

Now for the scattering cross-section $\sigma_{\boldsymbol{r}_0}$, defined in (10), after lengthy calculations we obtain

$$\begin{aligned} \sigma_{\boldsymbol{r}_0} \;=\;& \pi a^2\left\{\tfrac{2}{3}\tau^4 + \tfrac{2}{3}(ka)^2\tau^2\left[5 + \tfrac{4}{15}\tau^4\right]\right. \\ &\left. + (ka)^4\left(\tfrac{10}{3} + \tau^2\left[\tfrac{4}{9} + \tfrac{19}{90}\tau^2 + \tfrac{1}{30}\tau^4 + \tfrac{2}{175}\tau^6\right]\right)\right\} + O((ka)^6) \end{aligned} \tag{22}$$

as $ka \to 0$. In the special case $r_0 \to \infty$ ($\tau \to 0$), i.e. for a plane incident wave of arbitrary polarization $\widehat{\boldsymbol{b}}$ with $\widehat{\boldsymbol{b}}\cdot\widehat{\boldsymbol{r}}_0 = 0$, we obtain

$$\sigma = \tfrac{10}{3}(\pi a^2)(ka)^4 + O((ka)^6), \quad ka \to 0.$$

This is a well-known result, first obtained by Mie and Debye in 1909; see (10.202) of [9, p. 406], [7, p. 1884], [6, pp. 417 and 775] or (7.92) of [5].

6 Inverse scattering problems

In this section we consider the inverse problem of determining a small spherical scatterer from a number of measurements of either a far-field, or a near-field quantity.

6.1 Measurements of the scattering cross-section: far-field data

This approach is similar to the one considered by Dassios and his co-workers for the acoustic case, with various boundary conditions.

Recall that for the scattering cross-section we know in view of (22) that

$$\sigma_{\boldsymbol{r}_0} = \tfrac{2}{3}\pi a^2\,(a/r_0)^4, \quad \text{as } ka \to 0.$$

Choose a Cartesian coordinate system $Oxyz$, and five point-source locations, namely $(0,0,0)$, $(\ell,0,0)$, $(0,\ell,0)$, $(0,0,\ell)$, and $(0,0,2\ell)$, which are at (unknown) distances r_0, r_1, r_2, r_3, and r_4, respectively, from the sphere's centre. The parameter ℓ is a chosen fixed length. For each location, measure the leading-order term in the low-frequency expansion of the scattering cross-section. Thus, our five measurements are

$$m_j = \tfrac{2}{3}\pi a^2\,(a/r_j)^4, \quad j = 0,1,2,3,4.$$

Dimensionless quantities related to m_j are

$$\gamma_j = \frac{\ell}{\sqrt{m_j}} = \sqrt{\frac{3}{2\pi}}\frac{\ell}{a}\left(\frac{r_j}{a}\right)^2, \quad j = 0,1,2,3,4. \tag{23}$$

There are six unknowns, namely r_0, r_1, r_2, r_3, r_4, and a. However, r_0, r_3, and r_4 are related using the cosine rule [3], $r_4^2 = 2\ell^2 + 2r_3^2 - r_0^2$, whence

$$\gamma_4 = (\ell/a)^3\sqrt{6/\pi} + 2\gamma_3 - \gamma_0. \tag{24}$$

Then, eliminating a^3 between (23) and (24) gives

$$(r_j/\ell)^2 = 2\gamma_j/(\gamma_4 - 2\gamma^3 + \gamma_0), \quad j = 0,1,2,3.$$

Hence the centre of the spherical scatterer is obtained from the intersection of the four spheres centred at $(0,0,0)$, $(\ell,0,0)$, $(0,\ell,0)$, and $(0,0,\ell)$, with corresponding radii r_0, r_1, r_2, r_3, respectively, while the radius a of the sphere is given by (24).

6.2 Measurements of the scattered field at the source point: near-field data

The scattered field at the source point is given by setting $\boldsymbol{r} = \boldsymbol{r}_0$ in (13):

$$\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{x}}) = \frac{i}{h_0(kr_0)} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left\{ \alpha_n h_n(kr_0)\, \boldsymbol{N}^3_{e1n}(\boldsymbol{r}_0) - \beta_n \widetilde{h}_n(kr_0)\, \boldsymbol{M}^3_{o1n}(\boldsymbol{r}_0) \right\}.$$

As $\boldsymbol{r}_0 = r_0\widehat{\boldsymbol{z}}$, this formula simplifies, using

$$\lim_{\theta\to 0} \frac{d}{d\theta} P^1_n(\cos\theta) = \frac{n(n+1)}{2} \quad \text{and} \quad \lim_{\theta\to 0} \frac{P^1_n(\cos\theta)}{\sin\theta} = \frac{n(n+1)}{2}.$$

We obtain

$$\boldsymbol{P}_{e1n}(\widehat{\boldsymbol{z}}) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{z}}) = \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{z}}) = \tfrac{1}{2}\sqrt{n(n+1)}\,\widehat{\boldsymbol{x}},$$

whence (15) and (16) give

$$\boldsymbol{M}^3_{o1n}(\boldsymbol{r}_0) = \tfrac{1}{2} n(n+1)\, h_n(kr_0)\, \widehat{\boldsymbol{x}}, \quad \boldsymbol{N}^3_{e1n}(\boldsymbol{r}_0) = \tfrac{1}{2} n(n+1)\, \widetilde{h}_n(kr_0)\, \widehat{\boldsymbol{x}}\ ,$$

and

$$\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{x}}) = \frac{-i\widehat{\boldsymbol{x}}}{2h_0(kr_0)} \sum_{n=1}^{\infty} (2n+1)(\alpha_n - \beta_n)\, h_n(kr_0)\, \widetilde{h}_n(kr_0). \tag{25}$$

This formula is exact. Let us evaluate it when ka is small. From (19) and (20), we obtain

$$\alpha_n - \beta_n \sim \frac{i\,(ka)^{2n+1}}{nc_n^2} \quad \text{and} \quad \frac{h_n(kr_0)\, \widetilde{h}_n(kr_0)}{[h_0(kr_0)]^2} \sim \frac{-nc_n^2}{(kr_0)^{2n+1}}$$

as $ka \to 0$ for fixed $\tau = a/r_0$. Hence, (25) gives

$$\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{b}}) \sim \frac{1}{2}\, h_0(kr_0)\, \widehat{\boldsymbol{b}} \sum_{n=1}^{\infty} (2n+1)\tau^{2n+1} = \frac{1}{2}\, h_0(kr_0)\, \widehat{\boldsymbol{b}}\, \frac{\tau^3(3-\tau^2)}{(1-\tau^2)^2},$$

after summing the infinite series (recall that $0 \le \tau < 1$). Finally, we obtain

$$|\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_0}(\boldsymbol{r}_0;\widehat{\boldsymbol{b}})| \sim \frac{1}{2ka}\, \frac{\tau^4(3-\tau^2)}{(1-\tau^2)^2} \quad \text{as } ka \to 0.$$

This gives the magnitude of the scattered field at the source, for a sphere that is small compared to a wavelength.

Let us now formulate a simple inverse problem. Thus, we consider measurements of the scattered field at the same five source points as in Section 6.1, and let

$$M_j = 2k\ell\, |\boldsymbol{E}^{\mathrm{sc}}_{\boldsymbol{r}_j}(\boldsymbol{r}_j;\widehat{\boldsymbol{b}}_j)| = \frac{\alpha^3(3\rho_j - \alpha^2)}{\rho_j(\rho_j - \alpha^2)^2}, \quad j = 0,1,2,3,4, \tag{26}$$

where $\rho_j = (r_j/\ell)^2$ and $\alpha = a/\ell$. Thus, as before, we have five measurements with six unknowns (ρ_0, ρ_1, ρ_2, ρ_3, ρ_4, and α) and the cosine-rule constraint,

$$\rho_4 = 2 + 2\rho_3 - \rho_0; \tag{27}$$

we also have $\rho_j > \alpha^2 > 0$. We can write (26) as

$$\rho_j^3 - 2\alpha^2\rho_j^2 + \alpha^3(\alpha - 3/M_j)\rho_j + \alpha^5/M_j = 0,$$

which is a cubic equation for ρ_j if α is known. If α is not known, one has to solve the six algebraic equations, (26) and (27), for the six unknowns.

Analytical progress is possible if α is known to be small, so that $a \ll \ell$. Then, one can approximate (26) by

$$M_j = 3\alpha^3/\rho_j^2, \quad j = 0,1,2,3,4.$$

It follows that α can then be obtained from (27). One can then proceed as in Section 6.1.

Here, we have discussed a simple but genuinely near-field inverse problem. This is perhaps a more natural and realisable experiment. It is similar to, but more complicated than, the analogous acoustic problem analysed in [1].

Appendix: Spherical vector wave functions

In this appendix we include for convenience the definitions of the spherical vector wave functions used in the paper. So, for $n = 1, 2, \ldots$, and $\sigma = e$ or o (even or odd) we have the spherical vector wave functions of the first kind

$$\boldsymbol{M}^1_{\sigma 1n}(\boldsymbol{r}) = \operatorname{curl}\left[\boldsymbol{r}\, j_n(kr)\, Y_{\sigma 1n}(\theta,\phi)\right] = \sqrt{n(n+1)}\, \boldsymbol{C}_{\sigma 1n}(\theta,\phi)\, j_n(kr)$$

and

$$\begin{aligned} \boldsymbol{N}^1_{\sigma 1n}(\boldsymbol{r}) &= k^{-1} \operatorname{curl} \boldsymbol{M}^1_{\sigma 1n}(\boldsymbol{r}) \\ &= n(n+1)\, \boldsymbol{P}_{\sigma 1n}(\theta,\phi)\, (kr)^{-1}\, j_n(kr) + \sqrt{n(n+1)}\, \boldsymbol{B}_{\sigma 1n}(\theta,\phi)\, (kr)^{-1}\, (d/dr)[r j_n(kr)]. \end{aligned}$$

Here, the spherical harmonics are defined by $Y_{e1n}(\theta,\phi) = P^1_n(\cos\theta)\, \cos\phi$ and $Y_{o1n}(\theta,\phi) = P^1_n(\cos\theta)\, \sin\phi$, where $P^1_n(w) = (1-w^2)^{1/2}\, P'_n(w)$ is an associated Legendre function and $P_n(w)$ is a Legendre polynomial. Moreover, for $\sigma = e$ or o, we have [7, pp. 1898–1899]

$$\begin{aligned} \boldsymbol{P}_{\sigma 1n}(\theta,\phi) &= \widehat{\boldsymbol{r}} Y_{\sigma 1n}(\theta,\phi), \\ \boldsymbol{B}_{\sigma 1n}(\theta,\phi) &= r\{n(n+1)\}^{-1/2} \operatorname{grad} Y_{\sigma 1n}(\theta,\phi), \\ \boldsymbol{C}_{\sigma 1n}(\theta,\phi) &= \{n(n+1)\}^{-1/2} \operatorname{curl}\left[\boldsymbol{r} Y_{\sigma 1n}(\theta,\phi)\right]. \end{aligned}$$

The corresponding spherical vector wave functions of the third kind, $\boldsymbol{M}^3_{\sigma 1n}$ and $\boldsymbol{N}^3_{\sigma 1n}$, are obtained by using the spherical Hankel functions $h_n(kr)$ instead of $j_n(kr)$ in the above definitions.

References

[1] C. Athanasiadis, P. A. Martin, and I. G. Stratis, On spherical-wave scattering by a spherical scatterer and related near-field inverse problems, IMA J. Appl. Math. **66**, 539–549 (2001).
[2] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd edition (Springer-Verlag, Berlin, 1998).
[3] G. Dassios and G. Kamvyssas, Point source excitation in direct and inverse scattering: the soft and the hard small sphere, IMA J. Appl. Math. **55**, 67–84 (1995).
[4] G. Dassios, M. Hadjinicolaou, and G. Kamvyssas, Direct and inverse scattering from point source fields. The penetrable small sphere, J. Appl. Math. Mech. (ZAMM) **79**, 303–316 (1999).
[5] G. Dassios and R. Kleinman, Low Frequency Scattering (Clarendon Press, Oxford, 2000).
[6] J. D. Jackson, Classical Electrodynamics, 2nd edition (Wiley, New York, 1975).
[7] P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953).
[8] R. Potthast, Point Sources and Multipoles in Inverse Scattering Theory (Chapman & Hall/CRC, Boca Raton, 2001).
[9] D. L. Sengupta, The sphere. In: Electromagnetic and Acoustic Scattering by Simple Shapes, edited by J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi (North Holland, Amsterdam, 1969) pp. 353–415.
[10] C. T. Tai, Dyadic Green Functions in Electromagnetic Theory, 2nd edition (IEEE Press, New York, 1994).