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On Green's function for a bimaterial elastic half-plane

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Abstract

The problem of a point force acting in a composite, two-dimensional, isotropic elastic half-plane is considered. An exact solution is obtained, using Mellin transforms and the Melan solution for a point force in a homogeneous half-plane.

Keywords: Green's function; Melan solution; Mellin transforms; Composite wedge

1. Introduction

In a famous 'discussion', Dundurs (1969) introduced the dimensionless parameters, α and β, that bear his name. The discussion was of a paper by Bogy (1968) on two isotropic elastic quarter-planes, bonded together. One might expect that the eigenfunctions for this problem would depend on three dimensionless parameters (such as the two Poisson ratios and the ratio of the two shear moduli), whereas Dundurs showed that only two are needed.

We are interested in the same configuration as Bogy, but loaded by a point force inside one of the quarter-planes. Indeed, as Dundurs (1969) remarked: 'One desirable new result ought to be …explicit results for some specific cases of loading. In particular, concentrated unit loads for which the fields are Green's functions …could be suggested for such an investigation'. Bogy has shown how problems involving bimaterial wedges, loaded on their faces, can be solved, in principle, using Mellin transforms (Bogy, 1968, 1970, 1971) but, as far as we know, this method has not been used to construct the Green's function, $\mathbf{G}$. Bogy (1970) did give results for a point force acting perpendicularly to the boundary of one of the quarter-planes. Tewary (1991) has constructed $\mathbf{G}$ using the Green's function for the bimaterial full-plane (and anisotropic materials). His method requires the inversion of a 6×6 matrix. We use a different method, limit ourselves to isotropic materials, and have to invert a 4×4 matrix, which we do explicitly. Our strategy begins by subtracting Melan's solution (1932) for a point force in a homogeneous half-plane; this ensures

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that $\mathbf{G}$ has the correct singular behaviour. The difference is then calculated using eigenfunction expansions and Mellin transforms.

The solution of the problem of a point force acting inside one of the quarter-planes can be used to study the effect of various defects in that quarter-plane, perhaps by setting up a boundary integral equation over the boundary of the defect. Use of $\mathbf{G}$ will mean that only the boundary of the defect has to be discretised: the effects of both the free surface and the interface have already been included, exactly, in $\mathbf{G}$. Such a Green's function will be especially useful when the defect is near the intersection of the free surface and the bimaterial interface. Problems of this kind, involving an edge crack perpendicular to the free surface, have been considered in two recent papers. Xu et al. (2001) used a combination of approximate expansions to estimate the stress-intensity factors. Bae and Krishnaswamy (2001) described experiments with thermal and mechanical loadings, and compared their results with finite-element computations. By making use of $\mathbf{G}$, we will be able to analyse cracks of any shape and location within one of the quarter-planes (using a hypersingular integral equation). Cracks lying *in* the interface require a different analysis; see, for example, the paper by Antipov et al. (1997), and references therein.

The plan of the paper is as follows. After formulating the problem in Section 2, we discuss the anti-plane problem in Section 3. The main purpose is to explain the method before it is used for the much more complicated plane-strain problem. The method makes use of separated solutions of the governing differential equations in wedge-shaped regions, solutions that are free of tractions on one boundary of the wedge; the plane-strain version of these solutions are constructed in Section 4. Two of these solutions are combined in Section 5 so as to construct eigenfunctions for composite wedges. The construction of $\mathbf{G}$ itself is given in Section 6.

2. Formulation

Consider an elastic half-plane $x > 0$, where x, y are Cartesian coordinates. The boundary $x = 0$ is free from tractions. A point force is acting at a point P with coordinates (x', y'); we assume that $x' > 0$ (so that P is in the solid) and $y' > 0$. We want to calculate the Green's function $\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}')$, where $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x}' = (x', y')$ are the position vectors of a typical point in the solid and the point-force location, respectively, with respect to the origin O. $\mathbf{G}$ has components G_{ij}; as usual, $G_{ij}(\boldsymbol{x}; \boldsymbol{x}')$ gives the ith component of the displacement at $\boldsymbol{x}$ due to a point force acting in the jth direction at $\boldsymbol{x}'$.

For a homogeneous isotropic half-plane, under plane-strain conditions, the Green's function is well known; it was found by Melan (1932) and so we write it as $\mathbf{G}^{\mathrm{M}}$. Detailed expressions for $\mathbf{G}^{\mathrm{M}}$ are given in Appendix A; see also (Telles and Brebbia, 1981).

We are interested in composite half-planes, made from two isotropic quarter-planes, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where $\mathcal{Q}_1$ is the first quadrant in the (x, y)-plane ($x > 0$ and $y > 0$) and $\mathcal{Q}_2$ is the fourth quadrant ($x > 0$ and $y < 0$). Suppose that the solid in $\mathcal{Q}_\ell$ has Lamé moduli λ_ℓ and μ_ℓ, and Poisson's ratio ν_ℓ, $\ell = 1, 2$. Recall that P is at $\boldsymbol{x}' \in \mathcal{Q}_1$. Write

$$\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}') = \begin{cases} \mathbf{G}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') + \mathbf{G}^1(\boldsymbol{x}; \boldsymbol{x}'), & \boldsymbol{x} \in \mathcal{Q}_1, \\ \mathbf{G}^2(\boldsymbol{x}; \boldsymbol{x}'), & \boldsymbol{x} \in \mathcal{Q}_2; \end{cases} \tag{2.1}$$

the problem is to calculate $\mathbf{G}^1$ and $\mathbf{G}^2$. We know that $\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}')$ is singular at $\boldsymbol{x} = \boldsymbol{x}'$, has zero tractions on the free surface $x = 0$, and has continuous displacements and tractions across the interface $y = 0$. By introducing $\mathbf{G}^{\mathrm{M}}$, we have removed the singularity ($\mathbf{G}$ and $\mathbf{G}^{\mathrm{M}}$ have the same singularities at $\boldsymbol{x}'$) and we have not changed the free-surface conditions. However, we have changed the interface conditions: new conditions relating $\mathbf{G}^1$ and $\mathbf{G}^2$ across $y = 0$ will be obtained. We then construct $\mathbf{G}^1$ and $\mathbf{G}^2$ using polar coordinates and eigenfunction expansions in each quadrant. (Consequently, our method should extend to (straight) interfaces that are not perpendicular to the free surface and to certain composite wedges.)

3. The anti-plane problem

The scalar, anti-plane problem is relatively easy to solve. We use it to explain our method of solution. We seek a function $G(\boldsymbol{x};\boldsymbol{x}')$ that has a logarithmic singularity at $\boldsymbol{x}=\boldsymbol{x}'$, satisfies Laplace's equation in $x>0$ and

$$\partial G/\partial x = 0 \quad \text{on } x=0, \tag{3.1}$$

and is such that G and the corresponding tractions are continuous across the interface $y=0$, $x>0$; in addition, the stresses must $\to 0$ as $x^2+y^2\to\infty$ with $x>0$.

The half-plane solution (satisfying Eq. (3.1)) is

$$G^{\mathrm{M}}(\boldsymbol{x};\boldsymbol{x}') = \log R + \log R_0,$$

where $R=|\boldsymbol{x}-\boldsymbol{x}'| = \{(x-x')^2+(y-y')^2\}^{1/2}$ and $R_0=\{(x+x')^2+(y-y')^2\}^{1/2}$. Define

$$G^0(\boldsymbol{x}') \equiv G^{\mathrm{M}}(\boldsymbol{0};\boldsymbol{x}') = 2\log r'$$

and

$$\widetilde{G}^{\mathrm{M}}(\boldsymbol{x};\boldsymbol{x}') \equiv G^{\mathrm{M}} - G^0 = \log(RR_0/r'^2).$$

We write

$$G(\boldsymbol{x};\boldsymbol{x}') = G^0(\boldsymbol{x}') + \begin{cases} \widetilde{G}^{\mathrm{M}}(\boldsymbol{x};\boldsymbol{x}') + G^1(\boldsymbol{x};\boldsymbol{x}'), & \boldsymbol{x}\in\mathcal{Q}_1, \\ G^2(\boldsymbol{x};\boldsymbol{x}'), & \boldsymbol{x}\in\mathcal{Q}_2, \end{cases} \tag{3.2}$$

where G^1 and G^2 are to be found.

The conditions on G lead to the following conditions on G^ℓ:

$$\nabla^2 G^\ell = 0 \quad \text{in } \mathcal{Q}_\ell, \quad \ell=1,2, \tag{3.3}$$

$$(\partial/\partial x)G^1(0,y;\boldsymbol{x}') = 0 \quad \text{for } y>0, \tag{3.4}$$

$$(\partial/\partial x)G^2(0,y;\boldsymbol{x}') = 0 \quad \text{for } y<0, \tag{3.5}$$

$$G^1(x,0+;\boldsymbol{x}') - G^2(x,0-;\boldsymbol{x}') = -\widetilde{G}^{\mathrm{M}}(x,0;\boldsymbol{x}'), \quad x>0, \tag{3.6}$$

$$\mu_1\frac{\partial G^1}{\partial y}(x,0+;\boldsymbol{x}') - \mu_2\frac{\partial G^2}{\partial y}(x,0-;\boldsymbol{x}') = -\mu_1\frac{\partial G^{\mathrm{M}}}{\partial y}(x,0;\boldsymbol{x}'), \quad x>0. \tag{3.7}$$

In each quarter-plane, we can write down separated solutions in plane polar coordinates, namely

$$r^\omega U^\ell_\omega(\theta) \quad \text{in } \mathcal{Q}_\ell, \quad \ell=1,2,$$

where $U^1_\omega(\theta)=\cos\{\omega(\theta-\frac{1}{2}\pi)\}$, $U^2_\omega(\theta)=\cos\{\omega(\theta+\frac{1}{2}\pi)\}$ and ω is an arbitrary parameter. These expressions automatically satisfy Eqs. (3.3)–(3.5). We now consider a superposition of these solutions, and write

$$G^\ell = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A^\ell(z) r^{-z} U^\ell_{-z}(\theta)\,\mathrm{d}z \quad \text{in } \mathcal{Q}_\ell, \quad \ell=1,2, \tag{3.8}$$

where $A^1(z)$ and $A^2(z)$ are to be found, and the contour in the complex z-plane will be chosen later.

Let us write the first interface condition, Eq. (3.6), as

$$[G](x) = f(x), \quad x>0, \tag{3.9}$$

where the left-hand side gives the jump, G^1-G^2, across $y=0$, and $f(x)$ is known. Substituting from Eq. (3.8) gives

$$[G](x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z}\{A^1(z) - A^2(z)\}\cos\frac{\pi z}{2}\,dz,$$

using $x = r$ on $\theta = 0$. This contour integral is in the form of an inverse Mellin transform (Bleistein and Handelsman, 1975). Thus, if $f(x)$ has a Mellin transform $F(z)$, defined by

$$F(z) = \int_0^\infty x^{z-1} f(x)\,dx,$$

its inverse is given by

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} F(z)\,dz.$$

So, Eq. (3.9) gives

$$\{A^1(z) - A^2(z)\}\cos\frac{1}{2}\pi z = F(z). \tag{3.10}$$

Similarly, write the second interface condition, Eq. (3.7), as $[T](x) = h(x)$ for $x > 0$, where $[T]$ denotes the jump in the tractions and h is given. As $\partial u/\partial y = r^{-1}\partial u/\partial\theta$ on $\theta = 0$, we obtain

$$\{\mu_1 A^1(z) + \mu_2 A^2(z)\} z \sin\frac{1}{2}\pi z = H(z+1). \tag{3.11}$$

Solving Eqs. (3.10) and (3.11) gives

$$A^1(z) = \left\{H(z+1)\cos\frac{1}{2}\pi z + \mu_2 F(z) z \sin\frac{1}{2}\pi z\right\}\Big/\Delta_a(z),$$
$$A^2(z) = \left\{H(z+1)\cos\frac{1}{2}\pi z - \mu_1 F(z) z \sin\frac{1}{2}\pi z\right\}\Big/\Delta_a(z),$$

where

$$\Delta_a(z) = \frac{1}{2}(\mu_1 + \mu_2) z \sin\pi z.$$

G^ℓ can then be found from Eq. (3.8). To evaluate these contour integrals, we have to select c, and this requires us to know the strips of analyticity of $F(z)$ and $H(z)$. As

$$f(x) = \begin{cases} O(x) & \text{as } x \to 0, \\ O(\log x) & \text{as } x \to \infty, \end{cases}$$

it follows that $F(z)$ is analytic for $-1 < \mathrm{Re}(z) < 0$. (If we had not subtracted G^0 from G^M, $F(z)$ would not have existed for any z.) Similarly, we find that

$$h(x) = \begin{cases} O(1) & \text{as } x \to 0, \\ O(x^{-1}) & \text{as } x \to \infty, \end{cases}$$

so that $H(z)$ is analytic for $0 < \mathrm{Re}(z) < 1$. Hence, the parameter c should be chosen with $-1 < c < 0$.

Let us evaluate $F(z)$ and $H(z+1)$. We have

$$f(x) = -\frac{1}{2}\log\left\{\left[(x-x')^2 + y'^2\right]\Big/r'^2\right\} - \frac{1}{2}\log\left\{\left[(x+x')^2 + y'^2\right]\Big/r'^2\right\}.$$

After an integration by parts,

$$F(z) = \frac{1}{z}\int_0^\infty \left\{ \frac{x - x'}{(x - x')^2 + y'^2} + \frac{x + x'}{(x + x')^2 + y'^2} \right\} x^z \, \mathrm{d}x$$
$$= \frac{1}{2z}\int_0^\infty \left(\frac{1}{x - w} + \frac{1}{x - \bar{w}} + \frac{1}{x + w} + \frac{1}{x + \bar{w}} \right) x^z \, \mathrm{d}x,$$

where $w = x' + \mathrm{i}y' = r'\mathrm{e}^{\mathrm{i}\theta'}$ and $\bar{w} = x' - \mathrm{i}y'$. Hence, using Eq. (B.3) four times,

$$F(z) = -\frac{\pi(r')^z}{z \sin \frac{1}{2}\pi z} \cos\left\{ \left(\frac{1}{2}\pi - \theta' \right) z \right\}.$$

Similarly,

$$H(z+1) = \frac{\mu_1 \pi (r')^z}{\cos \frac{1}{2}\pi z} \cos\left\{ \left(\frac{1}{2}\pi - \theta' \right) z \right\},$$

whence

$$A^\ell(z) = \gamma_\ell \frac{\pi(r')^z}{\Delta_a(z)} \cos\left\{ \left(\frac{1}{2}\pi - \theta' \right) z \right\},$$

where $\gamma_1 = \mu_1 - \mu_2$ and $\gamma_2 = 2\mu_1$.

Next, let us evaluate G^ℓ. From Eq. (3.8), we have

$$G^1 = \frac{\mu_1 - \mu_2}{\mathrm{i}(\mu_1 + \mu_2)} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left(\frac{r}{r'} \right)^{-z} \frac{\cos\{(\frac{1}{2}\pi - \theta')z\} \cos\{(\frac{1}{2}\pi - \theta)z\}}{z \sin \pi z} \, \mathrm{d}z, \tag{3.12}$$

with $-1 < c < 0$. The integrand has simple poles at $z = \pm N$, $N = 1, 2, \ldots$, with residues

$$\pm \frac{(-1)^N}{N\pi} \left(\frac{r}{r'} \right)^{\mp N} \cos\left\{ \left(\frac{1}{2}\pi - \theta' \right) N \right\} \cos\left\{ \left(\frac{1}{2}\pi - \theta \right) N \right\}$$
$$= \pm \frac{1}{2N\pi} \left(\frac{r}{r'} \right)^{\mp N} \Big[\cos\{N(\pi + \theta - \theta')\} + \cos\{N(\theta + \theta')\} \Big],$$

and a double pole at $z = 0$ with residue $\pi^{-1}\log(r'/r)$. Moving the inversion contour to the left, we pick up residue contributions from the poles at $z = -N$, giving

$$G^1 = -\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \sum_{N=1}^{\infty} \frac{1}{N} \left(\frac{r}{r'} \right)^N [\cos\{N(\pi + \theta - \theta')\} + \cos\{N(\theta + \theta')\}] \tag{3.13}$$

$$= \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \{\log R_1 + \log R_2 - 2\log r'\}, \tag{3.14}$$

where $R_1 = \{(x + x')^2 + (y + y')^2\}^{1/2}$, $R_2 = \{(x - x')^2 + (y + y')^2\}^{1/2}$ and we have summed the series in Eq. (3.13) using

$$\sum_{N=1}^{\infty} \frac{X^N}{N} \cos N\Phi = -\frac{1}{2} \log(1 - 2X\cos\Phi + X^2),$$

the series being convergent for $|X| < 1$. Exactly the same formula, Eq. (3.14), is obtained by moving the contour in Eq. (3.12) to the right instead. Similarly, we obtain

$$G^2 = \frac{2\mu_1}{\mu_1 + \mu_2} \{\log R + \log R_0 - 2\log r'\},$$

and then Eq. (3.2) gives

$$G(\boldsymbol{x};\boldsymbol{x}') = G_a(\boldsymbol{x}') + \begin{cases} \log(RR_0) + [(\mu_1-\mu_2)/(\mu_1+\mu_2)]\log(R_1R_2), & \boldsymbol{x}\in\mathscr{Q}_1, \\ [2\mu_1/(\mu_1+\mu_2)]\log(RR_0), & \boldsymbol{x}\in\mathscr{Q}_2, \end{cases}$$

where $G_a = -2[(\mu_1-\mu_2)/(\mu_1+\mu_2)]\log r'$ is an additive constant (that can be discarded). One can verify that this expression for G satisfies all the relevant conditions.

In the sequel, we shall use a similar method for the corresponding plane-strain problem.

4. Elastic wedges

In plane polar coordinates r, θ, the equilibrium equations for plane-strain deformations are (Graff, 1991, p. 600)

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r} = 0, \tag{4.1}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{2}{r}\tau_{r\theta} = 0, \tag{4.2}$$

in the absence of body forces. The stresses are given by Hooke's law as

$$\tau_{rr} = (\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right),$$

$$\tau_{\theta\theta} = \lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right),$$

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right),$$

where u_r and u_θ are the radial and angular components, respectively, of the displacement.

We look for solutions of the equilibrium equations in the form

$$u_r(r,\theta) = \mathscr{A}r^\omega e^{im(\theta-\theta_0)} \quad \text{and} \quad u_\theta(r,\theta) = i\mathscr{B}r^\omega e^{im(\theta-\theta_0)},$$

where $\mathscr{A}$, $\mathscr{B}$, ω, m and θ_0 are constants. Substitution in Eqs. (4.1) and (4.2) gives

$$[2(1-\nu)(\omega^2-1) - m^2(1-2\nu)]\mathscr{A} + m(3-4\nu-\omega)\mathscr{B} = 0, \tag{4.3}$$

$$m(3-4\nu+\omega)\mathscr{A} + [(1-2\nu)(\omega^2-1) - 2m^2(1-\nu)]\mathscr{B} = 0, \tag{4.4}$$

using $\lambda/\mu = 2\nu/(1-2\nu)$. Setting the determinant of this system to zero gives

$$m^4 - 2m^2(\omega^2+1) + (\omega^2-1)^2 = 0,$$

whence $m = \pm(\omega+1)$ and $m = \pm(\omega-1)$. Then, Eq. (4.3) gives the following results:

$$\text{if } m = \pm(\omega+1), \quad \text{then } \mathscr{A} = \pm\mathscr{B},$$

whereas

$$\text{if } m = \pm(\omega-1), \quad \text{then } (3-4\nu+\omega)\mathscr{A} \pm (3-4\nu-\omega)\mathscr{B} = 0.$$

Hence, combining these four solutions and writing in real form, we obtain

$$u_r = r^\omega\{A\cos(\omega+1)\psi + B\sin(\omega+1)\psi + C\cos(\omega-1)\psi + D\sin(\omega-1)\psi\},$$

$$u_\theta = r^\omega\{-A\sin(\omega+1)\psi + B\cos(\omega+1)\psi - C\Omega\sin(\omega-1)\psi + D\Omega\cos(\omega-1)\psi\},$$

where A, B, C and D are arbitrary constants,

$$\psi = \theta - \theta_0, \quad \Omega = (\omega+\kappa)/(\omega-\kappa) \quad \text{and} \quad \kappa = 3 - 4\nu.$$

We want solutions that also satisfy

$$\tau_{r\theta} = \tau_{\theta\theta} = 0 \quad \text{on } \theta = \theta_0 (\psi = 0). \tag{4.5}$$

These conditions will be satisfied if

$$(\omega-\kappa)A + (\omega+1)C = (\omega-\kappa)B + (\omega-1)D = 0.$$

Hence, renaming the remaining arbitrary constants, we obtain the following expressions,

$$u_r = r^\omega U_\omega(\psi) \quad \text{and} \quad u_\theta = r^\omega V_\omega(\psi), \tag{4.6}$$

where

$$U_\omega(\psi) = A\{(\omega+1)\cos(\omega+1)\psi + (\kappa-\omega)\cos(\omega-1)\psi\} \\ + B\{(\omega-1)\sin(\omega+1)\psi + (\kappa-\omega)\sin(\omega-1)\psi\}, \tag{4.7}$$

$$V_\omega(\psi) = A\{-(\omega+1)\sin(\omega+1)\psi + (\kappa+\omega)\sin(\omega-1)\psi\} \\ + B\{(\omega-1)\cos(\omega+1)\psi - (\kappa+\omega)\cos(\omega-1)\psi\}, \tag{4.8}$$

and A and B are arbitrary constants (that can vary with ω). The corresponding stress components are

$$\tau_{r\theta} = 2\mu\omega r^{\omega-1}S_\omega(\psi) \quad \text{and} \quad \tau_{\theta\theta} = 2\mu\omega r^{\omega-1}T_\omega(\psi),$$

where

$$S_\omega(\psi) = A\{-(\omega+1)\sin(\omega+1)\psi + (\omega-1)\sin(\omega-1)\psi\} - 2B(\omega-1)\sin\omega\psi\sin\psi,$$

$$T_\omega(\psi) = 2A(\omega+1)\sin\omega\psi\sin\psi + B\{-(\omega-1)\sin(\omega+1)\psi + (\omega+1)\sin(\omega-1)\psi\}.$$

Evidently, $S_\omega(0) = T_\omega(0) = 0$ in accordance with Eq. (4.5), for any choice of A, B and ω.

Later, we will also need the Cartesian components of the displacement. As $u_x = u_r\cos\theta - u_\theta\sin\theta$ and $u_y = u_r\sin\theta + u_\theta\cos\theta$, we obtain

$$u_x = r^\omega X_\omega(\psi,\theta) \quad \text{and} \quad u_y = r^\omega Y_\omega(\psi,\theta), \tag{4.9}$$

where

$$X_\omega(\psi,\theta) = A\Phi_1(\omega,\psi,\theta,\kappa) + B\Psi_1(\omega,\psi,\theta,\kappa), \tag{4.10}$$

$$Y_\omega(\psi,\theta) = A\Phi_2(\omega,\psi,\theta,\kappa) + B\Psi_2(\omega,\psi,\theta,\kappa), \tag{4.11}$$

$$\Phi_1(\omega,\psi,\theta,\kappa) = (\omega+1)\cos(\omega\psi+\psi-\theta) + \kappa\cos(\omega\psi-\psi+\theta) - \omega\cos(\omega\psi-\psi-\theta), \tag{4.12}$$

$$\Psi_1(\omega,\psi,\theta,\kappa) = (\omega-1)\sin(\omega\psi+\psi-\theta) + \kappa\sin(\omega\psi-\psi+\theta) - \omega\sin(\omega\psi-\psi-\theta), \tag{4.13}$$

$$\Phi_2(\omega,\psi,\theta,\kappa) = -(\omega+1)\sin(\omega\psi+\psi-\theta) + \kappa\sin(\omega\psi-\psi+\theta) + \omega\sin(\omega\psi-\psi-\theta), \tag{4.14}$$

$$\Psi_2(\omega,\psi,\theta,\kappa) = (\omega-1)\cos(\omega\psi+\psi-\theta) - \kappa\cos(\omega\psi-\psi+\theta) - \omega\cos(\omega\psi-\psi-\theta). \tag{4.15}$$

5. Composite elastic wedges

Let us return to the geometry of interest, namely two right-angled wedges, $\mathscr{Q}_1$ and $\mathscr{Q}_2$. We use the expansions (4.6), and write

$$u_r = r^\omega U^\ell_\omega(\psi_\ell) \quad \text{and} \quad u_\theta = r^\omega V^\ell_\omega(\psi_\ell) \quad \text{in } \mathscr{Q}_\ell, \;\; \ell = 1, 2,$$

where $\psi_1 = \theta - \frac{1}{2}\pi$, $\psi_2 = \theta + \frac{1}{2}\pi$, and U^ℓ_ω and V^ℓ_ω are defined by Eqs. (4.7) and (4.8), respectively, with coefficients $A^\ell(\omega)$, $B^\ell(\omega)$ and $\kappa_\ell = 3 - 4\nu_\ell$, $\ell = 1, 2$. The parameter ω is unspecified at present.

Next, we calculate the discontinuity in the displacement and traction across the interface at $\theta = 0$; later, these discontinuities will be prescribed, and this will lead to a determination of the coefficients A^ℓ and B^ℓ. The desired discontinuities are

$$[u_r] = r^\omega\{U^1_\omega(-\tfrac{1}{2}\pi) - U^2_\omega(\tfrac{1}{2}\pi)\},$$

$$[u_\theta] = r^\omega\{V^1_\omega(-\tfrac{1}{2}\pi) - V^2_\omega(\tfrac{1}{2}\pi)\},$$

$$[t_r] = 2\omega r^{\omega-1}\{\mu_1 S^1_\omega(-\tfrac{1}{2}\pi) - \mu_2 S^2_\omega(\tfrac{1}{2}\pi)\},$$

$$[t_\theta] = 2\omega r^{\omega-1}\{\mu_1 T^1_\omega(-\tfrac{1}{2}\pi) - \mu_2 T^2_\omega(\tfrac{1}{2}\pi)\}.$$

Straightforward calculation gives

$$\begin{aligned}[u_r] = {} & r^\omega\{(\kappa_1 - 1 - 2\omega)A^1 - (\kappa_2 - 1 - 2\omega)A^2\}\sin\tfrac{1}{2}\omega\pi \\ & + r^\omega\{(\kappa_1 + 1 - 2\omega)B^1 + (\kappa_2 + 1 - 2\omega)B^2\}\cos\tfrac{1}{2}\omega\pi,\end{aligned} \tag{5.1}$$

$$\begin{aligned}[u_\theta] = {} & r^\omega\{(\kappa_1 + 1 + 2\omega)A^1 + (\kappa_2 + 1 + 2\omega)A^2\}\cos\tfrac{1}{2}\omega\pi \\ & + r^\omega\{-(\kappa_1 - 1 + 2\omega)B^1 + (\kappa_2 - 1 + 2\omega)B^2\}\sin\tfrac{1}{2}\omega\pi,\end{aligned} \tag{5.2}$$

$$[t_r] = 4\omega r^{\omega-1}\{\omega(\mu_1 A^1 + \mu_2 A^2)\cos\tfrac{1}{2}\omega\pi - (\omega - 1)(\mu_1 B^1 - \mu_2 B^2)\sin\tfrac{1}{2}\omega\pi\}, \tag{5.3}$$

$$[t_\theta] = 4\omega r^{\omega-1}\{(\omega + 1)(\mu_1 A^1 - \mu_2 A^2)\sin\tfrac{1}{2}\omega\pi + \omega(\mu_1 B^1 + \mu_2 B^2)\cos\tfrac{1}{2}\omega\pi\}. \tag{5.4}$$

Setting these four quantities to zero leads to a determinantal equation for ω, and the construction of eigenfunctions for the bimaterial wedge. Explicitly, we obtain

$$\mathscr{D}(\omega)\mathbf{a} = \mathbf{0}, \tag{5.5}$$

where $\mathbf{a} = \begin{pmatrix} A^1 & A^2 & B^1 & B^2 \end{pmatrix}^{\mathrm{T}}$,

$$\mathscr{D}(\omega) = \begin{pmatrix} -(\kappa_1 - 1 - 2\omega)S & (\kappa_2 - 1 - 2\omega)S & (\kappa_1 + 1 - 2\omega)C & (\kappa_2 + 1 - 2\omega)C \\ (\kappa_1 + 1 + 2\omega)C & (\kappa_2 + 1 + 2\omega)C & (\kappa_1 - 1 + 2\omega)S & -(\kappa_2 - 1 + 2\omega)S \\ 4\omega^2\mu_1 C & 4\omega^2\mu_2 C & 4\omega(\omega - 1)\mu_1 S & -4\omega(\omega - 1)\mu_2 S \\ -4\omega(\omega + 1)\mu_1 S & 4\omega(\omega + 1)\mu_2 S & 4\omega^2\mu_1 C & 4\omega^2\mu_2 C \end{pmatrix}, \tag{5.6}$$

$S = -\sin\frac{1}{2}\omega\pi$ and $C = \cos\frac{1}{2}\omega\pi$. Tedious calculation shows that

$$\det\mathscr{D} = -4^4\omega^2\{(1 - \nu_1)\mu_2 + (1 - \nu_2)\mu_1\}^2\Delta(\omega), \tag{5.7}$$

where

$$\Delta(\omega) = (\beta^2 - 1)S^4 + \{1 + 2\omega^2(\alpha - \beta)\beta\}S^2 + \omega^2\{\omega^2(\alpha - \beta)^2 - \alpha^2\} \tag{5.8}$$

and α and β are the Dundurs parameters (Dundurs, 1969), defined by

$$\alpha = \frac{(1-\nu_1)\mu_2 - (1-\nu_2)\mu_1}{(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1} \quad \text{and} \quad \beta = \frac{(1-2\nu_1)\mu_2 - (1-2\nu_2)\mu_1}{2(1-\nu_1)\mu_2 + 2(1-\nu_2)\mu_1}.$$

Expressions for $\Delta(\omega)$ were found previously by Dundurs (1969) and by Bogy (1968, 1970, 1971).

We are interested in the zeros of $\Delta(\omega)$ because they lead to non-trivial solutions of Eq. (5.5) and they determine the behaviour near $r = 0$. Note that $\det \mathscr{D}$ and Δ are even functions of ω, so that we can write

$$\Delta(\pm\omega_n) = 0, \quad n = 1, 2, \ldots,$$

where $\mathrm{Re}(\omega_n) > 0$. Some of these zeros have been plotted by Bogy (1971, Fig. 4(c)) as functions of α and β. In addition, $\omega = 0$ is a double zero of $\Delta(\omega)$: for small ω, $\Delta(\omega) \simeq \{(\pi/2)^2 - \alpha^2\}\omega^2 > 0$ as $|\alpha| < 1$ (Dundurs, 1969). This zero at $\omega = 0$ corresponds to an eigenfunction that is logarithmically singular as $r \to 0$.

Inspection of Eq. (5.8) shows that $\Delta(\pm 1) = 0$. In fact, if $\omega = 1 + \varepsilon$, we have $\Delta(1 + \varepsilon) \simeq 2\varepsilon\alpha(\alpha - 2\beta)$ for small ε. Thus, ± 1 are double zeros if $\alpha(\alpha - 2\beta) = 0$; this condition was found by Dundurs (1969, Eq. (8a)). Moreover, if $\alpha(\alpha - 2\beta) > 0$, there must be a (real) zero of $\Delta(\omega)$ between 0 and 1 (because $\Delta(\omega)$ increases from $\omega = 0$ but $\Delta(1) = 0$ and the slope $\Delta'(1) > 0$).

6. Construction of G_{ij}

Let $\mathbf{G}^0(\boldsymbol{x}') = \mathbf{G}^{\mathrm{M}}(\mathbf{0}; \boldsymbol{x}')$ and $\widetilde{\mathbf{G}}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') = \mathbf{G}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') - \mathbf{G}^0(\boldsymbol{x}')$. ($\mathbf{G}^0$ is given explicitly in Appendix A.) Change the decomposition (2.1) to

$$\mathbf{G}(\boldsymbol{x}; \boldsymbol{x}') = \mathbf{G}^0(\boldsymbol{x}') + \begin{cases} \widetilde{\mathbf{G}}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') + \mathbf{G}^1(\boldsymbol{x}; \boldsymbol{x}'), & \boldsymbol{x} \in \mathscr{Q}_1, \\ \mathbf{G}^2(\boldsymbol{x}; \boldsymbol{x}'), & \boldsymbol{x} \in \mathscr{Q}_2 \end{cases} \tag{6.1}$$

(so that the definition of $\mathbf{G}^2$ has changed too). We have to calculate $\mathbf{G}^1$ and $\mathbf{G}^2$. Let the corresponding stress components be

$$T^{\ell}_{pqj} = \lambda_\ell \delta_{pq} \frac{\partial}{\partial x_k} G^{\ell}_{kj} + \mu_\ell \left(\frac{\partial}{\partial x_p} G^{\ell}_{qj} + \frac{\partial}{\partial x_q} G^{\ell}_{pj} \right).$$

The boundary and interface conditions are

$$T^1_{p1j}(0, y; \boldsymbol{x}') = 0, \quad y > 0, \tag{6.2}$$

$$T^2_{p1j}(0, y; \boldsymbol{x}') = 0, \quad y < 0, \tag{6.3}$$

$$G^1_{pj}(x, 0+; \boldsymbol{x}') - G^2_{pj}(x, 0-; \boldsymbol{x}') = -\widetilde{G}^{\mathrm{M}}_{pj}(x, 0; \boldsymbol{x}'), \quad x > 0, \tag{6.4}$$

$$T^1_{p2j}(x, 0+; \boldsymbol{x}') - T^2_{p2j}(x, 0-; \boldsymbol{x}') = -T^{\mathrm{M}}_{p2j}(x, 0; \boldsymbol{x}'), \quad x > 0, \tag{6.5}$$

for $p = 1, 2$ and $j = 1, 2$. We also require that all the stresses $T^{\ell}_{pqj} \to 0$ as $x^2 + y^2 \to \infty$ in $\mathscr{Q}_\ell$.

For $j = 1, 2$ and $\ell = 1, 2$, let $G^{\ell}_{rj}(\boldsymbol{x}; \boldsymbol{x}')$ and $G^{\ell}_{\theta j}(\boldsymbol{x}; \boldsymbol{x}')$ denote the radial and angular components, respectively, of the displacement at $\boldsymbol{x}$; we use this mixed formulation because the point force acts in the jth Cartesian direction at $\boldsymbol{x}'$ but the displacement is most conveniently written in polar coordinates. Thus, we write

$$G^{\ell}_{rj} = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-z} U^{\ell}_{-z}(\psi_{\ell})\,\mathrm{d}z, \tag{6.6}$$

$$G^{\ell}_{\theta j} = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-z} V^{\ell}_{-z}(\psi_{\ell})\,\mathrm{d}z, \tag{6.7}$$

in $\mathscr{Q}_{\ell}$, where the unknown coefficients in U^{ℓ}_{-z} and V^{ℓ}_{-z} are $A^{\ell}_j(z)$ and $B^{\ell}_j(z)$. These representations automatically satisfy Eqs. (6.2) and (6.3).

We determine $A^{\ell}_j(z)$ and $B^{\ell}_j(z)$ using Eqs. (6.4) and (6.5). Let

$$f_{pj}(x) = -\widetilde{G}^{\mathrm{M}}_{pj}(x,0;\boldsymbol{x}') \quad \text{and} \quad h_{pj}(x) = -T^{\mathrm{M}}_{p2j}(x,0;\boldsymbol{x}'), \quad x>0.$$

We know that $u_r = u_x$, $u_\theta = u_y$, $\tau_{r\theta} = \tau_{xy}$ and $\tau_{\theta\theta} = \tau_{yy}$ on $\theta = 0$. Hence, following the method of Section 3 and making use of Eqs. (5.1)–(5.4), we obtain

$$\mathscr{D}(-z)\mathbf{a}_j = \mathbf{b}_j, \quad j = 1,2, \tag{6.8}$$

where $\mathbf{a}_j = \begin{pmatrix} A^1_j & A^2_j & B^1_j & B^2_j \end{pmatrix}^{\mathrm{T}}$, $\mathbf{b}_j = \begin{pmatrix} F_{1j}(z) & F_{2j}(z) & H_{1j}(z+1) & H_{2j}(z+1) \end{pmatrix}^{\mathrm{T}}$ and the 4×4 matrix $\mathscr{D}$ is defined by Eq. (5.6). $F_{pj}(z)$ and $H_{pj}(z)$ are the Mellin transforms of $f_{pj}(x)$ and $h_{pj}(x)$, respectively; these are calculated in Appendix B.

The explicit entries in the inverse matrix, $[\mathscr{D}(-z)]^{-1}$, are given in Appendix C. [1] We can write

$$[\mathscr{D}(-z)]^{-1} = \frac{\mathscr{E}(z)}{8(1-\Gamma)\Gamma\Delta(z)}, \tag{6.9}$$

where $\Gamma = \mu_2/\mu_1$, $\Delta(z)$ is the Bogy determinant defined by Eq. (5.8), and $\mathscr{E}(z)$ is the 4×4 matrix defined in Appendix C. Thus, solving Eq. (6.8), we obtain

$$A^{\ell}_j(z) = \Sigma^{\ell}_j(z)/\Delta(z) \quad \text{and} \quad B^{\ell}_j(z) = \Sigma^{\ell+2}_j(z)/\Delta(z), \tag{6.10}$$

where $\ell = 1,2$, $j = 1,2$ and

$$\Sigma^{\ell}_j(z) = \frac{1}{8(1-\Gamma)\Gamma}\sum_{p=1}^{2}\{\mathscr{E}_{\ell p}(z)F_{pj}(z) + \mathscr{E}_{\ell,p+2}(z)H_{pj}(z+1)\}. \tag{6.11}$$

The matrix $\mathscr{E}(z)$ is complicated, but each entry is an analytic function of z, except that $\mathscr{E}_{i3}$ and $\mathscr{E}_{i4}$, $i = 1,2,3,4$, all have a factor of z^{-1}. However, these factors are cancelled by a factor of z in the expressions for $H_{pj}(z+1)$. It is convenient to make these cancellations, and to identify the singularities of $F_{pj}(z)$ and $H_{pj}(z+1)$. Thus, from the expressions in Appendix B, we can write

$$F_{pj}(z) = \Lambda(z)\widehat{F}_{pj}(z) \quad \text{and} \quad H_{pj}(z+1) = \mu_1 z\Lambda(z)\widehat{H}_{pj}(z),$$

where

$$\Lambda(z) = \frac{(r')^z}{8\mu_1(1-\nu_1)z\sin\pi z},$$

$x' = r'\cos\theta'$ and $y' = r'\sin\theta'$. Similarly, from the expressions in Appendix C, we can write

$$\mathscr{E}_{i,p+2}(z) = (\mu_1 z)^{-1}\widehat{\mathscr{E}}_{ip}(z), \quad i = 1,2,3,4.$$

[1] This calculation was done by W. Hereman, using *Mathematica*.

Hence, we obtain

$$A_j^\ell(z) = \frac{(r')^z \widehat{\Sigma}_j^\ell(z)}{z \sin \pi z \Delta(z)} \quad \text{and} \quad B_j^\ell(z) = \frac{(r')^z \widehat{\Sigma}_j^{\ell+2}(z)}{z \sin \pi z \Delta(z)}, \tag{6.12}$$

where $\ell = 1, 2$, $j = 1, 2$, and

$$\widehat{\Sigma}_j^\ell(z) = \frac{\mu_1}{64(\mu_1 - \mu_2)\mu_2(1 - \nu_1)} \sum_{p=1}^{2} \{\mathscr{E}_{\ell p}(z)\widehat{F}_{pj}(z) + \widehat{\mathscr{E}}_{\ell p}(z)\widehat{H}_{pj}(z)\}. \tag{6.13}$$

In the form of Eq. (6.12), all the singularities are given by zeros of the denominator. Thus, there are poles at $z = \pm N$ and at $z = \pm\omega_N$, where $N = 1, 2, \ldots$ and $\Delta(\pm\omega_N) = 0$. In particular, there are double poles at $z = \pm 1$ (triple poles if $\alpha(\alpha - 2\beta) = 0$). There is also a *double* pole at $z = 0$ (as $\widehat{\Sigma}_j^i(z) = O(z^2)$ as $z \to 0$).

Having determined A_j^ℓ and B_j^ℓ, we can construct G_{pj}^ℓ, the cartesian components of $\mathbf{G}^1$ and $\mathbf{G}^2$. They are given by (cf. Eqs. (6.6) and (6.7))

$$G_{1j}^\ell = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-z} X_{-z}^\ell(\psi_\ell, \theta)\, dz,$$
$$G_{2j}^\ell = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-z} Y_{-z}^\ell(\psi_\ell, \theta)\, dz,$$

in $\mathscr{Q}_\ell$, where X_ω^ℓ and Y_ω^ℓ are given by Eqs. (4.10) and (4.11), respectively, in which A, B and κ are replaced by A_j^ℓ, B_j^ℓ and κ_ℓ, respectively. Making these substitutions, we obtain

$$G_{pj}^\ell(\boldsymbol{x}; \boldsymbol{x}') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-z} \{A_j^\ell(z)\Phi_p(-z, \psi_\ell, \theta, \kappa_\ell) + B_j^\ell(z)\Psi_p(-z, \psi_\ell, \theta, \kappa_\ell)\}\, dz, \tag{6.14}$$

where Φ_p and Ψ_p are defined by Eqs. (4.12)–(4.15). This formula shows the dependence on A_j^ℓ and B_j^ℓ. If we want to display the singularities in the integrand, we find that we can write G_{pj}^ℓ concisely in the form

$$G_{pj}^\ell(\boldsymbol{x}; \boldsymbol{x}') = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{r'}{r}\right)^z \mathscr{G}_{pj}^\ell(z; \theta, \theta') \frac{z\, dz}{\sin \pi z \Delta(z)}, \tag{6.15}$$

where $\mathscr{G}_{pj}^\ell$ are known: they are complicated functions of θ and θ', and analytic functions of z.

To evaluate the contour integral in Eq. (6.15), we have to select c. We see that $F_{pj}(z)$ is analytic for $-1 < \mathrm{Re}(z) < 0$, $H_{1j}(z+1)$ is analytic for $-2 < \mathrm{Re}(z) < 1$, and $H_{2j}(z+1)$ is analytic for $-1 < \mathrm{Re}(z) < 1$; their common strip of analyticity is $-1 < \mathrm{Re}(z) < 0$. Within this strip, there may be a zero of $\Delta(z)$. Let $-\omega_1$ be the first zero of $\Delta(z)$ to the left of $z = 0$, so that $\Delta(z)$ is free of zeros in the strip $-\mathrm{Re}(\omega_1) < \mathrm{Re}(z) < 0$. (Recall that $\mathrm{Re}(\omega_1) \leqslant 1$.) We choose the inversion contour in this strip by choosing c so that

$$-\mathrm{Re}(\omega_1) < c < 0.$$

(We cannot choose c so that $-1 < c < -\mathrm{Re}(\omega_1)$ when $0 < \mathrm{Re}(\omega_1) < 1$ because this choice would lead to displacements that are algebraically large, $O(r^{\mathrm{Re}(\omega_1)})$, as $r \to \infty$.)

6.1. Asymptotic results

The contour integrals in Eqs. (6.14) or (6.15) could be evaluated numerically with the substitution $z = c + i\xi$, giving an infinite integral over ξ. Alternatively, as in Section 3, one can move the contour, picking up residue contributions from the various poles. To outline this approach, let us move the contour to the left, so that the first pole encountered is at $z = -\omega_1$. Let us assume that $\alpha(\alpha - 2\beta) > 0$ so that $\omega_1 \neq 1$. Substituting from Eq. (6.10) in Eq. (6.14), we obtain

$$G^{\ell}_{pj}(\boldsymbol{x};\boldsymbol{x}') \sim \frac{r^{\omega_1}}{8(1-\Gamma)\Gamma\Delta'(-\omega_1)}\left\{\Sigma^{\ell}_j(-\omega_1)\Phi_p(\omega_1,\psi_\ell,\theta,\kappa_\ell)+\Sigma^{\ell+2}_j(-\omega_1)\Psi_p(\omega_1,\psi_\ell,\theta,\kappa_\ell)\right\}$$

as $r \to 0$, where $\Sigma^i_j(z)$ is defined by Eq. (6.11). This formula gives the behaviour of $\mathbf{G}^{\ell}$ near the intersection of the interface and the free surface. Further terms can be calculated by moving the contour further to the left. If $\omega_1 = 1$, there is a double pole at $z = -1$, so that the leading term would then involve a term proportional to $r \log r$ and another proportional to r.

Instead of moving the contour to the left, we can move it to the right. The first singularity encountered is the double pole at $z = 0$. In order to compute the residue at this pole, we use the following formulas: $a^z = 1 + z \log a + \mathrm{O}(z^2)$, $\sin \pi z = \pi z + \mathrm{O}(z^3)$,

$$\Delta(z) = \{(\pi/2)^2 - \alpha^2\}z^2 + \mathrm{O}(z^4),$$

$$\Phi_p(-z,\psi_\ell,\theta,\kappa_\ell) = \Phi^{\ell}_{p0} + z\Phi^{\ell}_{p1} + \mathrm{O}(z^2),$$

$$\Psi_p(-z,\psi_\ell,\theta,\kappa_\ell) = \Psi^{\ell}_{p0} + z\Psi^{\ell}_{p1} + \mathrm{O}(z^2),$$

$$\widehat{\Sigma}^i_j(z) = z^2\widehat{\Sigma}^i_{j2} + z^3\widehat{\Sigma}^i_{j3} + \mathrm{O}(z^4),$$

as $z \to 0$, where Φ^{ℓ}_{p0}, Φ^{ℓ}_{p1}, Ψ^{ℓ}_{p0}, Ψ^{ℓ}_{p1}, $\widehat{\Sigma}^i_{j2}$ and $\widehat{\Sigma}^i_{j3}$ can be found by routine Maclaurin expansions. Hence

$$r^{-z}A^{\ell}_j(z) \simeq \frac{\widehat{\Sigma}^{\ell}_{j2} + z\left\{\log(r'/r) + \widehat{\Sigma}^{\ell}_{j3}\right\}}{\pi\{(\pi/2)^2 - \alpha^2\}z^2}$$

near $z = 0$, with a similar approximation for $B^{\ell}_j(z)$. Finally, Eq. (6.14) gives

$$G^{\ell}_{pj}(\boldsymbol{x};\boldsymbol{x}') \simeq \frac{-1}{\pi\{(\pi/2)^2 - \alpha^2\}}\left\{(\Phi^{\ell}_{p0} + \Psi^{\ell}_{p0})\log\frac{r'}{r} + \widehat{\Sigma}^{\ell}_{j2}\Phi^{\ell}_{p1} + \widehat{\Sigma}^{\ell}_{j3}\Phi^{\ell}_{p0} + \widehat{\Sigma}^{\ell+2}_{j2}\Psi^{\ell}_{p1} + \widehat{\Sigma}^{\ell+2}_{j3}\Psi^{\ell}_{p0}\right\}.$$

This formula shows that $\mathbf{G}^{\ell}$ grows logarithmically as $r \to \infty$, as expected. Again, further terms can be calculated by moving the contour further to the right; the next term will come from the pole at $z = \omega_1$.

7. Conclusion

We have shown how the problem of a point force acting in a bimaterial half-plane can be solved exactly; both anti-plane and plane-strain problems have been solved. For the plane-strain problem, the resulting formulas for $\mathbf{G}^{\ell}$ are complicated, but they are all computable. In practice, it may be adequate to compute a few residues. Having found $\mathbf{G}^{\ell}$, the Green's function itself is given by Eq. (6.1); note that the singularity in $\mathbf{G}$ is exactly the same as that for the (full-plane) Kelvin solution, $\mathbf{G}^{\mathrm{K}}$, which is itself one part of the (half-plane) Melan solution, $\mathbf{G}^{\mathrm{M}}$.

The method used was developed for a half-plane composed of two quarter-planes. However, the method will extend to problems involving any two wedges bonded together. This extension may be useful for problems such as a quarter-plane bonded to a half-plane. Extensions of this kind remain for the future.

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Appendix A. The Melan solution

The two-dimensional Kelvin solution for a point force in an infinite plane is $\mathbf{G}^{\mathrm{K}}$, with components

$$G_{ij}^{\mathrm{K}}(\boldsymbol{x};\boldsymbol{x}') = \eta\left\{(3-4\nu)\delta_{ij}\log\frac{1}{R} + \frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j}\right\},$$

where $R = |\boldsymbol{x}-\boldsymbol{x}'| = \{(x-x')^2+(y-y')^2\}^{1/2}$, $x_1 \equiv x$, $x_2 \equiv y$ and $\eta = [8\pi\mu(1-\nu)]^{-1}$. This gives the ith component of the displacement at $\boldsymbol{x}$ due to a point force acting at $\boldsymbol{x}'$ in the jth direction. The corresponding stress components are given by

$$\begin{aligned} T_{pqj}^{\mathrm{K}} &= \lambda\delta_{pq}\frac{\partial}{\partial x_k}G_{kj}^{\mathrm{K}} + \mu\left(\frac{\partial}{\partial x_p}G_{qj}^{\mathrm{K}} + \frac{\partial}{\partial x_q}G_{pj}^{\mathrm{K}}\right) \\ &= \frac{2\mu\eta}{R}\left\{(1-2\nu)\left(\delta_{pq}\frac{\partial R}{\partial x_j} - \delta_{qj}\frac{\partial R}{\partial x_p} - \delta_{pj}\frac{\partial R}{\partial x_q}\right) - 2\frac{\partial R}{\partial x_j}\frac{\partial R}{\partial x_p}\frac{\partial R}{\partial x_q}\right\}. \end{aligned} \tag{A.1}$$

The Melan solution for a point force in a half-plane, $\mathbf{G}^{\mathrm{M}}(\boldsymbol{x};\boldsymbol{x}')$, can be written as

$$\mathbf{G}^{\mathrm{M}} = \mathbf{G}^{\mathrm{K}} + \mathbf{G}^{\mathrm{c}},$$

where $\mathbf{G}^{\mathrm{c}}(\boldsymbol{x};\boldsymbol{x}')$ is a non-singular 'correction' to account for the traction-free boundary at $x=0$. The components of $\mathbf{G}^{\mathrm{c}}$ are defined by

$$G_{11}^{\mathrm{c}} = \eta\left\{\frac{\kappa}{R_0^2}(x+x')^2 + F_1\right\}, \quad G_{12}^{\mathrm{c}} = \eta\left\{\frac{\kappa}{R_0^2}(x-x')(y-y') - F_2\right\},$$

$$G_{21}^{\mathrm{c}} = \eta\left\{\frac{\kappa}{R_0^2}(x-x')(y-y') + F_2\right\}, \quad G_{22}^{\mathrm{c}} = \eta\left\{\frac{\kappa}{R_0^2}(y-y')^2 + F_1\right\},$$

where $\kappa = 3-4\nu$, $R_0 = \{(x+x')^2+(y-y')^2\}^{1/2}$,

$$F_1 = \mathscr{A}\log R_0 - \frac{2xx'}{R_0^2} + \frac{4xx'}{R_0^4}(x+x')^2,$$

$$F_2 = \frac{4xx'}{R_0^4}(x+x')(y-y') - \mathscr{B}\tan^{-1}\left(\frac{y-y'}{x+x'}\right),$$

$\mathscr{A} = \kappa - 8(1-\nu)^2$ and $\mathscr{B} = 4(1-\nu)(1-2\nu)$. Note that our G_{ij}^{c} is u_{ji}^{c} in (Telles and Brebbia, 1981), and that $G_{ij}^{\mathrm{c}}(\boldsymbol{x};\boldsymbol{x}') = G_{ji}^{\mathrm{c}}(\boldsymbol{x}';\boldsymbol{x})$.

When calculating the corresponding stresses, we use the following formulas:

$$\frac{\partial F_1}{\partial x} = \frac{1}{R_0^2}\left\{\mathscr{A}(x+x') - 2x' + \frac{4x'}{R_0^2}(x+x')(x'+4x) - \frac{16xx'}{R_0^4}(x+x')^3\right\},$$

$$\frac{\partial F_1}{\partial y} = \frac{y-y'}{R_0^2}\left\{\mathscr{A} + \frac{4xx'}{R_0^2} - \frac{16xx'}{R_0^4}(x+x')^2\right\},$$

$$\frac{\partial F_2}{\partial x} = \frac{y-y'}{R_0^2}\left\{\mathscr{B} + \frac{4x'}{R_0^2}(x'+2x) - \frac{16xx'}{R_0^4}(x+x')^2\right\},$$

$$\frac{\partial F_2}{\partial y} = \frac{x+x'}{R_0^2}\left\{-\mathscr{B} - \frac{12xx'}{R_0^2} + \frac{16xx'}{R_0^4}(x+x')^2\right\}.$$

Using a similar notation to Eq. (A.1), we obtain

$$T^{\mathrm{c}}_{121} = 2\mu\eta\frac{y-y'}{R_0^2}\left\{1-2v+\frac{2}{R_0^2}[x'^2 + x(4vx'-\kappa x)] - \frac{16xx'}{R_0^4}(x+x')^2\right\},$$

$$T^{\mathrm{c}}_{111} = \frac{2\mu\eta}{R_0^2}\left\{(1-2v)(x-x') + \frac{2}{R_0^2}(x+x')[-x'^2 + x\{2x'(1+2v)-\kappa x\}] - \frac{16xx'}{R_0^4}(x+x')^3\right\},$$

$$T^{\mathrm{c}}_{221} = \frac{2\mu\eta}{R_0^2}\left\{(10v-7)x + (6v-1)x' + \frac{2}{R_0^2}(x+x')[-3x'^2 + x\{\kappa x - 2x'(3+2v)\}] + \frac{16xx'}{R_0^4}(x+x')^3\right\},$$

$$T^{\mathrm{c}}_{122} = \frac{2\mu\eta}{R_0^2}\left\{(6v-5)x + (2v-3)x' + \frac{2}{R_0^2}(x+x')[x'^2 + x\{2x'(5-2v)+\kappa x\}] - \frac{16xx'}{R_0^4}(x+x')^3\right\},$$

$$T^{\mathrm{c}}_{112} = 2\mu\eta\frac{y-y'}{R_0^2}\left\{2v-1+\frac{2}{R_0^2}[x'^2 - x\{4(1-v)x'+\kappa x\}] + \frac{16xx'}{R_0^4}(x+x')^2\right\},$$

$$T^{\mathrm{c}}_{222} = 2\mu\eta\frac{y-y'}{R_0^2}\left\{6v-5+\frac{2}{R_0^2}[3x'^2 + x\{4(2-v)x'+\kappa x\}] - \frac{16xx'}{R_0^4}(x+x')^2\right\}.$$

The stresses corresponding to $\mathbf{G}^{\mathrm{M}}$ are given by $T^{\mathrm{M}}_{pqj} = T^{\mathrm{K}}_{pqj} + T^{\mathrm{c}}_{pqj}$. As $R_0 = R$ on $x = 0$, it is easy to verify that $T^{\mathrm{M}}_{p1j} = 0$ on the half-plane boundary, $x = 0$, for $p = 1, 2$ and $j = 1, 2$, as expected. Also, on the line $y = 0$, we find that

$$T^{\mathrm{M}}_{p2j}(x, 0; \boldsymbol{x}') = \begin{cases} \mathrm{O}(1) & \text{as } x \to 0, \\ \mathrm{O}(x^{-2}) & \text{as } x \to \infty, \end{cases} \quad \text{for } p = 1, 2 \quad \text{and} \quad j = 1, 2.$$

Let $\mathbf{G}^0(\boldsymbol{x}') = \mathbf{G}^{\mathrm{M}}(\mathbf{0}; \boldsymbol{x}')$. We have

$$G^0_{11} = (2\pi\mu)^{-1}\{\cos^2\theta' - 2(1-v)\log r'\}, \quad G^0_{12} = (2\pi\mu)^{-1}\left\{\tfrac{1}{2}\sin 2\theta' - (1-2v)\theta'\right\},$$
$$G^0_{21} = (2\pi\mu)^{-1}\left\{\tfrac{1}{2}\sin 2\theta' + (1-2v)\theta'\right\}, \quad G^0_{22} = (2\pi\mu)^{-1}\{\sin^2\theta' - 2(1-v)\log r'\},$$

where $x' = r'\cos\theta'$ and $y' = r'\sin\theta'$. For each fixed $\boldsymbol{x}'$, and for each j, $G^0_{1j}(\boldsymbol{x}')$ and $G^0_{2j}(\boldsymbol{x}')$ are the components of a constant displacement vector. Evidently,

$$\widetilde{\mathbf{G}}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') \equiv \mathbf{G}^{\mathrm{M}}(\boldsymbol{x}; \boldsymbol{x}') - \mathbf{G}^0(\boldsymbol{x}')$$

vanishes at $\boldsymbol{x} = \mathbf{0}$, a fact that will be useful later. In fact, on the line $y = 0$, we have

$$\mathbf{G}^{\mathrm{M}}(x, 0; \boldsymbol{x}') = \begin{cases} \mathrm{O}(1) & \text{as } x \to 0, \\ \mathrm{O}(\log x) & \text{as } x \to \infty, \end{cases}$$

whereas

$$\widetilde{\mathbf{G}}^{\mathrm{M}}(x,0;\boldsymbol{x}') = \begin{cases} \mathrm{O}(x) & \text{as } x \to 0, \\ \mathrm{O}(\log x) & \text{as } x \to \infty; \end{cases}$$

this difference will enable us to use Mellin-transform techniques.

Appendix B. Some Mellin transforms

Let us begin with $f_{11}(x) = G^0_{11}(\boldsymbol{x}') - G^{\mathrm{M}}_{11}(x,0;\boldsymbol{x}')$. From the formulas in Appendix A, we have

$$f_{11}(x) = \eta \tilde{f}_{11}(x) + 2\eta x x' \mathscr{R}_0^{-4}\{y'^2 - (x+x')^2\},$$

where $\eta = [8\pi\mu_1(1-\nu_1)]^{-1}$,

$$\tilde{f}_{11}(x) = \kappa_1 \log\mathscr{R} - \mathscr{A}\log\mathscr{R}_0 - 8(1-\nu_1)^2\log r' + 4(1-\nu_1)\cos^2\theta' - (x-x')^2\mathscr{R}^{-2} - \kappa_1(x+x')^2\mathscr{R}_0^{-2},$$

$$\kappa_1 = 3-4\nu_1, \quad \mathscr{A} = \kappa_1 - 8(1-\nu_1)^2, \quad \mathscr{R} = \{(x-x')^2 + y'^2\}^{1/2} \quad \text{and} \quad \mathscr{R}_0 = \{(x+x')^2 + y'^2\}^{1/2}.$$

As $\tilde{f}_{11}(0) = 0$, we can integrate by parts, giving

$$\int_0^\infty x^{z-1}\tilde{f}_{11}(x)\,\mathrm{d}x = -\frac{1}{z}\int_0^\infty x^z \frac{\mathrm{d}}{\mathrm{d}x}\tilde{f}_{11}(x)\,\mathrm{d}x, \quad -1 < \mathrm{Re}(z) < 0. \tag{B.1}$$

Hence, the Mellin transform of $f_{11}(x)$ is $F_{11}(z)$, where

$$F_{pj}(z) = \frac{\eta}{z}\int_0^\infty x^z \mathscr{F}_{pj}(x)\,\mathrm{d}x, \tag{B.2}$$

$$\begin{aligned}\mathscr{F}_{11}(x) &= -\kappa_1\frac{x-x'}{\mathscr{R}^2} + \mathscr{A}\frac{x+x'}{\mathscr{R}_0^2} + 2y'^2\frac{x-x'}{\mathscr{R}^4} + 2\kappa_1 y'^2\frac{x+x'}{\mathscr{R}_0^4} + 2z\frac{x'}{\mathscr{R}_0^4}\{y'^2 - (x+x')^2\} \\ &= -\frac{\kappa_1}{2}\left\{\frac{1}{x-w} + \frac{1}{x-\bar{w}}\right\} + \frac{\mathscr{A}}{2}\left\{\frac{1}{x+w} + \frac{1}{x+\bar{w}}\right\} + \frac{y'}{2i}\left\{\frac{1}{(x-w)^2} - \frac{1}{(x-\bar{w})^2}\right\} \\ &\quad - \frac{\kappa_1 y'}{2i}\left\{\frac{1}{(x+w)^2} - \frac{1}{(x+\bar{w})^2}\right\} - zx'\left\{\frac{1}{(x+w)^2} + \frac{1}{(x+\bar{w})^2}\right\},\end{aligned}$$

$w = x' + \mathrm{i}y' = r'\mathrm{e}^{\mathrm{i}\theta'}$ and $\bar{w} = x' - \mathrm{i}y'$. Then, we can evaluate the integral for F_{11}, Eq. (B.2), using two standard integrals, namely

$$\int_0^\infty \frac{x^z}{x+X}\,\mathrm{d}x = -\frac{\pi|X|^z}{\sin\pi z}\mathrm{e}^{\mathrm{i}z\Phi}, \quad -1 < \mathrm{Re}(z) < 0, \tag{B.3}$$

and

$$\int_0^\infty \frac{x^z}{(x+X)^2}\,\mathrm{d}x = \frac{\pi z|X|^{z-1}}{\sin\pi z}\mathrm{e}^{\mathrm{i}(z-1)\Phi}, \quad -1 < \mathrm{Re}(z) < 1, \tag{B.4}$$

where $X = |X|\mathrm{e}^{\mathrm{i}\Phi}$ and $|\Phi| < \pi$. We obtain

$$\begin{aligned}F_{11}(z) = \Lambda(z)\{&\kappa_1\cos[z(\pi-\theta')] - \mathscr{A}\cos z\theta' - z\sin[(z-1)(\pi-\theta')]\sin\theta' - \kappa_1 z\sin[(z-1)\theta']\sin\theta' \\ &- 2z^2\cos[(z-1)\theta']\cos\theta'\},\end{aligned}$$

where $-1 < \mathrm{Re}(z) < 0$ and

$$\Lambda(z) = \frac{\eta\pi(r')^z}{z\sin\pi z}.$$

Next, we have

$$f_{21}(x) = \eta\tilde{f}_{21}(x) + 4\eta x x' y'(x+x')\mathscr{R}_0^{-4},$$

where

$$\tilde{f}_{21}(x) = y'(x-x')\{\mathscr{R}^{-2} + \kappa_1\mathscr{R}_0^{-2}\} + 2(1-v_1)\sin 2\theta' + \mathscr{B}\{\theta' - \tan^{-1}[y'/(x+x')]\}$$

and $\mathscr{B} = 4(1-v_1)(1-2v_1)$. Integrating by parts as in Eq. (B.1), we obtain $F_{21}(z)$ in the form of Eq. (B.2), where

$$\begin{aligned}\mathscr{F}_{21}(x) &= y'\left\{\frac{(x-x')^2 - y'^2}{\mathscr{R}^4} - \frac{\mathscr{B}}{\mathscr{R}_0^2} - \kappa_1\left[\frac{1}{\mathscr{R}_0^2} - 2\frac{x^2 - x'^2}{\mathscr{R}_0^4}\right] + 4zx'\frac{x+x'}{\mathscr{R}_0^4}\right\}\\
&= \frac{y'}{2}\left\{\frac{1}{(x-w)^2} + \frac{1}{(x-\bar{w})^2}\right\} + \frac{\mathscr{B}}{2i}\left\{\frac{1}{x+w} - \frac{1}{x+\bar{w}}\right\} - \frac{\mathrm{i}\kappa_1}{2}\left\{\frac{x'+w}{(x+w)^2} - \frac{x'+\bar{w}}{(x+\bar{w})^2}\right\}\\
&\quad + \mathrm{i}zx'\left\{\frac{1}{(x+w)^2} - \frac{1}{(x+\bar{w})^2}\right\}.\end{aligned}$$

Then

$$\begin{aligned}F_{21}(z) = \Lambda(z)\{&-\mathscr{B}\sin z\theta' + z\cos[(z-1)(\pi-\theta')]\sin\theta' + 2z(\kappa_1 - z)\sin[(z-1)\theta']\cos\theta'\\
&+ \kappa_1 z\cos[(z-1)\theta']\sin\theta'\}, \quad -1 < \mathrm{Re}(z) < 0.\end{aligned}$$

Similarly,

$$\begin{aligned}F_{12}(z) = \Lambda(z)\{&\mathscr{B}\sin z\theta' + z\cos[(z-1)(\pi-\theta')]\sin\theta' + 2z(\kappa_1 + z)\sin[(z-1)\theta']\cos\theta'\\
&+ \kappa_1 z\cos[(z-1)\theta']\sin\theta'\},\end{aligned}$$

$$\begin{aligned}F_{22}(z) = \Lambda(z)\{&\kappa_1\cos[z(\pi-\theta')] - \mathscr{A}\cos z\theta' + z\sin[(z-1)(\pi-\theta')]\sin\theta' + \kappa_1 z\sin[(z-1)\theta']\sin\theta'\\
&- 2z^2\cos[(z-1)\theta']\cos\theta'\}.\end{aligned}$$

Let us now examine the tractions on the interface, $h_{pj}(x)$. Making use of the formulas in Appendix A, we obtain

$$\begin{aligned}h_{11}(x) &= -T^{\mathrm{K}}_{121}(x,0;\boldsymbol{x}') - T^{\mathrm{c}}_{121}(x,0;\boldsymbol{x}')\\
&= 2\mathrm{i}\mu_1\eta(1-v_1)\left\{\frac{1}{x-w} - \frac{1}{x-\bar{w}} - \frac{1}{x+w} + \frac{1}{x+\bar{w}}\right\} - \mu_1\eta y'\left\{\frac{1}{(x-w)^2} + \frac{1}{(x-\bar{w})^2}\right\}\\
&\quad - \mathrm{i}\mu_1\eta\left\{\frac{3x' - \kappa_1 w}{(x+w)^2} - \frac{3x' - \kappa_1\bar{w}}{(x+\bar{w})^2}\right\} + 4\mathrm{i}\mu_1\eta x'\left\{\frac{w}{(x+w)^3} - \frac{\bar{w}}{(x+\bar{w})^3}\right\}.\end{aligned}$$

To compute $H_{11}(z+1)$, we use Eqs. (B.3), (B.4) and

$$\int_0^\infty \frac{x^z}{(x+X)^3}\,\mathrm{d}x = -\frac{\pi z(z-1)|X|^{z-2}}{2\sin\pi z}\mathrm{e}^{i(z-2)\Phi}, \quad -1 < \mathrm{Re}(z) < 2.$$

We find that

$$H_{11}(z+1) = -2\mu_1 z\Lambda(z)\{2(1-\nu_1)(\sin[z(\pi-\theta')] + \sin z\theta') + z\cos[(z-1)(\pi-\theta')]\sin\theta' + \kappa_1 z\sin z\theta' - z(2z+1)\sin[(z-1)\theta']\cos\theta'\}.$$

Similarly,

$$h_{21}(x) = -\mu_1\eta(1-2\nu_1)\left\{\frac{1}{x-w}+\frac{1}{x-\bar{w}}-\frac{1}{x+w}-\frac{1}{x+\bar{w}}\right\} + \mathrm{i}\mu_1\eta y'\left\{\frac{1}{(x-w)^2}-\frac{1}{(x-\bar{w})^2}\right\} + \mu_1\eta\left\{\frac{\kappa_1 w - x'}{(x+w)^2}+\frac{\kappa_1\bar{w}-x'}{(x+\bar{w})^2}\right\} + 4\mu_1\eta x'\left\{\frac{w}{(x+w)^3}+\frac{\bar{w}}{(x+\bar{w})^3}\right\},$$

$$h_{12}(x) = \mu_1\eta(1-2\nu_1)\left\{\frac{1}{x-w}+\frac{1}{x-\bar{w}}-\frac{1}{x+w}-\frac{1}{x+\bar{w}}\right\} - \mathrm{i}\mu_1\eta y'\left\{\frac{1}{(x-w)^2}-\frac{1}{(x-\bar{w})^2}\right\} + \mu_1\eta\left\{\frac{\kappa_1 w + 3x'}{(x+w)^2}+\frac{\kappa_1\bar{w}+3x'}{(x+\bar{w})^2}\right\} - 4\mu_1\eta x'\left\{\frac{w}{(x+w)^3}+\frac{\bar{w}}{(x+\bar{w})^3}\right\},$$

$$h_{22}(x) = 2\mathrm{i}\mu_1\eta(1-\nu_1)\left\{\frac{1}{x-w}-\frac{1}{x-\bar{w}}-\frac{1}{x+w}+\frac{1}{x+\bar{w}}\right\} + \mu_1\eta y'\left\{\frac{1}{(x-w)^2}+\frac{1}{(x-\bar{w})^2}\right\} - \mathrm{i}\mu_1\eta\left\{\frac{\kappa_1 w + x'}{(x+w)^2}-\frac{\kappa_1\bar{w}+x'}{(x+\bar{w})^2}\right\} + 4\mathrm{i}\mu_1\eta x'\left\{\frac{w}{(x+w)^3}-\frac{\bar{w}}{(x+\bar{w})^3}\right\},$$

whence

$$H_{21}(z+1) = 2\mu_1 z\Lambda(z)\{(1-2\nu_1)(\cos[z(\pi-\theta')] - \cos z\theta') + z\sin[(z-1)(\pi-\theta')]\sin\theta' + \kappa_1 z\cos z\theta' - z(2z-1)\cos[(z-1)\theta']\cos\theta'\},$$

$$H_{12}(z+1) = 2\mu_1 z\Lambda(z)\{-(1-2\nu_1)(\cos[z(\pi-\theta')] - \cos z\theta') - z\sin[(z-1)(\pi-\theta')]\sin\theta' + \kappa_1 z\cos z\theta' + z(2z+1)\cos[(z-1)\theta']\cos\theta'\},$$

$$H_{22}(z+1) = 2\mu_1 z\Lambda(z)\{-2(1-\nu_1)(\sin[z(\pi-\theta')] + \sin z\theta') + z\cos[(z-1)(\pi-\theta')]\sin\theta' + \kappa_1 z\sin z\theta' + z(2z-1)\sin[(z-1)\theta']\cos\theta'\}.$$

(As a simple check, one can verify that the residue of $H_{pj}(z+1)$ at $z=-1$ equals $h_{pj}(0)$.)

Appendix C. An inverse matrix

We require the inverse of the 4×4 matrix $\mathscr{D}(-z)$, where $\mathscr{D}(\omega)$ is defined by Eq. (5.6). Let $S=\sin\frac{1}{2}z\pi$, $C=\cos\frac{1}{2}z\pi$ and $\Gamma=\mu_2/\mu_1$. We find that $[\mathscr{D}(-z)]^{-1}$ can be written as Eq. (6.9), where the matrix $\mathscr{E}$ is given by

$$\mathscr{E}_{11} = 4(\alpha-\beta)\Gamma^2 S\widetilde{\mathscr{E}}_{11}, \quad \mathscr{E}_{12} = 4(\alpha-\beta)\Gamma^2 C\widetilde{\mathscr{E}}_{12},$$
$$\mathscr{E}_{13} = (\mu_1 z)^{-1}\Gamma C\widetilde{\mathscr{E}}_{13}, \quad \mathscr{E}_{14} = -(\mu_1 z)^{-1}\Gamma S\widetilde{\mathscr{E}}_{14},$$
$$\mathscr{E}_{21} = 4(\alpha-\beta)\Gamma S\widetilde{\mathscr{E}}_{21}, \quad \mathscr{E}_{22} = -4(\alpha-\beta)\Gamma C\widetilde{\mathscr{E}}_{22},$$
$$\mathscr{E}_{23} = -(\mu_1 z)^{-1} C\widetilde{\mathscr{E}}_{23}, \quad \mathscr{E}_{24} = -(\mu_1 z)^{-1} S\widetilde{\mathscr{E}}_{24},$$
$$\mathscr{E}_{31} = -4(\alpha-\beta)\Gamma^2 C\widetilde{\mathscr{E}}_{31}, \quad \mathscr{E}_{32} = 4(\alpha-\beta)\Gamma^2 S\widetilde{\mathscr{E}}_{32},$$
$$\mathscr{E}_{33} = (\mu_1 z)^{-1}\Gamma S\widetilde{\mathscr{E}}_{33}, \quad \mathscr{E}_{34} = (\mu_1 z)^{-1}\Gamma C\widetilde{\mathscr{E}}_{34},$$
$$\mathscr{E}_{41} = 4(\alpha-\beta)\Gamma C\widetilde{\mathscr{E}}_{41}, \quad \mathscr{E}_{42} = 4(\alpha-\beta)\Gamma S\widetilde{\mathscr{E}}_{42},$$
$$\mathscr{E}_{43} = -(\mu_1 z)^{-1} S\widetilde{\mathscr{E}}_{43}, \quad \mathscr{E}_{44} = (\mu_1 z)^{-1} C\widetilde{\mathscr{E}}_{44},$$

and the matrix $\widetilde{\mathscr{E}}$ is given by

$$\widetilde{\mathscr{E}}_{11} = S^2\{z(\beta-1)+\beta\} + z\{z^2(\alpha-\beta) - z\beta - \alpha + 1\},$$

$$\widetilde{\mathscr{E}}_{12} = S^2\{z(\beta-1)-1\} + z^2\{z(\alpha-\beta)+\alpha\},$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{13} = {} & S^2\{-2(\alpha-\beta)\Gamma + (\Gamma-1)(1+\alpha)(1-\beta) - 2z(\alpha-\beta)(1-\beta)\} + 2z^3(\alpha-\beta)^2 \\ & + z^2(\alpha-\beta)\{2\alpha\Gamma - (\Gamma-1)(1+\alpha)\} - z\alpha(1-\alpha)(\Gamma-1),\end{aligned}$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{14} = {} & S^2\{2\beta(\alpha-\beta)\Gamma - (\Gamma-1)(1-\beta)(2\beta+1-\alpha) - 2z(\alpha-\beta)(1-\beta)\} + 2z^3(\alpha-\beta)^2 \\ & + z^2(\alpha-\beta)\{-2\beta\Gamma + (\Gamma-1)(2\beta+1-\alpha)\} + z(1-\alpha)\{2(\alpha-\beta)\Gamma - \alpha(\Gamma-1)\} + (1-\alpha)(\Gamma-1),\end{aligned}$$

$$\widetilde{\mathscr{E}}_{21} = S^2\{z(\beta+1)+\beta\} + z\{z^2(\alpha-\beta) - z\beta - \alpha - 1\},$$

$$\widetilde{\mathscr{E}}_{22} = S^2\{z(\beta+1)+1\} + z^2\{z(\alpha-\beta)+\alpha\},$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{23} = {} & S^2\{2(\alpha-\beta)\Gamma - (\Gamma-1)(1-\beta)(1+\alpha) + 2z(\alpha-\beta)(1+\beta)\Gamma\} + 2z^3(\alpha-\beta)^2\Gamma \\ & + z^2(\alpha-\beta)\{2\alpha\Gamma - (\Gamma-1)(1+\alpha)\} - z\alpha(1+\alpha)(\Gamma-1),\end{aligned}$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{24} = {} & S^2\{2\beta(\alpha-\beta)\Gamma + (\Gamma-1)(1-\beta)(1+\alpha) + 2z(\alpha-\beta)(1+\beta)\Gamma\} + 2z^3(\alpha-\beta)^2\Gamma \\ & + z^2(\alpha-\beta)\{-2\beta\Gamma + (\Gamma-1)(1+\alpha)\} + z(1+\alpha)\{-2(\alpha-\beta)\Gamma + (\Gamma-1)(\alpha-2\beta)\} - (1+\alpha)(\Gamma-1),\end{aligned}$$

$$\widetilde{\mathscr{E}}_{31} = S^2\{z(\beta-1)+1\} + z^2\{z(\alpha-\beta)-\alpha\},$$

$$\widetilde{\mathscr{E}}_{32} = S^2\{z(\beta-1)-\beta\} + z\{z^2(\alpha-\beta) + z\beta + 1 - \alpha\},$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{33} = {} & S^2\{-2\beta(\alpha-\beta)\Gamma + (\Gamma-1)(1-\beta)(2\beta+1-\alpha) - 2z(\alpha-\beta)(1-\beta)\} + 2z^3(\alpha-\beta)^2 \\ & + z^2(\alpha-\beta)\{2\beta\Gamma - (\Gamma-1)(2\beta+1-\alpha)\} + z(1-\alpha)\{2(\alpha-\beta)\Gamma - \alpha(\Gamma-1)\} - (1-\alpha)(\Gamma-1),\end{aligned}$$

$$\begin{aligned}\widetilde{\mathscr{E}}_{34} = {} & S^2\{2(\alpha-\beta)\Gamma - (\Gamma-1)(1-\beta)(1+\alpha) - 2z(\alpha-\beta)(1-\beta)\} + 2z^3(\alpha-\beta)^2 \\ & + z^2(\alpha-\beta)\{-2\alpha\Gamma + (\Gamma-1)(1+\alpha)\} - z\alpha(1-\alpha)(\Gamma-1),\end{aligned}$$

$$\widetilde{\mathscr{E}}_{41} = S^2\{z(\beta+1)-1\} + z^2\{z(\alpha-\beta)-\alpha\},$$

$$\widetilde{\mathscr{E}}_{42} = S^2\{z(\beta+1)-\beta\} + z\{z^2(\alpha-\beta)+z\beta-1-\alpha\},$$

$$\widetilde{\mathscr{E}}_{43} = S^2\{2\beta(\alpha-\beta)\Gamma + (\Gamma-1)(1-\beta)(1+\alpha) - 2z(\alpha-\beta)(1+\beta)\Gamma\} - 2z^3(\alpha-\beta)^2\Gamma$$
$$+ z^2(\alpha-\beta)\{-2\beta\Gamma + (\Gamma-1)(1+\alpha)\} + z(1+\alpha)\{2(\alpha-\beta)\Gamma + (\Gamma-1)(2\beta-\alpha)\} - (1+\alpha)(\Gamma-1),$$

$$\widetilde{\mathscr{E}}_{44} = S^2\{2(\alpha-\beta)\Gamma - (\Gamma-1)(1-\beta)(1+\alpha) - 2z(\alpha-\beta)(1+\beta)\Gamma\} - 2z^3(\alpha-\beta)^2\Gamma$$
$$+ z^2(\alpha-\beta)\{2\alpha\Gamma - (\Gamma-1)(1+\alpha)\} + z\alpha(1+\alpha)(\Gamma-1).$$

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