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The method of fundamental solutions for scattering and radiation problems

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Abstract

The development of the method of fundamental solutions (MFS) and related methods for the numerical solution of scattering and radiation problems in fluids and solids is described and reviewed. A brief review of the developments and applications in all areas of the MFS over the last five years is also given. Future possible areas of applications in fields related to scattering and radiation problems are identified.

Keywords: Acoustics; Boundary collocation; Fundamental solutions; Nonlinear least squares; Radiation problems; Scattering problems

1. Introduction

The method of fundamental solutions (MFS) is a meshless technique for the numerical solution of certain elliptic boundary value problems which falls in the class of methods generally called boundary methods. Like the boundary element method (BEM), it is applicable when a fundamental solution of the differential equation in question is known. It shares the same advantages of the BEM over domain discretization methods, and also has certain advantages over the BEM.

In order to introduce the MFS and to put it into context, let us consider a specific problem from acoustics. Thus, consider the radiation of time-harmonic sound waves in a compressible fluid, caused by the vibrations of an immersed, three-dimensional, bounded obstacle. This problem can be reduced to an exterior boundary-value problem for the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0 \qquad \text{in } \Omega,$$

where ∇^2 is the Laplacian and Ω is the fluid domain. On the boundary of the obstacle, $\partial\Omega$, there is a boundary condition, which we take as

$$\partial u/\partial n = f \qquad \text{on } \partial\Omega, \tag{1}$$

so that the normal derivative of u is required to equal f on $\partial\Omega$ (exterior Neumann problem). (Other boundary conditions and problems are considered below.). Finally, the behaviour of u at infinity must be specified: we impose the Sommerfeld radiation condition (Eq. (10) below).

How can we solve this problem for u? If $\partial\Omega$ is a sphere, with centre at O, we can use the method of separation of variables, giving

$$u(P) = \sum_{n,m} c_n^m \psi_n^m(\mathbf{r}_P), \qquad P \in \Omega,$$

where the coefficients c_n^m are determined by the boundary condition on $\partial\Omega$, and $\mathbf{r}_P$ is the position vector of P with respect to O. Each $\psi_n^m(\mathbf{r})$ is a separated solution of the Helmholtz equation in spherical polar coordinates; it satisfies the Sommerfeld radiation condition and is singular at the origin. Specifically, we have

$$\psi_n^m(\mathbf{r}) = h_n(kr)Y_n^m(\hat{\mathbf{r}}),$$

where h_n is a spherical Hankel function, Y_n^m is a spherical harmonic, $r = |\mathbf{r}|$ and $\mathbf{r} = r\hat{\mathbf{r}}$.

The simplest ψ_n^m is ψ_0^0; it represents a spherically symmetric source at the origin. More generally, we write

$$G(P,Q) = \psi_0^0(\mathbf{r}_P - \mathbf{r}_Q) = R^{-1}\,\mathrm{e}^{\mathrm{i}kR},$$

for the field at P due to a source at Q, where $R = |\mathbf{r}_P - \mathbf{r}_Q|$ is the distance between P and Q. G is the simplest example of

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a *fundamental solution* for the three-dimensional Helmholtz equation.

Suppose now that $\partial\Omega$ does not have a simple shape. Then, we can attempt to use G as follows. Consider an auxiliary closed surface S in Ω^c, the interior of the obstacle. We may think of S as being a closed surface similar to but smaller than $\partial\Omega$, or we may choose S as a simple surface such as a sphere. Then, we seek the solution in the form of a source distribution over S,

$$u(P) = \int_S \mu(Q) G(P,Q)\,\mathrm{d}s(Q), \qquad P \in \Omega, \tag{2}$$

where μ is a function to be found. Evidently, this expression for u satisfies the Helmholtz equation in Ω and the Sommerfeld radiation condition, for any reasonable choice of μ, so that it only remains to satisfy the boundary condition on $\partial\Omega$. (Classically, one takes $S = \partial\Omega$, leading to a boundary integral equation for μ.) If S is strictly inside Ω^c, the boundary condition gives

$$\int_S \mu(Q) \frac{\partial}{\partial n_p} G(P,Q)\,\mathrm{d}s(Q) = f(P), \qquad P \in \partial\Omega, \tag{3}$$

which is an equation to solve for μ.

A difficulty with methods of this type is that $u(P)$ may not be representable as Eq. (2). For we know that when the solution of the exterior Neumann problem is continued analytically from Ω through $\partial\Omega$ into the portion of Ω^c between $\partial\Omega$ and S, singularities may be encountered. Indeed, there *must* be singularities within Ω^c, but we do not know a priori if they are all inside S. On the other hand, the representation (2) is not singular outside S. For more information on this difficulty, including computational aspects, see Ref. [1, Chapter IV, Section 2.1].

As S and μ will have to be discretized in some way, it is natural to replace the integral in Eq. (2) by a sum: this leads to the MFS. Thus, the solution u is approximated by a function of the form

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} c_j G(P, Q_j), \qquad P \in \Omega, \tag{4}$$

where $\mathbf{c} = (c_1, c_2, \dots, c_N) \in \mathbb{C}^N$ and $\mathbf{Q}$ is a $3N$-vector containing the coordinates of the singularities Q_j, which all lie inside Ω^c.

More generally, we could use an approximation of the form

$$u_N(\mathbf{c}; P) = \sum_{j=1}^{N} c_j \chi_j(P), \qquad P \in \Omega, \tag{5}$$

where $\{\chi_j(P)\}$ is a set of radiating solutions of the Helmholtz equation. These solutions must have singularities in Ω^c; they could be sources, dipoles or multipoles located at points $Q_j \in \Omega^c$. Numerical methods based on Eq. (5) are sometimes called Generalised Multipole Techniques (GMT), although many other names are also used [1: Chapter IV; 2: Chapter 8]. Electromagnetic applications of Eq. (5) have been reviewed recently in Ref. [3].

Having selected a representation for u, Eq. (4) or Eq. (5), the next step is to find the coefficients $\mathbf{c}$. A set of observation points $\{P_i\}_{i=1}^M$ is selected on $\partial\Omega$. One then applies the boundary condition at each of these points.

For the MFS when the locations of the singularities are fixed and preassigned, this process yields the equations

$$Bu_N(\mathbf{c}, \mathbf{Q}; P_i) = 0, \qquad i = 1, 2, \dots, M, \tag{6}$$

where we have written the boundary condition as $Bu = 0$ on $\partial\Omega$. When $M = N$, Eq. (6) is a linear system of N equations in N unknowns, whereas, when $M > N$, we have a linear least-squares problem.

Christiansen [4] refers to the use of Eq. (3) as Method I. He gives a thorough review of the literature (up to about 1980), and shows that the condition numbers of the corresponding linear systems worsen as the distance between S and $\partial\Omega$ increases. More recently, Ochmann [5] has reviewed the use of Eq. (3) in the context of acoustic radiation problems.

With the MFS, there is another option: the locations of the singularities can be determined along with the coefficients $\mathbf{c}$. In this case, there are $4N$ unknowns, comprising $\mathbf{c}$ and the Cartesian coordinates of the N singularities $\mathbf{Q}$; they are to be determined by minimizing the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M} |Bu_N(\mathbf{c}, \mathbf{Q}; P_i)|^2, \tag{7}$$

which is nonlinear in the coordinates of the Q_j. The minimization of this functional is commonly done using readily available nonlinear least-squares software such as the MINPACK routines LMDIF and LMDER [6], the Harwell subroutine VA07AD [7], and the NAG routine E04UPF [8]. The relative merits of these codes are examined in Refs. [9,10]. The constrained optimization features of E04UPF are particularly useful for ensuring that the singularities remain outside the domain of the problem. More details about available least-squares routines as well as various algorithmic features of the MFS may be found in Ref. [11].

The initial placement of the singularities can influence the convergence of a least-squares routine significantly. Usually the singularities are distributed uniformly around the domain of the problem at a fixed distance from its boundary. For problems with boundary singularities, Cisilino and Sensale [12,13] studied the placement of the singularities via a simulated annealing algorithm. Saavedra and Power [14] introduced an MFS algorithm with adaptive refinement in the case in which the singularities are fixed and their number is less than the number of boundary points. In the solution of the resulting least-squares problem, the distribution of singularities is selectively improved by taking into account the intensities of the fundamental solutions in the MFS expansion, i.e. the coefficients of the fundamental solutions, and using them as parameters in a multiple linear regression model. The positioning of the singularities has also been investigated via a genetic algorithm [15,16]. Adaptive GMT are discussed in Ref. [2, p. 275].

When the MFS with an equal number of fixed singularities and boundary points, both of which are uniformly distributed, is applied to certain problems in circular domains, it leads to linear systems with coefficient matrices which are circulant or block matrices with circulant blocks. Ways of exploiting the properties of such systems for the efficient implementation of the MFS applied to harmonic and biharmonic problems are investigated in Refs. [17,18].

1.1. A brief review

Early uses of the MFS were for the solution of various linear potential problems in two and three space variables. The method has since been applied to a variety of situations such as plane potential problems involving nonlinear radiation-type boundary conditions, free boundary problems, biharmonic problems, problems in elastostatics and in the analysis of wave scattering in fluids and solids. A survey of the MFS and related methods for the numerical solution of elliptic boundary value problems is presented in Ref. [11]. Material on the MFS may also be found in the recent books by Golberg and Chen [19] and Kolodziej [20]. In this paper, we describe the development of the MFS and related methods for the numerical solution of scattering and radiation problems in fluids and solids. We also give a brief review of the developments and applications in all areas of the MFS over the last five years and identify future possible areas of applications in fields related to scattering and radiation problems.

It is unclear who first used the MFS with fixed singularities; see the references in Refs. [4,11]. Certainly, the idea of representing potential flow past axisymmetric bodies using singularities on the axis of symmetry is very old; see, for example, the discussion of Rankine's work in Refs. [21: Section 97; 22: Section 6.8].

The MFS with moving singularities was first proposed by Mathon and Johnston [23]. Since the approximate solution in the MFS automatically satisfies the partial differential equation in question, the method may be viewed as a Trefftz method [24]. A survey of Trefftz methods is given in Ref. [25] and more recent studies of these methods include Refs. [26–30]. However, for exterior acoustic problems, it is imperative to use functions that satisfy both the governing differential equation *and* the radiation condition at infinity.

Lately, much attention has been devoted to the extension of the MFS to inhomogeneous problems, especially diffusion problems [31–36], often in conjunction with the dual reciprocity and radial basis function methods for treating the inhomogeneous terms [37,38]. The MFS combined with an operator splitting–radial basis function technique was used for the solution of transient nonlinear Poisson problems in Ref. [39]. The MFS combined with radial basis functions has also been used recently for the solution of linear [40] and nonlinear [41,42] Poisson problems, and for Stokes flow [43]. The MFS in conjunction with compactly supported radial basis functions has been considered in Ref. [44] for the solution of Poisson problems and in Ref. [45] for the solution of three-dimensional Helmholtz-type problems; see also [46]. In Ref. [47], the integrations involved in the evaluation of the particular solutions in Poisson problems are treated by a quasi-Monte Carlo method. The MFS is used in Ref. [48] for the solution of Poisson problems in combination with fundamental solutions of the modified Helmholtz equation instead of radial basis functions. A comparison of the MFS combined with radial basis functions and another meshless method, called Kansa's method, is presented in Ref. [49].

Christiansen [4] and Lazarashvili and Zakradze [50] investigated the dependence of the accuracy of the MFS solution of the Dirichlet problem for Laplace's equation on the location of the auxiliary boundary and the number of boundary points (which in their case is the same as the number of singularities). The same problem was also studied in Ref. [51]. Balakrishnan and Ramachandran [52] applied the MFS to singular problems governed by the modified Helmholtz equation and discussed the positioning of the boundary points and singularities to yield improved accuracy. In Refs. [17,18], certain aspects of the MFS related to the positioning of the singularities when the method is applied to harmonic and biharmonic Dirichlet problems in a disk are investigated.

Cisilino and Pardo [53] used an MFS-type approximation with a functional integral method for the solution of the Dirichlet problem for Laplace's equation in a disk. This approach introduces a regularization parameter which can be adjusted to reduce the error.

In Ref. [54], the MFS is used for the solution of anisotropic problems in elasticity and in Ref. [55] for the solution of anisotropic thin-plate bending problems. Fenner [56] applied the MFS to linear isotropic elasticity problems and also discussed a domain decomposition technique. The MFS is used for the calculation of the eigenvalues of the Helmholtz equation in Ref. [57]. Recently, the MFS was also applied to three-dimensional shape recognition problems by Kanali et al. [58,59].

In recent years, the MFS has also been widely used for problems in electrostatics. In particular, Ismail and Abu-Gammaz [60] applied the MFS to determine the electric field resulting from high voltage transmission systems and Vlad et al. [61] used the MFS to calculate the electric field in plate-type electrostatic separators; see also [62,63].

In all of these studies, the MFS with fixed singularities was employed. The MFS with moving singularities was implemented recently for the solution of three-dimensional Signorini problems [64], three-dimensional elasticity problems [65] and anisotropic single material and bimaterial problems in combination with a domain decomposition technique [66–68].

1.2. Irregular frequencies

It is known that Eq. (3) suffers from irregular frequencies (fictitious eigen-frequencies): uniqueness is lost whenever k^2 is an eigenvalue of the interior Dirichlet problem for S. This fact is stated clearly in Refs. [69,70] and in the review by Benthien and Schenck [71]. However, numerical experiments reported in Refs. [72–74] for the MFS solution of various scattering problems do not indicate the presence of irregular frequencies. A possible resolution of this situation may involve the difference between continuous and discrete representations: clearly, a discrete set of points in Ω^c does not define an internal surface S uniquely. Thus, this topic may warrant further careful computations.

2. Scattering and radiation problems

Because of their advantages over domain discretization methods for the solution of scattering and radiation problems in acoustics and elastodynamics, boundary integral methods have been used for such problems for some time [75,76]. Also, various MFS-type formulations have been proposed for time-harmonic acoustic and electromagnetic scattering problems [77–81]. In Refs. [82,83], the MFS with fixed singularities is applied to two- and three-dimensional acoustic wave scattering problems in fluids involving fluid scatterers. This method is extended in Ref. [84] to two-dimensional acoustic scattering from periodic fluid scatterers, and in Ref. [85] to the treatment of three-dimensional acoustic scattering from doubly periodic fluid scatterers. The MFS approach of Ref. [82] was also extended by Murphy et al. [86] to the solution of short-range ocean acoustics problems. Erez and Leviatan [87] used a modified version of the MFS to analyse acoustic scattering from two adjacent spheres of different sizes. The problem of scattering of a time-harmonic wave in a thin elastic plate by a small patch made of a different material using the MFS was addressed by Leviatan et al. [88]. Essentially the same method was applied by Song et al. [89–91] to two- and three-dimensional acoustic radiation problems and also by Jeans and Mathews [92] to spherical and spheroidal acoustic radiation and scattering problems. In Ref. [93], the MFS is combined with a singular value decomposition technique for treating radiation from a circular cylinder. It is also used in Ref. [73] to study the scattering/transmission of time-harmonic waves by an elastic obstacle embedded in a three-dimensional infinite elastic medium. In Ref. [74], acoustic wave scattering in fluids and wave scattering from rigid/vacuous obstacles in elastic regions are analysed using the MFS with moving singularities and fixed singularities, respectively; see also Refs. [94,95]. In Refs. [96,97], an MFS-type method with fixed singularities for the Helmholtz equation in two and three dimensions is discussed.

An MFS-type method with fixed singularities was also used by Stepanishen and Ramakrishna [98,99] for two-dimensional acoustic scattering from a cylinder with a plane of symmetry. The same authors [100] applied an MFS-type method to acoustic radiation problems in combination with singular value decomposition techniques. The MFS was applied by Johnson et al. [101] to acoustic problems in which the scattering obstacle includes internal scattering objects.

Ochmann [102] has used the MFS with moving singularities for three-dimensional acoustic radiation problems. Numerical results for a cube and a finite circular cylinder were presented. He found that the increase in computer time (compared to the MFS with fixed singularities) was compensated by 'an enormous gain of accuracy' [102, p. 1188].

The MFS with fixed singularities has also been used extensively for the solution of electromagnetic scattering problems, principally by Doicu, Eremin and Wriedt [103–109], Leviatan [110], and Maystre [111], together with their co-workers. Further references on MFS-type methods for acoustic and electromagnetic scattering can be found in the book by Doicu et al. [1].

2.1. Scattering in fluids

2.1.1. Single fluid case

Consider a time-harmonic acoustic wave in an unbounded, homogeneous, compressible fluid. The fluid domain is Ω. The wave is incident upon a rigid, fixed obstacle occupying the bounded region $\Omega^c \in \mathbb{R}^3$, the complement of Ω. The fluid particle velocity at a point $P \in \Omega$ is given by $\mathbf{v}(P,t) = -\nabla\phi(P,t)$, where ϕ is the velocity potential. For time-harmonic waves,

$$\phi(P,t) = Re\{\Phi(P)\exp(-\mathrm{i}\omega t)\},$$

where $\Phi(P)$ is the complex amplitude and ω is the circular frequency. The scattered wave is defined by $\Phi^{\mathrm{S}} = \Phi - \Phi^{\mathrm{I}}$, where Φ^{I} represents the incident wave. The function Φ satisfies the Helmholtz equation

$$\nabla^2\Phi(P) + k^2\Phi(P) = 0, \qquad P \in \Omega, \tag{8}$$

where $k = \omega/c$ is the wave number and c the wave speed.

On $\partial\Omega$, the boundary of the rigid scatterer, $\mathbf{v}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal to $\partial\Omega$; that is, $\partial\Phi/\partial n = 0$.

Rewriting the problem in terms of Φ^{S} gives

$$\begin{aligned} &\nabla^2\Phi^{\mathrm{S}}(P) + k^2\Phi^{\mathrm{S}}(P) = 0, \qquad P \in \Omega, \\ &\frac{\partial\Phi^{\mathrm{S}}}{\partial n}(P) + \frac{\partial\Phi^{\mathrm{I}}}{\partial n}(P) = 0, \qquad P \in \partial\Omega. \end{aligned} \tag{9}$$

In addition, Φ^{S} must satisfy the (three-dimensional) Sommerfeld radiation condition at infinity,

$$\lim_{r\to\infty} r\left(\frac{\partial \Phi^{\mathrm{S}}}{\partial r} - \mathrm{i}k\Phi^{\mathrm{S}}\right) = 0, \tag{10}$$

uniformly in all directions, where r is the distance from P to some fixed origin in the vicinity of Ω^c.

If P and Q are two points, a fundamental solution of the three-dimensional Helmholtz equation is

$$G(P,Q) = \frac{\mathrm{e}^{\mathrm{i}kR}}{R},$$

where R is the distance between P and Q. This fundamental solution satisfies the radiation condition as $r\to\infty$, for any fixed Q. In the MFS [74], the scattered field Φ^{S} at each point P is approximated by

$$\Phi_N^{\mathrm{S}}(\mathbf{c},\mathbf{Q};P) = \sum_{j=1}^{N} c_j G(P,Q_j), \tag{11}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)^{\mathrm{T}} \in \mathbb{C}^N$ is a vector of unknown coefficients and $\mathbf{Q}$ is a $3N$-vector containing the coordinates of the singularities Q_j, which lie in Ω^c. A set of points $\{P_i\}_{i=1}^M$ is selected on $\partial\Omega$, and the coefficients $\mathbf{c}$ and the locations of the singularities $\mathbf{Q}$ are determined by minimizing the functional

$$F(\mathbf{c},\mathbf{Q}) = \sum_{i=1}^{M}\left|\frac{\partial \Phi_N^{\mathrm{S}}}{\partial n_i}(\mathbf{c},\mathbf{Q};P_i) + \frac{\partial \Phi^{\mathrm{I}}}{\partial n_i}(P_i)\right|^2. \tag{12}$$

Applications of this formulation to this type of problem are given in Refs. [74,94,95].

Radiation problems can be treated in a similar way by taking the incident wave to be zero, by replacing the scattered wave with the radiated wave and by prescribing appropriate boundary conditions on $\partial\Omega$ (Eq. (1)).

2.1.2. Fluid–fluid case

In the case where the rigid obstacle Ω^c is replaced by a homogeneous fluid body with properties different to those of the infinite fluid body Ω, the situation is slightly different. In addition to the incident and scattered waves Φ^{I} and Φ^{S}, there is a transmitted wave Φ^{T} inside the obstacle Ω^c. The total wave in Ω, Φ, satisfies, as before, the Helmholtz equation (8) and $\Phi = \Phi^{\mathrm{S}} + \Phi^{\mathrm{I}}$. The transmitted wave Φ^{T} satisfies

$$\nabla^2\Phi^{\mathrm{T}}(P) + k_c^2\Phi^{\mathrm{T}}(P) = 0, \qquad P\in\Omega^c,$$

where k_c is the wave number in Ω^c. Under the assumption that the two fluid surfaces remain in contact, we must have continuity of pressure across the interface; this yields $\rho\Phi = \rho_c\Phi^{\mathrm{T}}$ on $\partial\Omega$, where ρ and ρ_c are the densities of the fluids in Ω and Ω^c, respectively. We also require continuity of normal velocity across the interface; this gives

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi^{\mathrm{T}}}{\partial n} \qquad \text{on } \partial\Omega.$$

Thus the scattered wave Φ^{S} satisfies Eq. (9), the radiation condition (10), and the interface conditions

$$\rho(\Phi^{\mathrm{S}} + \Phi^{\mathrm{I}}) = \rho_c\Phi^{\mathrm{T}} \qquad \text{and} \qquad \frac{\partial\Phi^{\mathrm{S}}}{\partial n} + \frac{\partial\Phi^{\mathrm{I}}}{\partial n} = \frac{\partial\Phi^{\mathrm{T}}}{\partial n}$$

on $\partial\Omega$.

In the MFS, the scattered field Φ^{S} is approximated as in Eq. (11), whereas the transmitted field Φ^{T} is approximated by

$$\Phi_{N^c}^{\mathrm{T}}(\mathbf{d},\mathbf{Q}^c;P) = \sum_{j=1}^{N^c} d_j G_c(P,Q_j^c), \tag{13}$$

where $\mathbf{d} = (d_1, d_2, \ldots, d_{N^c})^{\mathrm{T}} \in \mathbb{C}^{N^c}$ is another vector of unknown coefficients, $\mathbf{Q}^c$ is a $3N^c$-vector containing the coordinates of the singularities Q_j^c which lie in Ω, and $G_c = R^{-1}\exp(\mathrm{i}k_cR)$. As before, we choose points $\{P_i\}_{i=1}^M$ on $\partial\Omega$ and determine $\mathbf{c}$, $\mathbf{d}$ and the locations of the singularities $\mathbf{Q}$ and $\mathbf{Q}^c$ by minimizing the functional

$$\begin{aligned} F(\mathbf{c},\mathbf{d},\mathbf{Q},\mathbf{Q}^c) = {}& \sum_{i=1}^{M}\left|\Phi_N^{\mathrm{S}}(\mathbf{c},\mathbf{Q};P_i) + \Phi^{\mathrm{I}}(P_i) - \left(\frac{\rho_c}{\rho}\right)\Phi_{N^c}^{\mathrm{T}}(\mathbf{d},\mathbf{Q}^c;P_i)\right|^2 \\ & + a\sum_{i=1}^{M}\left|\frac{\partial\Phi_N^{\mathrm{S}}}{\partial n_i}(\mathbf{c},\mathbf{Q};P_i) + \frac{\partial\Phi^{\mathrm{I}}}{\partial n_i}(P_i) \right. \\ & \left. - \frac{\partial\Phi_{N^c}^{\mathrm{T}}}{\partial n_i}(\mathbf{d},\mathbf{Q}^c;P_i)\right|^2, \end{aligned} \tag{14}$$

where a is a typical length-scale for the problem (such as k^{-1} or the diameter of Ω^c). See Ref. [94] for applications of this approach to fluid–fluid problems.

2.2. Scattering in elastic solids

2.2.1. Single solid case

Suppose now that Ω is an unbounded homogeneous isotropic elastic solid. The obstacle is either rigid and fixed, or is a cavity, and is impinged upon by a time-harmonic wave. The field variable of interest is the displacement vector $\mathbf{u}$. Stresses and tractions can be obtained from $\mathbf{u}$ by Hooke's law; in particular, we denote the surface traction vector by $\mathbf{t}$. The governing system of differential equations in terms of the displacement vector is

$$(c_1^2 - c_2^2)\nabla(\nabla\cdot\mathbf{u}) + c_2^2\nabla^2\mathbf{u} + \omega^2\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{15}$$

where c_1 and c_2 are the dilatational and shear wave speeds, respectively, in Ω. The boundary conditions for the two types of obstacles are

$$\mathbf{u} = \mathbf{u}^{\mathrm{I}} + \mathbf{u}^{\mathrm{S}} = 0, \qquad \text{(rigid obstacle)},$$

and

$$\mathbf{t} = \mathbf{t}^{\mathrm{I}} + \mathbf{t}^{\mathrm{S}} = 0, \qquad \text{(cavity)},$$

where $\mathbf{u}^{\mathrm{I}}, \mathbf{u}^{\mathrm{S}}, \mathbf{t}^{\mathrm{I}}$ and $\mathbf{t}^{\mathrm{S}}$ are the incident and scattered displacements and tractions, respectively. For this problem, the fundamental solution is Stokes' displacement fundamental tensor [112]. If we denote the components of this tensor by U_{ij}, $i,j = 1,2,3$, and the components of the Stokes' traction fundamental tensor by T_{ij}, $i,j = 1,2,3$, then

$$U_{ij}(P,Q) = \frac{1}{4\pi\rho\omega^2 R^3}\{[\delta_{ij}(k_2R)^2 e_2 + D] + CR_{,i}R_{,j}\}, \tag{16}$$

$i,j = 1,2,3,$

where $k_j = \omega/c_j$, $e_j = \mathrm{e}^{\mathrm{i}k_jR}$, $j = 1,2$,

$$D = \Gamma_2 e_2 - \Gamma_1 e_1, \qquad C = \Delta_2 e_2 - \Delta_1 e_1,$$

$$\Gamma_j = -1 + \mathrm{i}k_jR, \qquad \Delta_j = 3 - 3\mathrm{i}k_jR - k_j^2R^2,\ \ j = 1,2,$$

and

$$T_{ij}(P,Q) = \frac{1}{4\pi\rho\omega^2 R^4}\Bigg\{\lambda e_1(k_1R)^2\Gamma_1 R_{,j}n_1 + \mu e_2(k_2R)^2\Gamma_2 \times\left(\delta_{ij}\frac{\partial R}{\partial n} + R_{,i}n_j\right) + 2\mu\Bigg[C\left(\delta_{ij}\frac{\partial R}{\partial n} + R_{,j}n_i + R_{,i}n_j\right) + FR_{,i}R_{,j}\frac{\partial R}{\partial n}\Bigg]\Bigg\}, \qquad i,j = 1,2,3, \tag{17}$$

where λ and μ are the Lamé moduli,

$$F = H_1e_1 - H_2e_2,$$

$$H_j = 15 - 15\mathrm{i}k_jR - 6k_j^2R^2 + \mathrm{i}k_j^3R^3,\ \ j = 1,2.$$

In Eqs. (16) and (17), commas indicate partial differentiation with respect to the Cartesian coordinates of P.

In the MFS [74], the scattered field $u_i^{\mathrm{S}}(P)$ is approximated by

$$u_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P) = \sum_{j=1}^{N}\sum_{n=1}^{3} C_{nj}U_{in}(P,Q_j), \qquad i = 1,2,3; \tag{18}$$

the corresponding tractions, $t_i^{\mathrm{S}}(P)$, are approximated by

$$t_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P) = \sum_{j=1}^{N}\sum_{n=1}^{3} C_{nj}T_{in}(P,Q_j), \qquad i = 1,2,3. \tag{19}$$

The unknowns are now the $3N$ complex coefficients C_{nj}, $n = 1,2,3$, $j = 1,\dots,N$, and the $3N$ coordinates of the Q_j.

Again, a set of points $\{P_j\}_{j=1}^{M}$ is selected on $\partial\Omega$, and the coefficients $\mathbf{C}$ and the locations of the singularities $\mathbf{Q}$ are determined by minimizing the functional

$$F_1(\mathbf{C},\mathbf{Q}) = \sum_{i=1}^{3}\sum_{j=1}^{M}|u_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P_j) + u_i^{\mathrm{I}}(P_j)|^2, \tag{20}$$

for a rigid obstacle, and

$$F_2(\mathbf{C},\mathbf{Q}) = \sum_{i=1}^{3}\sum_{j=1}^{M}|t_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P_j) + t_i^{\mathrm{I}}(P_j)|^2, \tag{21}$$

for a cavity. Applications of this formulation with both fixed and moving singularities are given in Refs [74,94,95].

In the case of radiation problems, the same formulation is used with the corresponding radiation boundary conditions and by taking the incident wave to be zero and by replacing the scattered wave with the radiated wave.

2.2.2. Solid–solid case

We next consider the case of wave-scattering in an infinite elastic region $\Omega \in \mathbb{R}^3$ with an elastic obstacle with different material properties occupying Ω^c, its complement. In addition to the incident and scattered waves in Ω, there is a transmitted wave in Ω^c whose displacement and traction are denoted by $\mathbf{u}^{\mathrm{T}}$ and $\mathbf{t}^{\mathrm{T}}$, respectively. Each of the displacement fields satisfies Eq. (15). On the interface $\partial\Omega$, the continuity of the displacements yields

$$\mathbf{u}^{\mathrm{T}} = \mathbf{u}^{\mathrm{I}} + \mathbf{u}^{\mathrm{S}},$$

and the continuity of the tractions gives

$$\mathbf{t}^{\mathrm{T}} = \mathbf{t}^{\mathrm{I}} + \mathbf{t}^{\mathrm{S}}.$$

As before, the scattered fields $\mathbf{u}^{\mathrm{S}}$ and $\mathbf{t}^{\mathrm{S}}$ are approximated by Eqs. (18) and (19), respectively, and the transmitted fields $\mathbf{u}^{\mathrm{T}}$ and $\mathbf{t}^{\mathrm{T}}$ by

$$u_{i,N}^{\mathrm{T}}(\mathbf{D},\mathbf{Q}^c;P) = \sum_{j=1}^{N^c}\sum_{n=1}^{3} D_{nj}U_{in}^c(P,Q_j^c), \qquad i = 1,2,3,$$

and

$$t_{i,N}^{\mathrm{T}}(\mathbf{D},\mathbf{Q}^c;P) = \sum_{j=1}^{N^c}\sum_{n=1}^{3} D_{nj}T_{in}^c(P,Q_j^c), \qquad i = 1,2,3,$$

where C_{nj} and D_{nj} ($n = 1,2,3$, $j = 1,\dots,N^c$) are unknown complex coefficients, U_{in}^c is the Stokes' displacement fundamental tensor for the solid in Ω^c, T_{in}^c are the corresponding tractions, and $\mathbf{Q}^c$ is a $3N^c$-vector containing the coordinates of the singularities Q_j^c, which lie in Ω. We choose points $\{P_j\}_{i=j}^{M}$ on $\partial\Omega$ and determine the coefficients C_{nj}, D_{nj} and the locations of the singularities $\mathbf{Q}$ and $\mathbf{Q}^c$ by minimizing the functional

$$F(\mathbf{C},\mathbf{D},\mathbf{Q},\mathbf{Q}^c) = \sum_{i=1}^{3}\sum_{j=1}^{M}|u_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P_j) + u_i^{\mathrm{I}}(Q_j) - u_{i,N}^{\mathrm{T}}(\mathbf{D},\mathbf{Q}^c;P_j)|^2 + \frac{a}{\mu}\sum_{i=1}^{3}\sum_{j=1}^{M}|t_{i,N}^{\mathrm{S}}(\mathbf{C},\mathbf{Q};P_j) + t_i^{\mathrm{I}}(P_j) - t_{i,N}^{\mathrm{T}}(\mathbf{D},\mathbf{Q}^c;P_j)|^2, \tag{22}$$

where a is a typical length-scale. Applications of this MFS formulation with fixed singularities are given in Refs. [73,94].

2.2.3. Semi-infinite bodies

We next consider the solution of half-space elastodynamic time-harmonic problems. Let $\Omega \in \mathbb{R}^3$ be a linear, isotropic and homogeneous elastic half-space and let the inclusion Ω^c be either rigid or a cavity and be at least partially embedded in Ω. The common boundary of the half-space and the inclusion is denoted by $\partial\Omega$ and the remaining part of the boundary of the half-space, which is traction-free, is denoted by $\partial\Omega_H$. The incident wave is now scattered by both Ω^c and the free-surface of the half-space, which renders the problem considerably more difficult. As before, both the incident and the scattered waves satisfy Eq. (15) and the boundary conditions

$$\mathbf{u}^{\mathrm{I}} + \mathbf{u}^{\mathrm{S}} = 0, \qquad \mathbf{t}^{\mathrm{I}} + \mathbf{t}^{\mathrm{S}} = 0,$$

on the boundary $\partial\Omega$ in the cases of a rigid obstacle and a cavity, respectively. In each case, on the free surface $\partial\Omega_H$ of the half-space, we have the condition

$$\mathbf{t}^{\mathrm{I}} + \mathbf{t}^{\mathrm{S}} = 0. \tag{23}$$

In the MFS, the scattered fields $\mathbf{u}^{\mathrm{S}}$ and $\mathbf{t}^{\mathrm{S}}$ are again approximated by Eqs. (18) and (19), respectively, and the boundary conditions on $\partial\Omega$ and $\partial\Omega_H$ are satisfied in a least-squares sense by minimizing an appropriate functional. Applications of this MFS formulation with fixed singularities can be found in Ref. [94].

An alternative approach is to use Lamb's tensors instead of Stokes' tensors. Lamb's tensors [113,114] automatically satisfy the boundary conditions on the free surface of the half-space, thus avoiding the imposition of condition (23).

3. Axisymmetric acoustic scattering and radiation problems

Consider an axisymmetric region $\Omega' \in \mathbb{R}^3$, which means that Ω' is formed as the exterior of a figure of revolution by rotating a plane region Ω, with boundary $\partial\Omega$, about the z-axis. If we want to solve the Helmholtz equation in Ω', certain simplifications can be achieved by integrating in the azimuthal direction. If the forcing is also axisymmetric, we only need simple ring-sources.

Thus, let (r_P, θ_P, z_P) and (r_Q, θ_Q, z_Q) be the cylindrical coordinates of two points P and Q in Ω', so that

$$R^2(P,Q) = r_Q^2 + r_P^2 - 2r_Q r_P \cos\theta + (z_Q - z_P)^2,$$

where $\theta = \theta_Q - \theta_P$. Then, a fundamental solution of the axisymmetric version of the Helmholtz equation is given by [115]

$$G(P,Q) = \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}kR(P,Q)}}{R(P,Q)}\mathrm{d}\theta. \tag{24}$$

This G also satisfies the radiation condition (10). Applications of this G in the MFS to axisymmetric acoustic scattering and radiation problems are described in Ref. [116].

Now, we can write

$$G(P,Q) = G_1(P,Q) + G_2(P,Q),$$

where

$$G_1(P,Q) = \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}kR(P,Q)} - 1}{R(P,Q)}\mathrm{d}\theta$$

and

$$G_2(P,Q) = \int_0^{2\pi} \frac{1}{R(P,Q)}\mathrm{d}\theta.$$

The integral $G_1(P,Q)$ can be evaluated numerically using a standard quadrature rule since its integrand is nonsingular. It can be shown [117,118] that

$$G_2(P,Q) = 4\mathscr{R}^{-1}K(\kappa), \tag{25}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind,

$$K(\kappa) = \int_0^{\pi/2} [1 - \kappa^2 \sin^2\theta]^{-1/2}\mathrm{d}\theta,$$

with $\kappa^2 = 4r_P r_Q/\mathscr{R}^2$ and $\mathscr{R}^2 = (r_Q + r_P)^2 + (z_Q - z_P)^2$.

The normal derivative of the fundamental solution is

$$\frac{\partial G(P,Q)}{\partial n_P} = \int_0^{2\pi} \frac{\partial}{\partial n_P}\left(\frac{\mathrm{e}^{\mathrm{i}kR(P,Q)}}{R(P,Q)}\right)\mathrm{d}\theta \equiv L_1(P,Q) + L_2(P,Q),$$

where

$$L_1(P,Q) = \int_0^{2\pi} \frac{\partial}{\partial n_P}\left(\frac{\mathrm{e}^{\mathrm{i}kR(P,Q)} - 1}{R(P,Q)}\right)\mathrm{d}\theta,$$

which can be evaluated using standard quadrature, and

$$L_2(P,Q) = \int_0^{2\pi} \frac{\partial}{\partial n_P}\left(\frac{1}{R(P,Q)}\right)\mathrm{d}\theta = \frac{\partial}{\partial n_P}\left(\frac{4K(\kappa)}{\mathscr{R}}\right).$$

In Ref. [115], it is shown that

$$L_2(P,Q) = \frac{2\{\mathscr{R}^2[E(\kappa) - K(\kappa)(1-\kappa^2)] - 2r_P(r_P + r_Q)E(\kappa)\}}{r_P\mathscr{R}^3(1-\kappa^2)} \times n_r - \frac{4(z_P - z_Q)E(\kappa)}{\mathscr{R}^3(1-\kappa^2)}n_z,$$

where $E(\kappa)$ is the complete elliptic integral of the second kind defined by

$$E(\kappa) = \int_0^{\pi/2} [1 - \kappa^2 \sin^2\theta]^{1/2}\mathrm{d}\theta,$$

and n_r and n_z are the components of the outward normal vector $\mathbf{n}$ to the boundary $\partial\Omega$ in the r and z directions, respectively.

Having obtained expressions for the fundamental solution of the axisymmetric version of the Helmholtz equation and the normal derivative of the fundamental solution, we now approximate radiating solutions by

$$\Phi_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} c_j G(P, Q_j), \qquad P \in \bar{\Omega},$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{C}^n$ and $\mathbf{Q}$ is a $2N$-vector containing the coordinates of the singularities Q_j, which lie outside $\bar{\Omega}$. A set of points $\{P_i\}_{i=1}^{M}$ is selected on $\partial\Omega$, and the coefficients $\mathbf{c}$ and the locations of the singularities $\mathbf{Q}$ are determined by minimizing the functional

$$F(\mathbf{c}, \mathbf{Q}) = \sum_{i=1}^{M} |B\Phi_N(\mathbf{c}, \mathbf{Q}; P_i)|^2,$$

where $B\Phi = 0$ is the boundary condition to be imposed. This method can be used for axisymmetric radiation problems or for axisymmetric scattering problems; an example of the latter occurs when the incident field is a plane wave propagating in the z-direction.

In the case of semi-infinite axisymmetric problems, namely when we have acoustic radiation and scattering from axisymmetric bodies in the half-space $z > 0$, say, a similar approach can be adopted. Thus, if $z = 0$ is a rigid plane, we can use the (modified) fundamental solution

$$G^H(P, Q) = \int_0^{2\pi} \frac{e^{ikR(P,Q)}}{R(P, Q)} d\theta + \int_0^{2\pi} \frac{e^{ikR(P,Q')}}{R(P, Q')} d\theta,$$

where the point Q' is the image point of Q in the plane. This fundamental solution satisfies the (Neumann) boundary condition on the free surface automatically. The application of the MFS based on this fundamental solution to several axisymmetric acoustic scattering and radiation problems can be found in Ref. [116].

The MFS has also been applied recently to axisymmetric potential problems [119] and axisymmetric elasticity problems [120].

4. Concluding remarks and future work

An attractive feature of the MFS is that one avoids potentially troublesome integrations (such as are present in BEMs). Also, in comparison with other boundary methods, the MFS is very easy to implement and requires very little data preparation. One of the most important features of the MFS with moving singularities is its adaptivity. By permitting the singularities to move, the method is able to adapt the approximation automatically to reflect any bad behaviour in the solution, and produces a uniform distribution of the error on the boundary regardless of its shape. In particular, the method can handle irregular obstacles which frequently arise in scattering and radiation problems. The non-linearity of the problem resulting from this adaptivity can be viewed as a disadvantage of the method. However, with the availability of good software, this is no longer a serious drawback in most cases.

As in other boundary methods, when applying the MFS, there is no need to truncate infinite or semi-infinite domains present in scattering and radiation problems. Thus complications arising from wave reflections from these artificial boundaries are avoided.

From the work performed on the application of the MFS to radiation and scattering problems, there appears to be some practical difficulties in the use of the MFS with moving singularities for certain problems, such as acoustic scattering in a fluid with a fluid obstacle with different properties or the scattering of elastic waves. Clearly, further investigation of this matter is required.

When applying the MFS with fixed singularities to certain scattering problems, it has been observed that, as the singularities are positioned far from the surface of the scatterer, the resulting coefficient matrix becomes ill-conditioned. This occurs because as the singularities move away from the surface they tend to cluster very close to each other (since we are approximating the solution outside the scatterer). On the other hand, experimental evidence reveals that the accuracy of the solution improves as the singularities move away from the boundary. These observations should be studied further.

So far, the MFS has only been applied to half-space elastodynamic time-harmonic problems using the full-space fundamental solution, the Stokes' tensor, instead of Lamb's tensor, the half-space fundamental solution. A comparison of the performance of the two MFS formulations would be of interest. A similar comparison for the BEM is described in Ref. [114].

The MFS has not yet been applied to fluid–solid wave interaction problems nor to transient problems in elastodynamics. Also, the application of the MFS to axisymmetric scattering and radiation problems in solids has yet to be addressed.

It would also be of interest to investigate the application of the MFS to problems governed by axisymmetric and helically symmetric Helmholtz-type equations whose fundamental solutions are analysed in Refs. [121–123]. Also, the MFS has not yet been applied to problems governed by general second order linear elliptic partial differential equations with variable coefficients for which fundamental solutions are studied in Refs. [124,125]. Finally, potential applications of the MFS can be found in the recent work of Rencis and Huang [126] on the fundamental solutions of two-dimensional microelastic bodies, and of Westphal et al. [127] on the fundamental solutions of the Kirchhoff, Reissner and Mindlin plate bending models.

References

[1] Doicu A, Eremin Yu, Wriedt T. Acoustic and electromagnetic scattering analysis using discrete sources. New York: Academic Press; 2000.

[2] Hafner C. Post-modern electromagnetics. Chichester: Wiley; 1999.

[3] Kaklamani DI, Anastassiu HT. Aspects of the method of auxiliary sources (MAS) in computational electromagnetics. IEEE Antennas Propag Mag 2002;44(3):48–64.

[4] Christiansen S. Condition number of matrices derived from two classes of integral equations. Math Meth Appl Sci 1981;3: 364–92.

[5] Ochmann M. The source simulation technique for acoustic radiation problems. Acustica 1995;81:512–27.

[6] Garbow BS, Hillstrom KE, Moré JJ. MINPACK project. Argonne National Laboratory; 1980.

[7] Hopper MJ, editor. Harwell subroutine library catalogue. AERE, Harwell, UK: Theoretical Physics Division; 1973.

[8] Numerical Algorithms Group Library, NAG(UK) Ltd, Wilkinson House, Jordan Hill Road, Oxford, UK.

[9] Karageorghis A, Fairweather G. The method of fundamental solutions for the numerical solution of the biharmonic equation. J Comput Phys 1987;69:434–59.

[10] Poullikkas A, Karageorghis A, Georgiou G. Methods of fundamental solutions for harmonic and biharmonic boundary value problems. Comput Mech 1998;21:416–23.

[11] Fairweather G, Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. Adv Computat Math 1998;9:69–95.

[12] Cisilino AP, Sensale B. Optimal placement of the source points for singular problems in the method of fundamental solutions. In: Denda M, Aliabadi MH, Charafi A, editors. Advances in boundary element techniques II. Geneva: Hoggar; 2001. p. 67–74.

[13] Cisilino AP, Sensale B. Application of a simulated annealing algorithm in the optimal placement of the source points in the method of the fundamental solutions. Comput Mech 2002;28: 129–36.

[14] Saavedra I, Power H. Adaptive refinement in method of fundamental solutions for bidimensional Laplace problems. Bound Elem Commun 2001;12:3–11.

[15] Nishimura R, Nishimori K, Ishihara N. Determining the arrangement of fictitious charges in charge simulation method using genetic algorithms. J Electrostat 2000;49:95–105.

[16] Nishimura R, Nishimori K, Ishihara N. Automatic arrangement of fictitious charges and contour points in charge simulation method for polar coordinate system. J Electrostat 2001;51/52:618–24.

[17] Smyrlis YS, Karageorghis A. Some aspects of the method of fundamental solutions for certain biharmonic problems. J Sci Comput 2001;16:341–71.

[18] Smyrlis YS, Karageorghis A. Some aspects of the method of fundamental solutions for certain biharmonic problems. Technical report TR/02/2002, Cyprus: Department of Mathematics and Statistics, University of Cyprus; 2002.

[19] Golberg MA, Chen CS. Discrete projection methods for integral equations. Southampton: Computational Mechanics Publications; 1996.

[20] Kolodziej JA. Applications of the boundary collocation method in applied mechanics. Poznan: Wydawnictwo Politechniki Poznanskiej; 2001. In Polish.

[21] Lamb H. Hydrodynamics, 6th ed. Cambridge: University Press; 1932.

[22] Batchelor GK. An introduction to fluid dynamics. Cambridge: University Press; 1967.

[23] Mathon R, Johnston RL. The approximate solution of elliptic boundary-value problems by fundamental solutions. SIAM J Numer Anal 1977;14:638–50.

[24] Khokhlov AV. Trefftz-like numerical method for linear boundary-value problems. Commun Numer Meth Engng 1993;9:607–12.

[25] Kita E, Kamiya N. Trefftz method: an overview. Adv Engng Software 1995;24:3–12.

[26] Abou-Dina MS. Implementation of Trefftz method for the solution of some elliptic boundary value problems. Appl Math Comput 2002; 127:125–47.

[27] Akella MR, Kotamraju GR. Trefftz indirect method applied to nonlinear potential problems. Engng Anal Bound Elem 2000;24: 459–65.

[28] Herrera I. Trefftz method: a general theory. Numer Meth Partial Differ Eq 2000;16:561–80.

[29] Reutskiy S. A boundary method of Trefftz type with approximate trial functions. Engng Anal Bound Elem 2002;26:341–53.

[30] Reutskiy SY, Tirozzi B. Quasi-Trefftz spectral method for separable linear elliptic equations. Comput Math Appl 1999;37(2):47–74.

[31] Chen CS, Golberg MA, Hon YC. The method of fundamental solutions and quasi-Monte Carlo method for diffusion equations. Int J Numer Meth Engng 1998;43:1421–36.

[32] Chen CS, Golberg MA, Rashed YF. A mesh free method for linear diffusion equations. Numer Heat Transfer Part B 1998;33:469–86.

[33] Golberg MA, Chen CS. The method of fundamental solutions for potential, Helmholtz and diffusion problems. In: Golberg MA, editor. Boundary integral methods—numerical and mathematical aspects. Boston: Computational Mechanics Publications; 1999. p. 103–76.

[34] Golberg MA, Chen CS. A mesh free method for solving non-linear reaction–diffusion equations. Comput Model Engng Sci 2001;2: 87–95.

[35] Golberg MA, Chen CS, Muleshkov AS. The method of fundamental solutions for time-dependent problems. In: Chen CS, Brebbia CA, Pepper DW, editors. Boundary element technology XIII. Southampton: WIT Press; 1999. p. 377–86.

[36] Partridge PW, Sensale B. The method of fundamental solutions with dual reciprocity for diffusion and diffusion–convection using subdomains. Engng Anal Bound Elem 2000;24:633–41.

[37] Li J. Mathematical justification for RBF-MFS. Engng Anal Bound Elem 2001;25:897–901.

[38] Ramachandran PA, Balakrishnan K. Radial basis functions as approximate particular solutions: review of recent progress. Engng Anal Bound Elem 2000;24:575–82.

[39] Balakrishnan K, Sureshkumar R, Ramachandran PA. An operator splitting–radial basis function method for the solution of transient nonlinear Poisson problems. Comput Math Appl 2002;43: 289–304.

[40] Elansari M, Ouazar D, Cheng AHD. Boundary solution of Poisson's equation using radial basis function collocated on Gaussian quadrature nodes. Commun Numer Meth Engng 2001; 17:455–64.

[41] Balakrishnan K, Ramachandran PA. A particular solution Trefftz method for non-linear Poisson problems in heat and mass transfer. J Comput Phys 1999;150:239–67.

[42] Balakrishnan K, Ramachandran PA. Osculatory interpolation in the method of fundamental solutions for nonlinear Poisson problems. J Comput Phys 2001;172:1–18.

[43] Tsai CC, Young DL, Cheng AHD. Meshless BEM for three-dimensional Stokes flows. Comput Model Engng Sci 2002;3: 117–28.

[44] Chen CS, Brebbia CA, Power H. Dual reciprocity method using compactly supported radial basis functions. Commun Numer Meth Engng 1999;15:137–50.

[45] Golberg MA, Chen CS, Ganesh M. Particular solutions of 3D Helmholtz-type equations using compactly supported radial basis functions. Engng Anal Bound Elem 2000;24:539–47.

[46] Chen CS, Ganesh M, Golberg MA, Cheng AHD. Multilevel compact radial functions based computational schemes for some elliptic problems. Comput Math Appl 2002;43:359–78.

[47] Chen CS, Golberg MA, Hon YC. Numerical justification of fundamental solutions and the quasi-Monte Carlo method for Poisson-type equations. Engng Anal Bound Elem 1998;22:61–9.

[48] Alves CJS, Chen CS. The method of fundamental solutions adapted for a non homogeneous equation. In: Proceedings of the International Conference in Engineering and Science 2001 (ICES '01). Tech Science Press on CD-ROM.

[49] Li J, Hon YC, Chen CS. Numerical comparisons of two meshless methods using radial basis functions. Engng Anal Bound Elem 2002; 26:205–25.

[50] Lazarashvili L, Zakradze M. On the method of fundamental solutions and some aspects of its application. Proc Razmadze Math Inst 2000; 123:41–51.

[51] Gallopoulos E, Lee D. Fast Laplace solver by boundary integral based domain decomposition methods. In: Rodrigue G, editor. Parallel processing for scientific computing. Philadelphia: SIAM; 1989. p. 141–5.

[52] Balakrishnan K, Ramachandran PA. The method of fundamental solutions for linear diffusion–reaction equations. Math Comput Model 2000;31(2-3):221–37.

[53] Cisilino AP, Pardo E. A functional integral formulation for the method of fundamental solutions. In: Denda M, Aliabadi MH, Charafi A, editors. Advances in Boundary Element Techniques II, Geneva: Hoggar; 2001, p. 13–20.

[54] Khatiashvili G, Silagadze G. On a numerical solution of two-dimensional problems of elasticity for anisotropic medium by the method of fundamental solutions. Reports of the Enlarged Sessions of Seminars of the I. Vekua Institute of Applied Mathematics, Tbilisi State University, vol. XIV, No. 3; 1999.

[55] Rajamohan C, Ramachandran J. Bending of anisotropic plates in charge simulation method. Adv Engng Software 1999;30:369–73.

[56] Fenner RT. A force superposition approach to plane elastic stress and strain analysis. J Strain Anal 2001;36:517–29.

[57] Karageorghis A. The method of fundamental solutions for the calculation of the eigenvalues of the Helmholtz equation. Appl Math Lett 2001;14:837–42.

[58] Kanali C, Murase H, Honami N. Three-dimensional shape recognition using a charge simulation method to process primary image features. J Agri Engng Res 1998;70:195–208.

[59] Kanali C, Murase H, Honami N. Shape identification using a charge simulation retina model. Math Comput Simul 1998;48: 103–18.

[60] Ismail HM, Abu-Gammaz AR. Electric field and right-of-way analysis of Kuwait high-voltage transmission systems. Electr Power Syst Res 1999;50:213–8.

[61] Vlad S, Mihailescu M, Rafiroiu D, Iuga A, Dascalescu L. Numerical analysis of the electric field in plate-type electrostatic separators. J Electrostat 2000;48:217–29.

[62] Pawłowski S. Solutions of boundary value problem approximated by fundamental solutions. J Tech Phys (Warsaw) 1993;34:161–72.

[63] Xu W, Zhao Y, Du Y, Kagawa Y, Wakatsuki N, Park KC. Electrical impedance evaluation in a conductive-dielectric half space by the charge simulation method. Engng Anal Bound Elem 2002;26: 583–90.

[64] Poullikkas A, Karageorghis A, Georgiou G. The numerical solution of three dimensional Signorini problems with the method of fundamental solutions. Engng Anal Bound Elem 2001;25: 221–7.

[65] Poullikkas A, Karageorghis A, Georgiou G. The method of fundamental solutions for three dimensional elastostatics problems. Comput Struct 2002;80:365–70.

[66] Berger JR, Karageorghis A. The method of fundamental solutions for layered elastic materials. Engng Anal Bound Elem 2001;25: 877–86.

[67] Skilowitz JL. Static and transient Green's functions for diffusion in anisotropic bimaterials. MSc Thesis. Division of Engineering, Colorado School of Mines; 1998.

[68] Berger JR, Tewary VK. Greens functions for boundary element analysis of anisotropic bimaterials. Engng Anal Bound Elem 2001; 25:279–88.

[69] Hwang JY, Chang SC. A retracted boundary integral equation for exterior acoustic problem with unique solution for all wave numbers. J Acoust Soc Am 1991;90:1167–80.

[70] Wilton DT, Mathews IC, Jeans RA. A clarification of nonexistence problems with the superposition method. J Acoust Soc Am 1993;94: 1676–80.

[71] Benthien W, Schenck HA. Nonexistence and nonuniqueness problems associated with integral equation methods in acoustics. Comput Struct 1997;65:295–305.

[72] Miller RD, Moyer ET, Huang H, Überall H. A comparison between the boundary element method and the wave superposition approach for the analysis of the scattered fields from rigid bodies and elastic shells. J Acoust Soc Am 1991;89:2185–96.

[73] Kondapalli PS, Shippy DJ, Fairweather G. The method of fundamental solutions for transmission and scattering of elastic waves. Comput Meth Appl Mech Engng 1992;96:255–69.

[74] Kondapalli PS, Shippy DJ, Fairweather G. Analysis of acoustic scattering in fluids and solids by the method of fundamental solutions. J Acoust Soc Am 1992;91:1844–54.

[75] Von Estorff O, editor. Boundary elements in acoustics. Southampton: WIT Press; 2000.

[76] Wu TW, editor. Boundary element acoustics: fundamentals and computer codes. Southampton: WIT Press; 2000.

[77] Chow YL, Chaudhury SK. Non-linear optimization for field scattering and SEM problems. IEE Conf Publ 1981;195: 378–82.

[78] Komiyama A. Multiple scattering in electromagnetic scattering by many perfectly conducting circular cylinders. Electron Commun Jpn 1981;67:37–45.

[79] Kress R, Mohsen A. On the simulation source technique for exterior problems in acoustics. Math Meth Appl Sci 1986;8:585–97.

[80] Limić N. The exterior Neumann problem for the Helmholtz equation. Glasnik Matematički Ser III 1981;36:51–64.

[81] Müller C, Kersten H. Zwei Klassen vollständiger Funktionensysteme zur Behandlung von Randwertaufgaben der Schwingungsgleichung $\Delta U + k^2 U = 0$. Math Meth Appl Sci 1980;2:48–67.

[82] Boag A, Leviatan Y, Boag A. Analysis of acoustic scattering from fluid cylinders using a multifilament source model. J Acoust Soc Am 1988;83:1–8.

[83] Boag A, Leviatan Y, Boag A. Analysis of acoustic scattering from fluid bodies using a multipoint source model. IEEE Trans Ultrason Ferroelectr Freq Control 1989;36:119–28.

[84] Boag A, Leviatan Y, Boag A. Analysis of two-dimensional acoustic scattering from periodic structures using a hybrid source model. J Acoust Soc Am 1989;86:387–94.

[85] Boag A, Leviatan Y, Boag A. Analysis of three-dimensional acoustic scattering from doubly periodic structures using a source model. J Acoust Soc Am 1992;91:572–80.

[86] Murphy JE, Li G, Chin-Bing SA, King DB. Multifilament source model for short-range underwater acoustic problems involving penetrable ocean bottoms. J Acoust Soc Am 1996;99: 845–50.

[87] Erez E, Leviatan Y. Analysis of scattering from structures containing a variety of length scales using a source-model technique. J Acoust Soc Am 1993;93:3027–31.

[88] Leviatan Y, Erez E, Beran MJ. A source model technique for analysis of flexural wave scattering in a heterogeneous thin plate. Quart J Mech Appl Math 1992;45:499–514.

[89] Koopman GH, Song L, Fahnline JB. A method for computing acoustic fields based on the principle of wave superposition. J Acoust Soc Am 1989;86:2433–8.

[90] Song L, Koopman GH, Fahnline JB. Numerical errors associated with the method of superposition for computing acoustic fields. J Acoust Soc Am 1991;89:2625–33.

[91] Song L, Koopman GH, Fahnline JB. Active control of the acoustic radiation of a vibrating structure using a superposition formulation. J Acoust Soc Am 1991;89:2786–92.

[92] Jeans R, Mathews IC. The wave superposition method as a robust technique for computing acoustic fields. J Acoust Soc Am 1992;92: 1156–66.

[93] Fahnline JB, Koopman GH. A numerical solution for the general radiation problem based on the combined methods of superposition and singular-value decomposition. J Acoust Soc Am 1991;90: 2808–19.

[94] Kondapalli PS. Time-harmonic solutions in acoustics and elastodynamics by the method of fundamental solutions. PhD Thesis. Department of Engineering Mechanics, University of Kentucky; 1991.

[95] Shippy DJ, Kondapalli PS, Fairweather G. Analysis of acoustic scattering in fluids and solids by the method of fundamental solutions. Math Comput Model 1990;14:74–9.

[96] Caruthers JE, French JC, Raviprakash GK. Green's function discretization for numerical solution of the Helmholtz equation. J Sound Vibr 1995;187:553–68.

[97] Caruthers JE, French JC, Raviprakash GK. Recent developments concerning a new discretization method for the Helmholtz equation. Proceedings of the First CEAS/AIAA Aeroacoustics Conference, Munich, Germany, vol. II; 1995. p. 819–826.

[98] Stepanishen PR, Ramakrishna S. Acoustic radiation from cylinders with a plane of symmetry using internal multipole line source distributions. I. J Acoust Soc Am 1993;93:658–72.

[99] Ramakrishna S, Stepanishen PR. Acoustic scattering from cylinders with a plane of symmetry using internal multipole line source distributions. II. J Acoust Soc Am 1993;93:673–82.

[100] Stepanishen PR, Ramakrishna S. Acoustic radiation impedances and impulse responses for elliptical cylinders using internal source density and singular value decomposition methods. J Sound Vibr 1994;176:49–68.

[101] Johnson ME, Elliot SJ, Baek KH, Garcia-Bonito J. An equivalent source technique for calculating the sound field inside an enclosure containing scattering objects. J Acoust Soc Am 1998;104:1221–31.

[102] Ochmann M. Calculation of sound radiation from complex structures using the multipole radiator synthesis with optimized source locations. In: Crocker M, editor. Proceedings of the Second International Congress on Recent Developments in Air- and Structure-Borne Sound and Vibration, Auburn; 1992. p. 1187–94.

[103] Doicu A, Eremin Y, Wriedt T. Non-axisymmetric models for light scattering from a particle on or near a plane surface. Opt Commun 2000;182:281–8.

[104] Doicu A, Eremin Y, Wriedt T. Scattering of evanescent waves by a particle on or near a plane surface. Comput Phys Commun 2001;134: 1–10.

[105] Doicu A, Eremin Y, Wriedt T. Scattering of evanescent waves by a sensor tip near a plane surface. Opt Commun 2001;190:5–12.

[106] Eremin Y, Orlov NV, Rozenberg VI. Multiple electromagnetic scattering by a linear array of electrified raindrops. J Atmosph Terrest Phys 1995;57:311–9.

[107] Eremin Y, Orlov N. Modeling of light scattering by non-spherical particles based on discrete sources method. J Quant Spectrosc Radiat Transfer 1998;60:451–62.

[108] Eremin Y, Stover JC, Grishina NV. Discrete sources method for light scattering analysis from 3D asymmetrical features on a substrate. J Quant Spectrosc Radiat Transfer 2001;70:421–31.

[109] Eremin Y, Sveshnikov AG. Discrete sources method in electromagnetic scattering problems. Electromagnetics 1993;13:1–22.

[110] Shifman Y, Friedman M, Leviatan Y. Analysis of electromagnetic scattering by cylinders with edges using a hybrid moment method. IEE Proc Microwaves, Antennas Propag 1997;144:235–40.

[111] Maystre D, Saillard M, Tayeb G. Special methods of wave diffraction. In: Pike R, Sabatier P, editors. Scattering. London: Academic Press; 2002. p. 366–83.

[112] Pao YH, Varatharajulu V. Huygens' principle, radiation conditions, and integral formulas for the scattering of elastic waves. J Acoust Soc Am 1976;59:1361–71.

[113] Kobayashi S, Nishimura N. Green's tensors for elastic half-spaces—an application of boundary integral equation method. Memoirs of the Faculty of Engineering, Kyoto University, vol. 42; 1980. p. 228–41.

[114] Pan L, Rizzo F, Martin PA. Some efficient boundary integral strategies for time-harmonic wave problems in an elastic halfspace. Comput Meth Appl Mech Engng 1998;164:207–21.

[115] Seybert AF, Soenarko B, Rizzo FJ, Shippy DJ. A special integral equation formulation for acoustic radiation and scattering for axisymmetric bodies and boundary conditions. J Acoust Soc Am 1986;80:1241–7.

[116] Karageorghis A, Fairweather G. The method of fundamental solutions for axisymmetric acoustic scattering and radiation problems. J Acoust Soc Am 1998;104:3212–8.

[117] Gupta AK. The boundary integral equation method for potential problems involving axisymmetric geometry and arbitrary boundary conditions. MS Thesis. Department of Engineering Mechanics, University of Kentucky; 1979.

[118] Rizzo FJ, Shippy DJ. A boundary integral approach to potential and elasticity problems for axisymmetric bodies with arbitrary boundary conditions. Mech Res Commun 1979;6:99–103.

[119] Karageorghis A, Fairweather G. The method of fundamental solutions for axisymmetric potential problems. Int J Numer Meth Engng 1999;44:1653–69.

[120] Karageorghis A, Fairweather G. The method of fundamental solutions for axisymmetric elasticity problems. Comput Mech 2000;25:524–32.

[121] Igarashi H, Honma T. On axially and helically symmetric fundamental solutions to modified Helmholtz-type equations. Appl Math Model 1992;16:314–9.

[122] Tezer-Sezgin M, Dost S. On the fundamental solutions of the axisymmetric Helmholtz-type equations. Appl Math Model 1993;17: 47–51.

[123] Tsuchimoto M, Miya K, Honma T, Igarashi H. Fundamental solutions of the axisymmetric Helmholtz-type equations. Appl Math Model 1990;14:605–11.

[124] Ang WT, Kusuma J, Clements D. A boundary element method for a second order elliptic partial differential equation with variable coefficients. Engng Anal Bound Elem 1996;18:311–6.

[125] Clements DL. Fundamental solutions for second order linear elliptic partial differential equations. Comput Mech 1998;22:26–31.

[126] Rencis JJ, Huang Q. Fundamental functions for a two-dimensional microelastic body. Engng Anal Bound Elem 1999;23:787–90.

[127] Westphal T, Andrä H, Schnack E. Some fundamental solutions for the Kirchhoff, Reissner and Mindlin plates and a unified BEM formulation. Engng Anal Bound Elem 2001;25:129–39.