

## ACOUSTIC SCATTERING BY INHOMOGENEOUS OBSTACLES\*

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**Abstract.** Acoustic scattering problems are considered when the material parameters (density and speed of sound) are functions of position within a bounded region. An integro-differential equation for the pressure in this region is obtained. It is proved that solving this equation is equivalent to solving the scattering problem. Problems of this kind are often solved by regarding the effects of the inhomogeneity as an unknown source term driving a Helmholtz equation, leading to an equation of Lippmann–Schwinger type. It is shown that this approach is incomplete when the density is discontinuous. Analogous scattering problems for elastic waves and for electromagnetic waves are also discussed briefly.

**Key words.** Bergmann’s equation, Lippmann–Schwinger equation

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**1. Introduction.** Time-harmonic acoustic waves in an inhomogeneous compressible fluid can be modelled using Bergmann’s equation (see (2.3) below). If the waves are generated by a point source located at  $\mathbf{r}'$ , the pressure at  $\mathbf{r}$ ,  $\mathcal{G}(\mathbf{r}; \mathbf{r}')$ , satisfies

$$(1.1) \quad \nabla^2 \mathcal{G}(\mathbf{r}; \mathbf{r}') - \rho^{-1}(\text{grad } \rho) \cdot \text{grad } \mathcal{G}(\mathbf{r}; \mathbf{r}') + k^2(\mathbf{r}) \mathcal{G}(\mathbf{r}; \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

where  $k^2(\mathbf{r}) = [\omega/c(\mathbf{r})]^2$ ,  $\omega$  is the frequency,  $c(\mathbf{r})$  is the speed of sound, and  $\rho(\mathbf{r})$  is the density. ( $\mathcal{G}$  is an exact Green’s function for the problem.) Equation (1.1) is supposed to hold everywhere in space, and it is to be solved subject to a radiation condition at infinity.

How does one solve (1.1)? According to a recent review article by Tourin, Fink, and Derode [39], “the solution of (1.1) can be written as”

$$(1.2) \quad \mathcal{G}(\mathbf{r}; \mathbf{r}') = G_e(\mathbf{r}; \mathbf{r}') + \int G_e(\mathbf{r}; \mathbf{r}_1) V(\mathbf{r}_1) \mathcal{G}(\mathbf{r}_1; \mathbf{r}') d\mathbf{r}_1,$$

where the integration is over all of space,  $V$  is a “potential operator”, defined by

$$(1.3) \quad V(\mathbf{r}) = k_e^2 - k^2(\mathbf{r}) + \rho^{-1}(\text{grad } \rho(\mathbf{r})) \cdot \text{grad},$$

$k_e$  is the wavenumber for a related homogeneous medium, and  $G_e$  is the (known) solution of the problem for that medium:  $G_e$  satisfies

$$(1.4) \quad \nabla^2 G_e(\mathbf{r}; \mathbf{r}_1) + k_e^2 G_e(\mathbf{r}; \mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_1)$$

and the radiation condition and is given explicitly by (3.1) below. Equation (1.2) is not derived in [39] and no indication of its range of validity is given. In fact, as we shall see, (1.2) is not valid when  $\rho(\mathbf{r})$  is discontinuous. (This is unfortunate, because most of the applications in [39] are to arrays of discrete scatterers, such as steel rods in water.)

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**1.1. A formal derivation.** Equations such as (1.1) are often treated by moving all the complicated terms to the right-hand side where they are regarded as a forcing term. Thus, write (1.1) as

$$\nabla^2 \mathcal{G}(\mathbf{r}; \mathbf{r}') + k_e^2 \mathcal{G}(\mathbf{r}; \mathbf{r}') = V(\mathbf{r})\mathcal{G}(\mathbf{r}; \mathbf{r}') + \delta(\mathbf{r} - \mathbf{r}').$$

Equation (1.2) then follows by noting that

$$(1.5) \quad u(\mathbf{r}) = \int G_e(\mathbf{r}; \mathbf{r}') f(\mathbf{r}') d\mathbf{r}' \quad \text{solves} \quad (\nabla^2 + k_e^2)u = f.$$

Formal derivations of this kind are often found in textbooks; see, for example, [9, sect. 8.9.1] or [18, eqn. (21.37)]. The result (1.5) can be justified readily if one assumes that  $f$  is (Hölder) continuous. However, for discrete scatterers, there will be interfaces across which  $k(\mathbf{r})$  and the normal derivative of  $\mathcal{G}$ ,  $\partial\mathcal{G}/\partial n$ , will be discontinuous (although  $\mathcal{G}$  and  $\rho^{-1}\partial\mathcal{G}/\partial n$  are both continuous across such interfaces).

**1.2. The present paper.** The formal derivation in section 1.1 is incomplete. It can be repaired so as to give the correct result. Thus, when considered as a distribution, we have

$$V\mathcal{G} = \{V\mathcal{G}\} + [\rho]\delta(S)(\rho^{-1}\partial\mathcal{G}/\partial n),$$

where  $\{V\mathcal{G}\}$  denotes the value of  $V\mathcal{G}$  anywhere but on the interfaces  $S$  and  $[\rho]$  denotes the discontinuity in  $\rho$  across  $S$  [41, sect. 1.13]. Then, (1.5) suggests an additional term on the right-hand side of (1.2), namely

$$(1.6) \quad \int_S G_e(\mathbf{r}; \mathbf{r}_s) [\rho](\mathbf{r}_s) \left( \frac{1}{\rho} \frac{\partial\mathcal{G}}{\partial n} \right) (\mathbf{r}_s; \mathbf{r}') ds(\mathbf{r}_s).$$

In this paper, we shall derive an equation, similar to (1.2), that respects the proper transmission conditions across interfaces: we do not use distribution theory. We prove that solving this equation is equivalent to solving the transmission problem for the acoustic pressure (Theorem 3.1).

We are mainly concerned with the following problem: acoustic scattering by a bounded inhomogeneity embedded in an unbounded homogeneous medium. The density and sound-speed are assumed to be functions of position within the inhomogeneity, and they can be discontinuous across the interface between the inhomogeneity and the surrounding fluid. Problems in which the inhomogeneity is *spherically symmetric*, so that  $\rho$  and  $c$  are assumed to be given functions of the spherical polar coordinate  $r$  (only), have been studied by several authors; see [28] for references.

The new equation is derived in section 3. It reduces to the well-known Lippmann–Schwinger equation when the density in the inhomogeneity is constant and equal to the density of the surrounding homogeneous fluid. It also reduces to the equation derived formally above, namely (1.2), *but only when there is no discontinuity in the density across the boundary of the inhomogeneity*. If there is such a discontinuity (as is typical in applications), an extra term is needed; see (1.6) and (4.4) below. Analogous scattering problems for electromagnetic waves and for elastic waves are discussed briefly in sections 4.4 and 4.5, respectively.

**2. Formulation.** Consider the scattering of time-harmonic sound waves in a homogeneous compressible fluid by an inhomogeneous obstacle. In the exterior fluid,  $B_e$ , we can write

$$(2.1) \quad p_e = p_{\text{inc}} + p_{\text{sc}},$$

where  $p_e$  is the (total) acoustic pressure,  $p_{\text{inc}}$  is the given incident field, and  $p_{\text{sc}}$  is the scattered field. The governing equation for  $p_{\text{sc}}$  is

$$(2.2) \quad (\nabla^2 + k_e^2)p_{\text{sc}} = 0 \quad \text{in } B_e,$$

where  $k_e = \omega/c_e$  is the wave number (assumed to be real and positive),  $\omega$  is the frequency, and  $c_e$  is the constant speed of sound. We assume that the incident field  $p_{\text{inc}}$  satisfies (2.2) everywhere, except possibly at some places in  $B_e$  (so that  $p_{\text{inc}}$  could correspond to a point source in  $B_e$ , for example). We require that  $p_{\text{sc}}$  satisfies the Sommerfeld radiation condition at infinity.

Within the obstacle,  $B$ , the governing equation is Bergmann's equation ([3, 28] and [29, p. 408])

$$(2.3) \quad \rho_i \operatorname{div} (\rho_i^{-1} \operatorname{grad} p_i) + k_i^2 p_i = 0 \quad \text{in } B,$$

where  $p_i$  is the pressure and  $k_i = \omega/c_i$ . The interior density  $\rho_i$  and speed of sound  $c_i$  can vary with position in  $B$ . At the interface  $S$  between  $B$  and  $B_e$ , we have a pair of transmission conditions, expressing continuity of pressure and normal velocity. These are

$$(2.4) \quad p_e = p_i \quad \text{and} \quad \frac{1}{\rho_e} \frac{\partial p_e}{\partial n} = \frac{1}{\rho_i} \frac{\partial p_i}{\partial n} \quad \text{on } S,$$

where  $\rho_e$  is the (constant) density of the fluid in  $B_e$ .

Summarizing, we have the following problem to solve.

*Scattering Problem.* Let  $p_{\text{inc}}$  be a given incident field. Find a pair of functions,  $\{p_e, p_i\}$ , where  $p_{\text{sc}} = p_e - p_{\text{inc}}$  satisfies (2.2) and the Sommerfeld radiation condition,  $p_i$  satisfies (2.3), and  $p_e$  and  $p_i$  satisfy the transmission conditions (2.4) across the interface  $S$ .

Werner wrote an important paper on the Scattering Problem in 1963 [43]. He reduced the problem to a system of coupled integral equations, using single-layer, double-layer, and volume potentials; see Appendix A. Werner's approach is an example of an *indirect method*, meaning that the unknown quantities do not have any physical relevance. He proved that the Scattering Problem has exactly one solution; his uniqueness result is in [42]. However, as far as we know, his system of integral equations has not been used in computations.

We shall use a *direct method*, meaning that the unknown quantity is recognized as a physical variable, namely  $p_i$ . Moreover, we shall use only volume potentials; this is convenient from a computational point of view, because one does not have to approximate a mixture of surface and volume contributions.

In solving the Scattering Problem, we seek classical solutions. We shall appeal to Werner's existence result, so we suppose that  $p_{\text{sc}} \in C^2(B_e) \cap C^1(\overline{B_e})$  and  $p_i \in C^2(B) \cap C^1(\overline{B})$ . We assume that  $\rho_i \in C^2(\overline{B})$  and  $c_i \in C^1(\overline{B})$  are both positive. Finally, we assume that  $S$  is smooth ( $C^2$ ). It is likely that these conditions can be weakened; for example, Werner's uniqueness theorem [42] requires that  $\rho_i$  and  $c_i$  be in Hölder spaces, with  $\rho_i \in C^{1,\alpha}(\overline{B})$  and  $c_i \in C^{0,\alpha}(\overline{B})$ .

Bergmann's equation (2.3) can be written in other ways. One alternative is

$$(2.5) \quad \nabla^2 p_i + \rho_i (\operatorname{grad} \rho_i^{-1}) \cdot \operatorname{grad} p_i + k_e^2 N p_i = 0 \quad \text{in } B,$$

where  $N = (k_i/k_e)^2 = (c_e/c_i)^2$  is the (square of the) *refractive index*. Another is

$$\nabla^2 p_i + k_e^2 p_i = V p_i \quad \text{in } B,$$

where (see (1.3))

$$(2.6) \quad Vu = k_e^2(1 - N)u + \rho_i^{-1}(\text{grad } \rho_i) \cdot \text{grad } u.$$

Bergmann's equation can also be reduced to an equation without first derivatives by introducing a new dependent variable [3],  $u = p_i \rho_i^{-1/2}$ :  $u$  is found to satisfy

$$(2.7) \quad \nabla^2 u + (k_i^2 + K)u = 0,$$

where

$$(2.8) \quad K = \frac{1}{2} \rho_i^{-1} \nabla^2 \rho_i - \frac{3}{4} \rho_i^{-2} |\text{grad } \rho_i|^2$$

$$(2.9) \quad = -\rho_i^{1/2} \nabla^2 (\rho_i^{-1/2}).$$

Equations (2.7) and (2.8) (but not (2.9)) can be found in [4, p. 171]. Equation (2.7) can also be written as *Schrödinger's equation* [32, eqn. (10.59)].

Much has been written on the case where  $\rho_i$  is constant, so that the second term on the left-hand side of (2.5) can be deleted [12, Chap. 8]; see [23] for a review of available point-source solutions (Green's functions) for various functional forms of  $N$ . Here, we do not make this assumption: we allow both  $\rho_i$  and  $N$  to vary with position. Note that if the material in  $B$  is actually homogeneous, so that  $\rho_i$  and  $k_i$  are both constants, boundary integral equations over  $S$  can be used; see [20] for a review.

Kriegsmann and Reiss [22] have given long-wave approximations to the solution of the Scattering Problem, assuming that  $\rho_i/\rho_e \simeq 1$ . Specifically, they take  $\rho_i/\rho_e = 1 + O(\varepsilon^2)$  and  $k_e a = O(\varepsilon)$ , where  $a$  is the diameter of  $B$  and  $0 < \varepsilon \ll 1$ .

Colton and Monk [13] have reviewed progress with inverse problems for (2.3), which they write as  $\text{div}(\rho_i^{-1} \text{grad } p_i) + k_e^2 \tilde{N} p_i = 0$ , where  $\tilde{N} = N/\rho_i$ . They also assume that  $\rho_i$  is constant near  $S$  so that the ratio  $\rho_e/\rho_i$  occurring in (2.4)<sub>2</sub> is constant; in general, this need not be true.

**2.1. Uniqueness.** The Scattering Problem has at most one solution [42]. This uniqueness theorem can be proved as follows. Set  $p_{\text{inc}} \equiv 0$  and then apply Green's theorem to  $p_e$  and its complex conjugate,  $\bar{p}_e$ , in the exterior, giving

$$(2.10) \quad 2ik_e \lim_{R \rightarrow \infty} \int_{S_R} |p_e|^2 ds + \int_S \left( p_e \frac{\partial \bar{p}_e}{\partial n} - \bar{p}_e \frac{\partial p_e}{\partial n} \right) ds = 0.$$

Here, the unit normal to  $S$ ,  $\mathbf{n}$ , points out of  $B$ ,  $S_R$  is a large sphere of radius  $R$  that encloses  $S$ , we have used the radiation condition, and we have assumed that  $k_e$  is real. Next, apply the divergence theorem in  $B$  to the vector field  $(\bar{p}_i/\rho_i) \text{grad } p_i$ , giving

$$\int_B (|\text{grad } p_i|^2 - k_i^2 |p_i|^2) \frac{dV}{\rho_i} = \int_S \frac{\bar{p}_i}{\rho_i} \frac{\partial p_i}{\partial n} ds,$$

where we have used (2.3). The imaginary part of this equation gives

$$\int_S \left( p_i \frac{\partial \bar{p}_i}{\partial n} - \bar{p}_i \frac{\partial p_i}{\partial n} \right) \frac{ds}{\rho_i} = \int_B (k_i^2 - \bar{k}_i^2) |p_i|^2 \frac{dV}{\rho_i}.$$

Then, making use of the transmission conditions (2.4), (2.10) gives

$$k_e \lim_{R \rightarrow \infty} \int_{S_R} |p_e|^2 ds + 2 \int_B \text{Re}(k_i) \text{Im}(k_i) |p_i|^2 \frac{\rho_e}{\rho_i} dV = 0.$$

Rellich's lemma [11, Lem. 3.11] then implies that  $p_e \equiv 0$  in  $B_e$ , provided that

$$\operatorname{Re}(k_i) \operatorname{Im}(k_i) \geq 0.$$

The transmission conditions then imply that  $p_i = 0$  and  $\partial p_i / \partial n = 0$  on  $S$ . Thus,  $p_i$  solves the Cauchy problem for the elliptic partial differential equation (2.3), in which the (positive) coefficients  $\rho_i^{-1}$  and  $k_i^2 / \rho_i$  are  $C^2$  and  $C^1$ , respectively. It follows that  $p_i \equiv 0$  in  $B$ , as required. Here, we have used a unique continuation result due to Müller [30] and Aronszajn [2]; see [21] for a brief review.

**3. An integral representation and an integro-differential equation.** We shall consider integral representations obtained using the free-space Green's function for the exterior fluid,

$$(3.1) \quad G_e(P, Q) = -\exp(ik_e R) / (4\pi R),$$

where  $P$  and  $Q$  are typical points in three-dimensional space and  $R = |\mathbf{r}_P - \mathbf{r}_Q|$  is the distance between  $P$  and  $Q$ .

An application of Green's second theorem in  $B_e$  to  $p_{sc}$  and  $G_e$  gives

$$\int_S \left\{ G_e(P, q) \frac{\partial p_{sc}}{\partial n_q} - p_{sc}(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q = \begin{cases} p_{sc}(P), & P \in B_e, \\ 0, & P \in B. \end{cases}$$

A similar application in  $B$  to  $p_{inc}$  and  $G_e$  gives

$$\int_S \left\{ G_e(P, q) \frac{\partial p_{inc}}{\partial n_q} - p_{inc}(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q = \begin{cases} 0, & P \in B_e, \\ -p_{inc}(P), & P \in B. \end{cases}$$

Adding these gives

$$(3.2) \quad \int_S \left\{ G_e(P, q) \frac{\rho_e}{\rho_i} \frac{\partial p_i}{\partial n_q} - p_i(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q = \begin{cases} p_{sc}(P), & P \in B_e, \\ -p_{inc}(P), & P \in B, \end{cases}$$

where we have used (2.1) and the transmission conditions (2.4). The first of these gives an integral representation for  $p_{sc}(P)$  in terms of a distribution of sources and dipoles over  $S$ . Such representations are common in scattering theory. However, it is not very convenient here because we do not know  $p_i$  or  $\partial p_i / \partial n$  on  $S$ .

To make progress, recall Green's first theorem,

$$\int_B \{ \phi \nabla^2 \psi + (\operatorname{grad} \phi) \cdot (\operatorname{grad} \psi) \} dV = \int_S \phi \frac{\partial \psi}{\partial n} ds,$$

where  $\phi$  and  $\psi$  are sufficiently smooth in  $B$ . Choose  $\phi(Q) = p_i(Q)$  and  $\psi(Q) = G_e(P, Q)$  with  $P \in B_e$ , whence

$$(3.3) \quad \int_S p_i(q) \frac{\partial}{\partial n_q} G_e(P, q) ds_q \\ = \int_B \{ (\operatorname{grad} p_i) \cdot (\operatorname{grad}_Q G_e) - k_e^2 p_i(Q) G_e(P, Q) \} dV_Q,$$

where we have used  $(\nabla^2 + k_e^2)G_e(P, Q) = 0$  for  $P \neq Q$ . Similarly, if we choose  $\psi(Q) = p_i(Q)$  and  $\phi(Q) = (\rho_e / \rho_i)G_e(P, Q)$  with  $P \in B_e$ , we obtain

$$(3.4) \quad \int_S \frac{\rho_e}{\rho_i} \frac{\partial p_i}{\partial n_q} G_e(P, q) ds_q \\ = \int_B \frac{\rho_e}{\rho_i} \{ (\operatorname{grad} p_i) \cdot (\operatorname{grad}_Q G_e) - k_e^2 N(Q) p_i(Q) G_e(P, Q) \} dV_Q,$$

where we have used (2.5). Subtracting (3.3) from (3.4) gives the left-hand side of (3.2) for  $P \in B_e$ , whence  $p_{sc}(P) = (\mathcal{L}p_i)(P)$  for  $P \in B_e$ , where

$$(3.5) \quad (\mathcal{L}v)(P) = \int_B \{(\alpha(Q) - 1) (\text{grad } v) \cdot (\text{grad}_Q G_e(P, Q)) + (1 - N\alpha) k_e^2 v(Q) G_e(P, Q)\} dV_Q$$

and

$$\alpha(P) = \rho_e / \rho_i(P).$$

We repeat the calculations for  $P \in B$ , having excised a small sphere centered at  $P$ . The singularity at  $P = Q$  has no effect on (3.4) but it causes  $-p_i(P)$  to be added to the left-hand side of (3.3). Then, (3.2) for  $P \in B$  becomes

$$-p_{inc}(P) = -p_i(P) + (\mathcal{L}p_i)(P), \quad P \in B.$$

At this stage, we have proved one half of the following theorem.

**THEOREM 3.1.** *Let the pair  $\{p_e, p_i\}$  solve the Scattering Problem. Then  $v(P) \equiv p_i(P) \in C^2(B)$  solves*

$$(3.6) \quad v(P) - (\mathcal{L}v)(P) = p_{inc}(P), \quad P \in B,$$

where  $\mathcal{L}v$  is defined by (3.5). Conversely, let  $v$  solve (3.6). Then the pair  $\{p_e, p_i\}$ , defined by

$$(3.7) \quad p_e(P) = p_{inc}(P) + (\mathcal{L}v)(P) \quad \text{for } P \in B_e$$

and  $p_i(P) = v(P)$  for  $P \in B$  solves the Scattering Problem.

*Proof.* We have to prove the second half of the theorem. From (3.7), we define  $p_{sc}$  using

$$(3.8) \quad p_{sc}(P) = (\mathcal{L}v)(P), \quad P \in B_e;$$

evidently,  $p_{sc}$  satisfies (2.2) and the Sommerfeld radiation condition, as it inherits these properties from  $G_e$ .

Next, let us show that  $p_i \equiv v$  satisfies (2.3). As  $p_{inc}$  satisfies (2.2) in  $B$ , (3.6) gives

$$(3.9) \quad (\nabla^2 + k_e^2)(v - \mathcal{L}v) = 0 \quad \text{in } B.$$

Now, from the definition (3.5), we have

$$(3.10) \quad (\mathcal{L}v)(P) = -\frac{\partial}{\partial x_j^P} \int_B (\alpha - 1) \frac{\partial v}{\partial x_j^Q} G_e(P, Q) dV_Q + k_e^2 \int_B (1 - N\alpha) v(Q) G_e(P, Q) dV_Q,$$

where  $P \equiv (x_1^P, x_2^P, x_3^P)$ ,  $Q \equiv (x_1^Q, x_2^Q, x_3^Q)$ , and summation over  $j$  is implied. The second integral in (3.10) is an acoustic volume potential, and the first term is the sum of three first derivatives of volume potentials. The properties of volume potentials are

summarized in Appendix B. In particular, the result of applying  $(\nabla^2 + k_e^2)$  is given by (B.1), so that we obtain

$$\begin{aligned} (\nabla^2 + k_e^2)(\mathcal{L}v) &= -\frac{\partial}{\partial x_j^P} \left\{ (\alpha - 1) \frac{\partial v}{\partial x_j^P} \right\} + k_e^2 (1 - N\alpha) v(P) \\ &= (\nabla^2 + k_e^2)v - \rho_e \operatorname{div} (\rho_i^{-1} \operatorname{grad} v) - k_i^2 (\rho_e / \rho_i) v, \quad P \in B, \end{aligned}$$

whence (3.9) gives the desired result.

To verify the transmission conditions, observe that (3.6) gives

$$(3.11) \quad p_i(P) - p_{\text{inc}}(P) = (\mathcal{L}v)(P), \quad P \in B.$$

However, as  $\mathcal{L}v$  comprises a volume potential and first derivatives of volume potentials, it follows that  $(\mathcal{L}v)(P)$  is continuous as  $P$  crosses  $S$  (see Appendix B). Thus, (3.8) and (3.11) show that the first transmission condition, (2.4)<sub>1</sub>, is satisfied.

For the second transmission condition, we take the normal derivative of (3.8) and (3.11) to give

$$(3.12) \quad \frac{\partial}{\partial n} \{p_{\text{sc}} - (p_i - p_{\text{inc}})\} = \left[ \frac{\partial}{\partial n} \mathcal{L}v \right] \quad \text{on } S,$$

where  $[f]$  is the discontinuity in  $f$  across  $S$ , defined by

$$(3.13) \quad [f(p)] = \lim_{P_e \rightarrow p} f(P_e) - \lim_{P \rightarrow p} f(P), \quad P_e \in B_e, \quad P \in B, \quad p \in S.$$

It is shown in Appendix B that

$$(3.14) \quad \left[ \frac{\partial}{\partial n} \mathcal{L}v \right] = \left( \frac{\rho_e}{\rho_i} - 1 \right) \frac{\partial v}{\partial n},$$

and then (3.12) and  $v \equiv p_i$  imply that (2.4)<sub>2</sub> is satisfied. This completes the proof of Theorem 3.1.  $\square$

#### 4. Discussion of the integro-differential equation (3.6).

**4.1. Solvability.** We have seen that solving the Scattering Problem is equivalent to solving (3.6), which is an integro-differential equation for  $v(P)$ ,  $P \in B$ . This equation is uniquely solvable. To see this, we appeal to Werner’s existence result [43]: the solution  $\{p_e, p_i\}$  of the Scattering Problem exists and, by the first half of Theorem 3.1,  $p_i$  solves (3.6). For uniqueness, suppose that  $v_0(P)$  solves (3.6) with  $p_{\text{inc}} \equiv 0$ . Construct  $p_e = (\mathcal{L}v_0)(P)$  for  $P \in B_e$  and  $p_i = v_0(P)$  for  $P \in B$ . By the second half of Theorem 3.1, these fields solve the homogeneous Scattering Problem; they must vanish identically by the uniqueness theorem for the Scattering Problem (section 2.1). In particular,  $v_0(P) \equiv 0$  for  $P \in B$ , as required.

We note that an integro-differential equation equivalent to (3.6) was derived by Gerjuoy and Saxon [15] in 1954. In fact, they derived a coupled system, involving the pressure and the velocity, which they regarded as preferable to a single equation for the pressure as they were motivated by a desire to obtain variational principles.

**4.2. The Lippmann–Schwinger equation.** As a special case of the Scattering Problem, suppose that  $\rho_i(Q) = \rho_e$  for all  $Q \in B$ , so that the density of the scatterer is

the same as that of the surrounding homogeneous fluid. Then, the integro-differential equation (3.6) reduces to the integral equation

$$(4.1) \quad v(P) - k_e^2 \int_B \{1 - N(Q)\} v(Q) G_e(P, Q) dV_Q = p_{\text{inc}}(P), \quad P \in B,$$

where  $N(Q) = (k_i/k_e)^2 = \{c_e/c_i(Q)\}^2$ . This integral equation and its numerical treatment have been discussed in [10, 5, 46, 8] and [9, sect. 8.9.1]

Let us define  $N(P) = 1$  for  $P \in B_e$  and

$$w(P) = \begin{cases} p_e(P), & P \in B_e, \\ p_i(P), & P \in B. \end{cases}$$

Then, we can combine (4.1) with the representation (3.7) to obtain

$$(4.2) \quad w(P) - k_e^2 \int \{1 - N(Q)\} w(Q) G_e(P, Q) dV_Q = p_{\text{inc}}(P)$$

for all  $P \in B \cup B_e$ , where the integration is over all  $Q$ . We recognize this equation as the *Lippmann-Schwinger equation* [24]; see, for example, [1], [12, sect. 8.2] [32, sect. 10.3], and [35, Thm. 9.4]. Notice that our derivation shows that the Lippmann-Schwinger equation is valid even when  $N(Q)$  is discontinuous as  $Q$  crosses  $S$ . This fact is implicit in [34] and explicit in [44].

**4.3. An alternative equation.** As we know that  $v \equiv p_i$  solves (2.5) in  $B$ , we can use this fact to rewrite the expression for  $\mathcal{L}v$ . Thus

$$(\alpha - 1) (\text{grad } v) \cdot (\text{grad}_Q G_e) = \text{div} \{(\alpha - 1) G_e \text{grad } v\} - G_e \text{div} \{(\alpha - 1) \text{grad } v\}$$

and

$$\begin{aligned} \text{div} \{(\alpha - 1) \text{grad } v\} &= (\alpha - 1) \nabla^2 v + \rho_e (\text{grad } \rho_i^{-1}) \cdot \text{grad } v \\ &= (\alpha - 1) \{ \nabla^2 v + \rho_i (\text{grad } \rho_i^{-1}) \cdot \text{grad } v \} + \rho_i (\text{grad } \rho_i^{-1}) \cdot \text{grad } v \\ &= (1 - \alpha) k_e^2 N v - \rho_i^{-1} (\text{grad } \rho_i) \cdot \text{grad } v. \end{aligned}$$

Hence, substituting in (3.5), we obtain

$$(\mathcal{L}v)(P) = \int_B G_e(P, Q) (Vv)(Q) dV_Q + (\mathcal{L}_E v)(P),$$

where  $Vv$  is defined by (2.6) and

$$(4.3) \quad \begin{aligned} (\mathcal{L}_E v)(P) &= \int_B \text{div} \{(\alpha(Q) - 1) G_e \text{grad } v\} dV_Q \\ &= \int_S \{ \alpha(q) - 1 \} G_e(P, q) \frac{\partial v}{\partial n} ds_q, \end{aligned}$$

by the divergence theorem. Thus, the Scattering Problem can be reduced to solving

$$(4.4) \quad p_i(P) = p_{\text{inc}}(P) + \int_B G_e(P, Q) (Vp_i)(Q) dV_Q + p_E(P), \quad P \in B,$$

where

$$p_E(P) = (\mathcal{L}_E p_i)(P) = \int_S \left( \frac{\partial p_e}{\partial n} - \frac{\partial p_i}{\partial n} \right) G_e(P, q) ds_q$$

and we have used (2.4)<sub>2</sub> in (4.3).

When both  $\rho_i$  and  $N$  are constants, (4.4) reduces to an equation obtained previously by Ramm [37]. However, in this situation, the scatterer is homogeneous and the problem can be reduced to boundary integral equations over  $S$ ; see [20].

If we had attempted to solve the Scattering Problem using the formal method described in section 1.1, we would have obtained precisely (4.4) but with  $p_E(P) \equiv 0$ . In general, this extra term is not zero, and its magnitude is difficult to estimate. Observe that, from (4.3),  $p_E$  does vanish if  $\rho_i(q) = \rho_e$  for all  $q \in S$ , which means that the density is continuous across  $S$ . Otherwise, the single-layer potential  $p_E(P)$  should be retained.

**4.4. Electromagnetic waves.** For Maxwell’s equations, we can encounter exactly the same difficulty as in acoustics. Thus, in an inhomogeneous medium, the electric field  $\mathbf{E}$  satisfies

$$\mu \operatorname{curl} \{ \mu^{-1} \operatorname{curl} \mathbf{E} \} - k^2 \mathbf{E} = \mathbf{0},$$

where  $\mu(\mathbf{r})$  is the magnetic permeability,  $k(\mathbf{r}) = \omega \sqrt{\mu \varepsilon}$ , and  $\varepsilon(\mathbf{r})$  is the electric permittivity. Moreover, the transmission conditions across an interface  $S$  are that  $\mathbf{n} \times \mathbf{E}$  and  $\mathbf{n} \times (\mu^{-1} \operatorname{curl} \mathbf{E})$  should both be continuous. (See, for example, [9, sect. 8.9.2]; for existence and uniqueness theorems, see [31].) We can then mimic the derivations in section 3 to show that discontinuities in  $\mu$  across  $S$  will lead to an extra term similar to  $p_E$  in (4.4).

Specifically, we find the following electromagnetic analogue of (3.6):

$$(4.5) \quad \left\{ 1 - \frac{1}{3} \left( 1 - \frac{\varepsilon_i(P)}{\varepsilon_e} \right) \right\} \mathbf{E}_i(P) - \int_B \mathbf{W}(P, Q) dV_Q = \mathbf{E}_{\text{inc}}(P), \quad P \in B.$$

Here,  $\mathbf{E}_i$  is the field in  $B$ ,  $\mathbf{E}_{\text{inc}}$  is the incident field,  $\varepsilon_e$  is the (constant) electric permittivity in  $B_e$ , and  $\varepsilon_i$  is the electric permittivity in  $B$ . The field  $\mathbf{W}$  is defined by

$$\begin{aligned} \mathbf{W}(P, Q) = & \left( 1 - \frac{\mu_e}{\mu_i} \right) (\operatorname{grad}_Q G_e) \times \operatorname{curl} \mathbf{E}_i(Q) \\ & + \left( 1 - \frac{\varepsilon_i}{\varepsilon_e} \right) \{ k_e^2 G_e \mathbf{E}_i - \operatorname{grad}_P (\mathbf{E}_i \cdot \operatorname{grad}_Q G_e) \}, \end{aligned}$$

where  $\mu_e$  is the (constant) magnetic permeability in  $B_e$ ,  $\mu_i$  is the magnetic permeability in  $B$ , and  $k_e = \omega \sqrt{\mu_e \varepsilon_e}$ . The integral in (4.5) is to be interpreted in the Cauchy principal-value sense with a spherical exclusion volume. In the special case that  $\mu_i(Q) \equiv \mu_e$ , (4.5) reduces to eqn. (2.1.41) in [40].

The electromagnetic analogue of (4.4) is

$$(4.6) \quad \mathbf{E}_i(P) = \mathbf{E}_{\text{inc}}(P) + \int_B G_e(P, Q) \mathbf{V}(Q) dV_Q + \int_S \mathbf{F}(P, q) ds_q,$$

where

$$\begin{aligned} \mathbf{V} = & (k_e^2 - k_i^2) \mathbf{E}_i - (\mu_i^{-1} \operatorname{grad} \mu_i) \times \operatorname{curl} \mathbf{E}_i + \operatorname{grad} \operatorname{div} \mathbf{E}_i, \\ \mathbf{F} = & \left( 1 - \frac{\varepsilon_i}{\varepsilon_e} \right) (\mathbf{n} \cdot \mathbf{E}_i) \operatorname{grad}_q G_e \\ & + \left\{ \left( 1 - \frac{\mu_e}{\mu_i} \right) (\mathbf{n} \times \operatorname{curl} \mathbf{E}_i) + \mathbf{n} (\varepsilon_i^{-1} \operatorname{grad} \varepsilon_i) \cdot \mathbf{E}_i \right\} G_e, \end{aligned}$$

and  $k_i = \omega \sqrt{\mu_i \varepsilon_i}$ . Notice that  $\mathbf{F}$  vanishes if  $\varepsilon_i(q) = \varepsilon_e$ ,  $\mu_i(q) = \mu_e$  and  $\varepsilon_i(P)$  is constant near  $S$ . Also, in the special case that  $\mu_i(Q) \equiv \mu_e$ , (4.6) reduces to eqn. (4.18) in [36].

**4.5. Elastic waves.** We can also consider analogous problems for elastic waves: scattering of elastic waves in a homogeneous solid by an inhomogeneous (and anisotropic) inclusion. It turns out that the formal method of section 1.1 (described in detail in [33, 17]) and the method of section 3 yield exactly the same equation for the displacement within the inclusion,  $\mathbf{u}$ . This is because the “extra term” analogous to  $p_E$  involves the discontinuity in the traction vector across the interface: this is zero for a perfect (welded) interface. (For imperfect interfaces [27], a nonzero contribution is obtained.) Some applications of the volume equation for  $\mathbf{u}$  can be found in [38, 6, 7]. A polarization approach (leading to a coupled system) has been developed by Willis [45]. Long-wave approximations can be found in [45, 38].

**Appendix A. Werner’s solution.** Werner [43] proved an existence theorem for a problem that is very similar to our Scattering Problem: he considered inhomogeneous forms of (2.2) and (2.3) but supposed that  $p_{\text{inc}} \equiv 0$ . His method leads to the following integral representations:

$$p_{\text{sc}}(P) = - \int_S \left\{ \alpha(q) \mu(q) G_e(P, q) - \nu(q) \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q, \quad P \in B_e,$$

$$p_i(P) = \int_S \left\{ \mu G_e(P, q) - \nu \frac{\partial}{\partial n_q} G_e(P, q) \right\} ds_q + \int_B \varphi(Q) G_e(P, Q) dV_Q, \quad P \in B.$$

The two surface densities,  $\mu(q)$  and  $\nu(q)$ , and the volume density,  $\varphi(Q)$ , satisfy the following system of integral equations:

$$(A.1) \quad \nu(p) + \int_S \mu(q) \{1 - \alpha(q)\} G_e(p, q) ds_q + \int_B \varphi(Q) G_e(p, Q) dV_Q = -p_{\text{inc}},$$

$$(A.2) \quad \frac{1 + \alpha}{2} \mu + \int_S \mu(1 - \alpha) \frac{\partial}{\partial n_p} G_e(p, q) ds_q + \int_B \varphi \frac{\partial}{\partial n_p} G_e(p, Q) dV_Q = -\frac{\partial p_{\text{inc}}}{\partial n_p},$$

$$(A.3) \quad \varphi + \int_S V_P \left\{ \mu G_e(P, q) - \nu \frac{\partial G_e}{\partial n_q} \right\} ds_q - \int_B \varphi (V_P G_e(P, Q)) dV_Q = 0.$$

Equations (A.1) and (A.2) hold for  $p \in S$ , whereas (A.3) holds for  $P \in B$ .  $V_P$  denotes the operator  $V$  (defined by (2.6)) applied with respect to  $P$ .

Werner’s proof [43] can be adapted to show that the system (A.1)–(A.3) is uniquely solvable.

**Appendix B. Volume potentials.** Define a *volume potential*  $W(P)$  by

$$W(P) = \int_B \varphi(Q) G_e(P, Q) dV_Q,$$

where  $G_e(P, Q) = -\exp(ik_e R)/(4\pi R)$ . The properties of such potentials are similar to those of *Newtonian potentials* for which  $k_e = 0$ : thus, define

$$W_0(P) = -\frac{1}{4\pi} \int_B \frac{\varphi(Q)}{R} dV_Q.$$

From [12, sect. 8.2], we have

$$(B.1) \quad (\nabla^2 + k_e^2)W = \begin{cases} 0, & P \in B_e, \\ \varphi(P), & P \in B, \end{cases}$$

where  $B_e$  is the region exterior to  $B$ . Also, if  $\varphi$  is piecewise continuous, then  $W(P)$  and its first partial derivatives are continuous everywhere in three-dimensional space;

see, for example, [19, Chapter VI, sect. 3], [14, Chapter IV, sect. 1.2], [41, sect. 3.9], [12, Thm. 8.1], and [16, sect. 4.2].

We also require the behavior of the second derivatives of  $W$  near the boundary of  $B$ ,  $S$ . As Kellogg remarks [19, p. 156], “in general, the derivatives of second order will not exist. It is clear that they cannot all be continuous, for as we pass from an exterior to an interior point through the boundary where  $\varphi$  is not 0,  $\nabla^2 W_0$  experiences a break of  $\varphi$ .” This discontinuous behavior is described in [25, p. 175] and [26, p. 125]:

$$\left[ \frac{\partial^2}{\partial n \partial x_i^P} W_0(P) \right] = -\varphi(p) n_i(p), \quad p \in S.$$

The same formula holds for  $W$  because the difference,  $W - W_0$ , is less singular. Finally, the result (3.14) is obtained from (3.10), using  $\varphi(Q) = (1 - \alpha)\partial v/\partial x_i^Q$ . Note that the second term in (3.10) is a volume potential: it does not contribute as its first derivatives are continuous across  $S$ .

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