On Webster's horn equation and some generalizations

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(Received 17 October 2003; revised 18 May 2004; accepted 2 June 2004)

Sound waves along a rigid axisymmetric tube with a variable cross-section are considered. The governing Helmholtz equation is solved using power-series expansions in a stretched radial coordinate, leading to a hierarchy of one-dimensional ordinary differential equations in the longitudinal direction. The lowest approximation for axisymmetric motion turns out to be Webster's horn equation. Fourth-order differential equations are obtained at the next level of approximation. Comparisons with existing asymptotic theories for waves in slender tubes are made. [DOI: 10.1121/1.1775272]

PACS numbers: 43.20.Mv [LLT] Pages: 1381–1388

I. INTRODUCTION

Webster's horn equation (1919) gives a one-dimensional approximation for low-frequency sound waves along a rigid tube with a variable cross-sectional area $A(z)$. The equation itself can be written as

$$\frac{1}{A}\frac{d}{dz}\left(A(z)\frac{dP}{dz}\right)+k^2P(z)=0, \tag{1}$$

where $k=\omega/c$, ω is the frequency, c is the (constant) speed of sound, and z is the coordinate along the tube. Equation (1) can be derived by considering a thin layer of the fluid at z, perpendicular to the z-direction, with the assumption that the acoustic pressure is constant over this layer; this pressure is $P(z)$. For a succinct derivation, see (Pierce, 1989, p. 360). Alternative derivations and extensions to other problems (such as with fluid flow through the tube) are available; see, for example, (Reinstra, 2002) and (Hélie, 2003). Exact solutions of Eq. (1) are available for various specific $A(z)$; for some recent results, see (Kumar and Sujith, 1997).

In 1967, Eisner published an excellent review of early work based on Eq. (1). On p. 1127, we read: "Eq. (1) is usually called 'Webster's horn equation,' but we see that there is little justification for this name. Daniel Bernoulli, Euler, and Lagrange all derived the equation and did most interesting work on its solution, more than 150 years before Webster."

We are interested in giving a systematic derivation of Webster's equation, and of higher-order variants. We limit our analysis to axisymmetric tubes, and obtain Webster's equation when we seek axisymmetric solutions. We also consider nonaxisymmetric motions. To be specific, we consider a tube of finite length, and seek the frequencies of free vibration of the compressible fluid within the closed tube.

In 1916 (three years before Webster's paper), Lord Rayleigh published a paper on axisymmetric motions in an axisymmetric tube. He began by writing the general solution of the axisymmetric Helmholtz equation in cylindrical polar coordinates (r,θ,z) as

$$u(r,\theta,z)=J_0\left(r\sqrt{\frac{d^2}{dz^2}+k^2}\right)u_0(z), \tag{2}$$

where $u_0(z)=u(0,\theta,z)$ is the value of u on the axis of the tube; as the Bessel function $J_0(w)$ has a power-series expansion in integer powers of w^2, this provides meaning to the right-hand side of Eq. (2) [see Eq. (26) below]. Rayleigh then obtained an equation for $u_0(z)$ by applying the boundary condition on the rigid wall of the tube; as a first approximation, he obtained Eq. (1) [see Eq. (8) in (Rayleigh, 1916)]. He also discussed some higher-order approximations. We shall adopt a similar approach, although we do not begin with an explicit representation such as Eq. (2): we shall use a Frobenius-type power-series expansion for the radial variation of $u(r,\theta,z)$.

After formulating the problem in Sec. II, we examine the special case of circular cylinders in Secs. III and IV. The exact solution for the eigenfrequencies is recalled in Sec. III. Then, the approximate method is developed in Sec. IV. It is based on some observations of Boström (2000) for the related axisymmetric problems of elastic waves in isotropic rods. In principle, we can obtain a heirarchy of approximations: We give explicit results for the first two members of this heirarchy. Apart from the merits of explaining the method for a simple case, we are also able to give a quantitative comparison with the exact solutions from Sec. III.

In Sec. V, we consider axisymmetric, noncylindrical tubes. We change variables in the governing Helmholtz equation from r and z to ρ and ζ, where ρ is a scaled version of r chosen to that the lateral boundary is mapped to $\rho=$constant; this has the effect of complicating the partial differential equation (via the chain rule) but has the virtue that the lateral boundary condition is applied on a coordinate surface. The complications bring the shape of the boundary into the differential equation [cf. Webster's equation (1)] but they can be dealt with readily because we then use a Frobenius-type expansion in the new "radial" variable ρ. Again, we obtain a hierarchy of approximations. The first approximation gives a second-order ordinary differential equation; it reduces to Webster's equation for axisymmetric motions. The second approximation gives a fourth-order ordinary differential

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equation. In order to assess these approximations, we compare with some results of Ting and Miksis (1983) and Geer and Keller (1983). These authors began with the governing elliptic boundary-value problem for waves in slender tubes, and then obtained various asymptotic approximations. The comparisons are made in Sec. VI. Some concluding remarks can be found in Sec. VII.

In summary, the approximate method described below has two virtues. First, the approximations can be improved. Second, the use of power series means that the basic method can be applied to much more complicated equations of motion, and to systems of such equations. For example, the propagation of elastic waves in nonuniform anisotropic rods can be studied; for some preliminary results in this direction, see (Martin, 2004).

II. FORMULATION

Consider a tube of circular cross-section and length L. Using cylindrical polar coordinates, $(\tilde{r},\theta,\tilde{z})$, the interior of the tube is specified by

$$0\leqslant\tilde{r}<a\tilde{R}(\tilde{z}),\quad 0\leqslant\theta<2\pi,\quad 0<\tilde{z}<L,$$

where $0<\tilde{R}(\tilde{z})\leqslant 1$, so that $2a$ is the maximum diameter of the tube. It will be convenient to define dimensionless variables, using L as our length scale. Thus, we put

$$r=\tilde{r}/L,\quad z=\tilde{z}/L,\quad \tilde{R}(\tilde{z})=R(z)\quad\text{and}\quad \varepsilon=a/L.$$

(Later, we shall regard ε as a small parameter.) Hence, the tube becomes

$$0\leqslant r<\varepsilon R(z),\quad 0\leqslant\theta<2\pi,\quad 0<z<1.$$

Inside the tube, the acoustic potential $U(r,\theta,z,t)$ satisfies the wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}+\frac{\partial^2 U}{\partial z^2}=\frac{L^2}{c^2}\frac{\partial^2 U}{\partial t^2},\tag{3}$$

where c is the speed of sound. On the lateral wall of the tube, the normal derivative of U vanishes

$$\frac{\partial U}{\partial r}-\varepsilon R'(z)\frac{\partial U}{\partial z}=0\quad\text{on}\quad r=\varepsilon R(z),\quad 0<z<1.\tag{4}$$

We assume that $R(0)$ and $R(1)$ are both positive, and close the two ends of the tube with rigid circular discs, giving

$$\partial U/\partial z=0\quad\text{at}\quad z=0\quad\text{and}\quad\text{at}\quad z=1.\tag{5}$$

We seek free vibrations of the compressible fluid within the axisymmetric tube. Thus, we put $U(r,\theta,z,t)=u(r,z)\cos m\theta\cos\omega t$, where m is a non-negative integer and ω is the frequency. The wave equation becomes

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+r^2\frac{\partial^2 u}{\partial z^2}+(k^2r^2-m^2)u=0,\tag{6}$$

where (now) $k=\omega L/c$ is a dimensionless wave number. We are interested in determining *eigenfrequencies* ω so that there is a nontrivial u that satisfies Eq. (6) and the boundary conditions.

III. CIRCULAR TUBE: EXACT SOLUTION

Consider a circular tube with $R(z)=1$ for $0<z<1$. We can write down separated solutions of Eq. (6), $u(r,z)=J_m(\lambda r)\cos\mu z$, where J_m is a Bessel function, and λ and μ are arbitrary constants satisfying $k^2=\lambda^2+\mu^2$. The boundary conditions give $\lambda\varepsilon=j'_{m,\ell}$ and $\mu=N\pi$, where $N\geqslant 0$ and $\ell\geqslant 1$ are integers and $j'_{m,\ell}$ is the ℓ-th zero of J'_m:$J'_m(j'_{m,\ell})=0$, $\ell=1$, 2,...,. Hence,

$$k^2=(N\pi)^2+(j'_{m,\ell})^2\varepsilon^{-2}.\tag{7}$$

This is the exact solution of the problem. It can be found in textbooks [for example, p. 221 of (Kinsler *et al.*, 1982)] and was given by Lord Rayleigh in the first edition of volume II of *The Theory of Sound*, published in 1878 (Rayleigh, 1945).

A. Axisymmetric modes ($m=0$)

The axisymmetric problem ($m=0$) is unusual, because we can take $\lambda=0$: both Eq. (6) and the lateral boundary condition on $r=\varepsilon$ are satisfied by $u(r,z)=\cos\mu z$, so that one set of eigenfrequencies is given by

$$k=N\pi\quad\text{for all}\quad\varepsilon>0.\tag{8}$$

This set is elementary; it can be found in Sec. 62 of Lamb's book (1960).

The next set comes by using $j'_{0,2}\simeq 3.832$

$$k^2=(N\pi)^2+14.68\varepsilon^{-2},\quad N=0,1,2,...,.\tag{9}$$

B. Flexural modes ($m=1$)

We are also interested in nonaxisymmetric motions. As an example, we consider the case $m=1$ ("flexural modes"). As $j'_{1,1}\simeq 1.841$ and $j'_{1,2}\simeq 5.331$, Eq. (7) gives the first two sets of eigenfrequencies as

$$k^2=(N\pi)^2+3.39\varepsilon^{-2}\quad\text{and}\quad k^2=(N\pi)^2+28.42\varepsilon^{-2}\tag{10}$$

for $N=0,1,2,...$, later, we will compare the coefficients 3.39 and 28.42 with those obtained by certain approximate theories.

IV. CIRCULAR TUBE: APPROXIMATE METHOD

If the tube is slender, $\varepsilon\equiv a/L\ll 1$, we expect to be able to derive one-dimensional theories. We shall do this using power series in r. Note that we do not limit ourselves to *polynomials* in r, and so we are not limited, in principle, to very long waves. Nevertheless, it turns out that the low-order truncations obtained below work best for longer waves.

We begin, as in the method of Frobenius, by writing

$$u(r,z)=\sum_{n=0}^{\infty}r^{2n+\alpha}u_n(z),\tag{11}$$

where α and $u_n(z)$ are to be found. Substitution in Eq. (6) gives

$$(\alpha^2-m^2)u_0(z)+\sum_{n=0}^{\infty}r^{2n+2}\{\sigma_n(\alpha)u_{n+1}+u''_n+k^2u_n\}=0,$$

where $\sigma_n(\alpha)=(2n+2+\alpha)^2-m^2$. Just as in the method of Frobenius for ordinary differential equations, we require that every coefficient of r^ℓ vanishes. For the first term to vanish, we obtain $\alpha^2=m^2$. As we want u to be bounded at $r=0$, we take $\alpha=+m$, and then we obtain

$$u_n''(z)+k^2u_n(z)=-4(n+1)(n+m+1)u_{n+1}(z),$$
$$n=0,1,2,...,.$$

Notice that this procedure does not determine $u_0(z)$. However, following Boström (2000), we note that u_1, $u_2,\dots,$ are all determined by u_0; for example, we have

$$u_1=-\frac{u_0''+k^2u_0}{4(m+1)} \tag{12}$$

and

$$u_2=-\frac{u_1''+k^2u_1}{8(m+2)}=\frac{u_0^{\text{iv}}+2k^2u_0''+k^4u_0}{32(m+1)(m+2)}.$$

Then, *regardless of the choice of* u_0, the infinite series in Eq. (11) will give an exact solution of Eq. (6), assuming that the series converges.

The end boundary conditions, Eq. (5), give $\sum_{n=0}^{\infty}r^{2n}u_n'(z)=0$ at the two ends, whence

$$u_n'(0)=u_n'(1)=0, \quad n=0,1,2,...,. \tag{13}$$

The lateral boundary condition reduces to $\partial u/\partial r=0$ on $r=\varepsilon$, and this gives

$$\sum_{n=0}^{\infty}(2n+m)\varepsilon^{2n}u_n(z)=0, \quad 0<z<1.$$

Eliminating u_n in favor of u_0, we obtain an equation for $u_0(z)$

$$\begin{aligned}0&=mu_0+(m+2)\varepsilon^2u_1+(m+4)\varepsilon^4u_2+\cdots\\&=mu_0-\frac{(m+2)\varepsilon^2}{4(m+1)}(u_0''+k^2u_0)\\&\quad+\frac{(m+4)\varepsilon^4}{32(m+1)(m+2)}(u_0^{\text{iv}}+2k^2u_0''+k^4u_0)+\cdots.\end{aligned} \tag{14}$$

At this stage, no approximations have been made. We obtain various approximations by truncating Eq. (14); this is done next.

A. First approximation

If we discard all terms with powers of ε greater than 2 in Eq. (14), we obtain

$$u_0''(z)+\mathcal{E}_m^{(1)}u_0(z)=0, \tag{15}$$

where

$$\mathcal{E}_m^{(1)}(k,\varepsilon)=k^2-[4m(m+1)/(m+2)]\varepsilon^{-2}. \tag{16}$$

From Eq. (13), we see that Eq. (15) is to be solved subject to $u_0'(0)=u_0'(1)=0$.

If we look for solutions of Eq. (15) in the form

$$u_0(z)=\cos\mu z, \tag{17}$$

we find that $\mu^2=\mathcal{E}_m^{(1)}(k,\varepsilon)$. Then, in order to satisfy the boundary conditions on the two ends of the tube, we obtain $\mu=N\pi$, whence

$$k^2=(N\pi)^2+[4m(m+1)/(m+2)]\varepsilon^{-2}.$$

This can be compared with the exact solution, Eq. (7). For $m=0$, we recover the exact set of solutions given by Eq. (8). For $m=1$, we can compare with Eq. (10); when $m=1$, we have $4m(m+1)/(m+2)=8/3\simeq 2.67$ which is in error by about 20%.

B. Second approximation

If we retain the terms in ε^4 in Eq. (14), we obtain

$$u_0^{\text{iv}}(z)+\mathcal{C}_mu_0''(z)+\mathcal{E}_m^{(2)}u_0(z)=0, \tag{18}$$

where

$$\mathcal{C}_m(k,\varepsilon)=2k^2-\frac{8(m+2)^2}{(m+4)\varepsilon^2}, \tag{19}$$

$$\mathcal{E}_m^{(2)}(k,\varepsilon)=k^4+\frac{32(m+1)(m+2)}{(m+4)\varepsilon^4}\left[m-\frac{(m+2)(k\varepsilon)^2}{4(m+1)}\right]. \tag{20}$$

The fourth-order Eq. (18) is to be solved subject to

$$u_0'(0)=u_0'''(0)=u_0'(1)=u_0'''(1)=0, \tag{21}$$

using Eq. (12) to express u_1' in terms of u_0''' and u_0', and Eq. (13).

Substituting Eq. (17) in Eq. (18) gives

$$\mu^4-\mu^2\mathcal{C}_m(k,\varepsilon)+\mathcal{E}_m^{(2)}(k,\varepsilon)=0, \tag{22}$$

and then a routine calculation yields

$$\mu^2=k^2-\nu_m/\varepsilon^2=\mu_m^2(k,\varepsilon), \tag{23}$$

say, where $\nu_m=\chi_m+4(m+2)^2/(m+4)$ and

$$\chi_m=\pm\left(1+\frac{1}{4}m\right)^{-1}\sqrt{(m+2)(8+4m-4m^2-m^3)}. \tag{24}$$

Finally, the end boundary conditions give

$$k^2=(N\pi)^2+\nu_m/\varepsilon^2. \tag{25}$$

1. Axisymmetric modes (m=0)

When $m=0$, Eq. (25) reduces to Eq. (8) when we take the minus sign in $\pm$. If we take the plus sign, we obtain $k^2=(N\pi)^2+8\varepsilon^{-2}$, where the coefficient 8 can be compared with the exact 14.68 in Eq. (9).

For this axisymmetric problem, we can compare with Rayleigh's approach. As $J_0(w)=1-1/4w^2+1/64w^4-\cdots$, Eq. (2) gives

$$\begin{aligned}u(r,z)&=u_0(z)-\tfrac{1}{4}r^2(u_0''+k^2u_0)\\&\quad+\tfrac{1}{64}r^4(u_0^{\text{iv}}+2k^2u_0''+k^4u_0)-\cdots.\end{aligned} \tag{26}$$

If we discard the higher-order terms and apply the boundary condition, $\partial u/\partial r=0$ on $r=\varepsilon$, we obtain precisely Eq. (18) with $m=0$. This is reassuring but not surprising: The representations given by Eqs. (2) and (11) are equivalent, although the latter can be used for nonaxisymmetric problems.

2. Flexural modes (m=1)

When $m=1$, we obtain

$$k^2=(N\pi)^2+\tfrac{4}{5}(9\pm\sqrt{21})\varepsilon^{-2}. \tag{27}$$

To obtain the lowest values, we take the minus sign, giving $k^2=(N\pi)^2+3.53\varepsilon^{-2}$; the coefficient 3.53 differs from the exact 3.39 by about 4%. Thus, the fourth-order model gives good accuracy for the lowest set of eigenfrequencies.

If we take the plus sign in Eq. (27), the coefficient multiplying ε^{-2} becomes 10.87, which can be compared with the exact value of 28.42 given in the second of Eq. (10). Thus, as when $m=0$, the second set of eigenfrequencies is not approximated well by the fourth-order model.

V. NONCYLINDRICAL TUBES

Once we move away from cylinders, we no longer have exact solutions. Therefore, an approximate method will be required. For this reason, we choose to make a simple change of the independent variables, from (r,z) to (ρ,ζ), so that the new geometry is a circular cylinder. The price of this change is that the new partial differential equation is more complicated.

Thus, define new variables ρ and ζ by

$$\rho=r/R(z) \quad \text{and} \quad \zeta=z, \tag{28}$$

so that the tube is mapped onto the circular tube, given by $0\leqslant\rho<\varepsilon$, $0<\zeta<1$. (Later, we will use z in place of ζ, but it is clearer to distinguish the two variables at this stage.) The chain rule gives

$$\frac{\partial u}{\partial r}=\frac{1}{R}\frac{\partial u}{\partial \rho}, \quad \frac{\partial^2 u}{\partial r^2}=\frac{1}{R^2}\frac{\partial^2 u}{\partial \rho^2},$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \zeta}-\rho\frac{R'}{R}\frac{\partial u}{\partial \rho},$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{\partial^2 u}{\partial \zeta^2}-2\rho\frac{R'}{R}\frac{\partial^2 u}{\partial\rho\partial\zeta}-\rho\frac{\partial u}{\partial\rho}\left(\frac{R'}{R}\right)'+\rho\left(\frac{R'}{R}\right)^2\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right).$$

Hence, Eq. (6) becomes

$$(1+\rho^2R'^2)\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\rho^3(R'^2-RR'')\frac{\partial u}{\partial\rho}+\rho^2R^2\frac{\partial^2 u}{\partial\zeta^2}-2\rho^3RR'\frac{\partial^2 u}{\partial\rho\partial\zeta}+[(k\rho R)^2-m^2]u=0, \tag{29}$$

the lateral boundary condition, Eq. (4), becomes

$$(1+\varepsilon^2R'^2)\frac{\partial u}{\partial\rho}-\varepsilon RR'\frac{\partial u}{\partial\zeta}=0 \quad \text{on } \rho=\varepsilon, \quad 0<\zeta<1, \tag{30}$$

and the end boundary conditions, Eq. (5), become

$$\frac{\partial u}{\partial\zeta}-\rho\frac{R'}{R}\frac{\partial u}{\partial\rho}=0 \quad \text{at } \zeta=0 \quad \text{and} \quad \text{at } \zeta=1. \tag{31}$$

To solve Eq. (29), we proceed as in Sec. IV, and write

$$u(\rho,\zeta)=\sum_{n=0}^{\infty}\rho^{2n+\alpha}u_n(\zeta). \tag{32}$$

Substitution in Eq. (29) gives

$$(\alpha^2-m^2)u_0(\zeta)+\sum_{n=0}^{\infty}\rho^{2n+2}\{\sigma_n(\alpha)u_{n+1}+\Lambda_n(\zeta;\alpha)\}=0,$$

where $\sigma_n(\alpha)=(2n+2+\alpha)^2-m^2$, and

$$\Lambda_n(z;\alpha)=R^2u_n''-2RR'(2n+\alpha)u_n'+[k^2R^2+(2n+\alpha)^2R'^2+(2n+\alpha)(R'^2-RR'')]u_n.$$

As before, we take $\alpha=+m$ and then

$$\Lambda_n(z;m)=-4(n+1)(n+m+1)u_{n+1}(z), \quad n=0,1,2,...,.$$

In particular, we find that

$$\begin{aligned}-4(m+1)u_1&=R^2u_0''-2mRR'u_0'+[k^2R^2+m(m+1)R'^2-mRR'']u_0,\\ -8(m+2)u_2&=R^2u_1''-2(m+2)RR'u_1'+[k^2R^2+(m+2)\{(m+3)R'^2-RR''\}]u_1.\end{aligned} \tag{33}$$

Eliminating u_1 from the last equation, using Eq. (33), gives

$$32(m+1)(m+2)u_2=\alpha_m u_0^{\mathrm{iv}}+\beta_m u_0'''+\gamma_m u_0''+\delta_m u_0'+\varepsilon_m u_0, \tag{34}$$

where

$$\alpha_m(z)=R^4, \quad \beta_m(z)=-4mR^3R',$$

$$\gamma_m(z)=2R^2\{k^2R^2+3m[(m+1)R'^2-RR'']\},$$

$$\delta_m(z)=-4mR\{k^2R^2R'+R^2R'''-(m+1)R'[3RR''-(m+2)R'^2]\},$$

$$\varepsilon_m(z)=k^4R^4+2mk^2R^2\{(m+1)R'^2-RR''\}-mR^3R^{\mathrm{iv}}+m(m+1)\{R^2(4R'R'''+3R''^2)-(m+2)R'^2[6RR''-(m+3)R'^2]\}.$$

When Eq. (32) is substituted in the lateral boundary condition Eq. (30), together with $\alpha=m$, we obtain

$$0=mu_0(z)+\sum_{n=0}^{\infty}\varepsilon^{2n+2}\{(2n+2+m)u_{n+1}(z)+(2n+m)R'^2u_n-RR'u_n'\}. \tag{35}$$

Similarly, the end boundary conditions Eq. (31) give

$$\sum_{n=0}^{\infty}\rho^{2n}\left[u_n'(z)-\frac{R'}{R}(2n+m)u_n(z)\right]=0 \quad \text{at} \quad z=0 \quad \text{and} \quad \text{at } z=1, \tag{36}$$

which immediately gives

$$R(z)u_n'(z)-R'(z)(2n+m)u_n(z)=0 \quad \text{at } z=0 \quad \text{and} \quad \text{at } z=1, \tag{37}$$

for $n=0,1,2,...,$.

Below, we shall use the following shorthand notation:

$$\mathcal{S}_1(z)=R'/R, \quad \mathcal{S}_2=R''/R, \quad \mathcal{S}_3=R'''/R, \quad \mathcal{S}_4=R^{\text{iv}}/R. \tag{38}$$

A. First approximation

If we retain only the terms up to ε^2 in Eq. (35), we find that

$$0=mu_0\varepsilon^{-2}+(m+2)u_1+mR'^2u_0-RR'u_0'.$$

Upon using Eq. (33), this gives

$$u_0''(z)+D_m^{(1)}(z)u_0'(z)+E_m^{(1)}(z)u_0(z)=0, \tag{39}$$

where

$$D_m^{(1)}(z)=2\mathcal{S}_1(2-m^2)/(m+2),$$

$$E_m^{(1)}(z)=\mathcal{E}_m^{(1)}(k,\varepsilon R)+\frac{m(m+1)(m-2)}{m+2}\mathcal{S}_1^2-m\mathcal{S}_2,$$

and $\mathcal{E}_m^{(1)}$ are defined by Eq. (16). From Eq. (37), Eq. (39) is to be solved subject to $Ru_0'-mR'u_0=0$ at $z=0$ and at $z=1$.

Equation (39) can be transformed so as to eliminate the first-derivative term. Thus, put

$$u_0(z)=R^\gamma U_0(z) \quad \text{with} \quad \gamma=(m^2-2)/(m+2). \tag{40}$$

Then, we find that $U_0(z)$ solves

$$U_0''(z)+[k^2-K(z)]U_0(z)=0, \tag{41}$$

where

$$K(z)=\frac{2(m+1)}{m+2}\left\{\frac{m}{m+2}\mathcal{S}_1^2+\mathcal{S}_2+\frac{2m}{\varepsilon^2R^2}\right\},$$

subject to $(m+2)U_0'-2(m+1)\mathcal{S}_1U_0=0$ at $z=0$ and at $z=1$. Notice that Eq. (41) has oscillatory solutions when $k^2>K$ but exponential solutions when $k^2<K$.

Equation (39) and its associated boundary conditions can also be written as a regular Sturm-Liouville problem. Thus,

$$(p(z)u_0')'+[q(z)+\lambda w(z)]u_0(z)=0,$$

where $p=w=R^{-2\gamma}$, $\lambda=k^2$, γ is defined by Eq. (40), and

$$q(z)=mR^{-2\gamma}\left\{\frac{m+1}{m+2}\left[(m-2)\mathcal{S}_1^2-\frac{4}{\varepsilon^2R^2}\right]-\mathcal{S}_2\right\}.$$

1. Axisymmetric modes (m=0)

When $m=0$, Eq. (39) reduces to

$$(Au_0')'+k^2Au_0=0, \tag{42}$$

where $A(z)=\pi a^2[R(z)]^2$ is the area of the (circular) cross-section at z; Eq. (42) is recognized as Webster's horn equation Eq. (1). The appropriate boundary conditions are $u_0'(0)=u_0'(1)=0$.

2. Flexural modes (m=1)

When $m=1$, Eq. (39) reduces to

$$u_0''(z)+\frac{2}{3}\mathcal{S}_1u_0'(z)+\left[k^2-\frac{2}{3}\mathcal{S}_1^2-\mathcal{S}_2-\frac{8}{3\varepsilon^2R^2}\right]u_0(z)=0,$$

subject to $Ru_0'-R'u_0=0$ at $z=0$ and at $z=1$.

B. Second approximation

If we retain the terms up to ε^4 in Eq. (35), we obtain an equation containing u_0, u_0', u_1, u_1', and u_2. If we eliminate u_1, u_1', and u_2, using Eqs. (33) and (34), we obtain a fourth-order equation for $u_0(z)$,

$$u_0^{\text{iv}}+B_m^{(2)}(z)u_0'''+C_m^{(2)}(z)u_0''+D_m^{(2)}(z)u_0'+E_m^{(2)}(z)u_0=0, \tag{43}$$

where

$$B_m^{(2)}=4\mathcal{S}_1(4-2m-m^2)/(m+4),$$

$$C_m^{(2)}=\mathcal{C}_m(k,\varepsilon R)-6m\{\mathcal{S}_2+(4-m-m^2)\mathcal{S}_1^2(m+4)\}, \tag{44}$$

$$D_m^{(2)}=\mathcal{D}_m(k,\varepsilon R)-4m\mathcal{S}_3+\frac{4m\mathcal{S}_1}{m+4}\{(m+1)(4-m^2)\mathcal{S}_1^2+3m(m+3)\mathcal{S}_2\},$$

$$\begin{aligned}E_m^{(2)}=&\mathcal{E}_m^{(2)}(k,\varepsilon R)-m\mathcal{S}_4+3m(m+1)\mathcal{S}_2^2\\&-2mk^2\left\{\frac{4-m-m^2}{m+4}\mathcal{S}_1^2+\mathcal{S}_2\right\}+\frac{4m^2(m+3)}{m+4}\mathcal{S}_1\mathcal{S}_3\\&-\frac{6m^2(m+1)(m+2)}{m+4}\mathcal{S}_1^2\mathcal{S}_2\\&+\frac{m(m+1)(m+2)}{m+4}(m^2-m-4)\mathcal{S}_1^4\\&-\frac{8m(m+2)}{(m+4)\varepsilon^2R^2}\{(m+1)(m-2)\mathcal{S}_1^2-(m+2)\mathcal{S}_2\},\end{aligned}$$

$\mathcal{C}_m$ and $\mathcal{E}_m^{(2)}$ are defined by Eqs. (19) and (20), respectively, and

$$\mathcal{D}_m(k,\varepsilon)=\frac{4\mathcal{S}_1}{m+4}\left\{(4-2m-m^2)k^2+\frac{4}{\varepsilon^2}(m+2)(m^2-2)\right\}. \tag{45}$$

Note that Eq. (43) reduces to Eq. (18) when $R(z)\equiv1$.

Equation (43) is to be supplemented with boundary conditions. From Eq. (37), these are

$$Ru_0'-mR'u_0=0 \quad \text{and} \quad Ru_1'-(m+2)R'u_1=0 \tag{46}$$

at $z=0$ and at $z=1$. Eliminating u_1 from the second of Eq. (46), using Eq. (33), and then using the first of Eq. (46), we obtain

$$u_0'''-3m\mathcal{S}_1u_0''+\{2(m^2-1)\mathcal{S}_1^2+3\mathcal{S}_2\}u_0'-m\mathcal{S}_3u_0=0; \tag{47}$$

both this condition and $u_0' = m\mathcal{S}_1 u_0$ are to be imposed at $z=0$ and at $z=1$. Notice that the boundary conditions do not involve k^2.

1. Axisymmetric modes (m=0)

When $m=0$, Eq. (43) reduces to

$$u_0^{\mathrm{iv}} + \frac{4R'}{R}u_0''' + 2\left(k^2 - \frac{4}{\varepsilon^2 R^2}\right)u_0'' + \frac{4R'}{R}\left(k^2 - \frac{4}{\varepsilon^2 R^2}\right)u_0' + \left(k^4 - \frac{8k^2}{\varepsilon^2 R^2}\right)u_0 = 0. \tag{48}$$

From Eq. (47), we see that Eq. (48) is to be solved subject to Eq. (21). We observe that Eq. (48) involves R and R', but not higher derivatives of R. Also, Eq. (48) can be derived directly by applying Eq. (4) to Eq. (26).

2. Flexural modes (m=1)

When $m=1$, we obtain

$$B_1^{(2)}(z) = \frac{4}{5}\mathcal{S}_1, \quad C_1^{(2)}(z) = 2k^2 - \frac{12}{5}\mathcal{S}_1^2 - 6\mathcal{S}_2 - \frac{72}{5}(\varepsilon R)^{-2},$$

$$D_1^{(2)}(z) = \frac{4}{5}\mathcal{S}_1\{k^2 + 6\mathcal{S}_1^2 + 12\mathcal{S}_2 - 12(\varepsilon R)^{-2}\} - 4\mathcal{S}_3,$$

$$E_1^{(2)}(z) = k^4 - \mathcal{S}_4 + 6\mathcal{S}_2^2 - \frac{2}{5}k^2\{2\mathcal{S}_1^2 + 5\mathcal{S}_2 + 36(\varepsilon R)^{-2}\} + \frac{2}{5}\mathcal{S}_1\{8\mathcal{S}_3 - 18\mathcal{S}_1\mathcal{S}_2 - 12\mathcal{S}_1^3\} + \frac{24}{5\varepsilon^2 R^2}\left\{2\mathcal{S}_1^2 + 3\mathcal{S}_2 + \frac{8}{\varepsilon^2 R^2}\right\}.$$

Then, the differential equation Eq. (43) (with $m=1$) is to be solved subject to

$$u_0' - \mathcal{S}_1 u_0 = 0 \quad \text{and} \quad u_0''' - 3\mathcal{S}_1 u_0'' + 3\mathcal{S}_2 u_0' - \mathcal{S}_3 u_0 = 0$$

at $z=0$ and at $z=1$.

VI. COMPARISON WITH ASYMPTOTIC APPROXIMATIONS

A number of formal asymptotic theories have been developed for waves in slender tubes. In the papers by Ting and Miksis (1983) and by Geer and Keller (1983), the tubes need not have circular cross-sections and they need not be straight. Here, we shall compare the results obtained from our one-dimensional theory with those obtained by specialising the analysis in (Ting and Miksis, 1983) and (Geer and Keller, 1983) to our axisymmetric geometries.

Two asymptotic regimes are of interest to us. In one, $k=O(1)$ as $\varepsilon\to 0$, so that the wavelength is comparable to L, the length of the tube. All solutions of this kind are axisymmetric. In a second regime, $k=O(\varepsilon^{-1})$ as $\varepsilon\to 0$, so that the wavelength is comparable to $a \ll L$. Both kinds of solution are seen in the exact solutions for circular tubes (Sec. III).

A. Wavelength comparable to *L*

For axisymmetric motion, we obtained the fourth-order Eq. (48), which we write here as

$$(R^2 u_0')' + k^2 R^2 u_0 = \tfrac{1}{8}\varepsilon^2 R^3(R u_0^{\mathrm{iv}} + 4R' u_0''' + 2k^2 R u_0'' + 4k^2 R' u_0' + k^4 R u_0). \tag{49}$$

In this equation, k and $u_0(z)$ are unknown; they are to be determined subject to the four boundary conditions, Eq. (21).

Let us look for solutions of the form

$$k^2 = k_0^2 + \varepsilon^2 k_2^2 + \cdots, \tag{50}$$

where k_0 and k_2 are to be found. Thus, we put $u_0(z) = v_0(z) + \varepsilon^2 v_1(z) + \cdots$ and Eq. (50) in Eq. (49). This is a classic singular perturbation, because Eq. (49) reduces to a second-order equation when $\varepsilon=0$, implying that the boundary conditions will have to be modified. Indeed, writing out the first two terms of the exact boundary condition, Eq. (36), we have

$$0 = u_0'(z) + \rho^2[u_1' - 2(R'/R)u_1] + \cdots$$
$$= u_0'(z) - \tfrac{1}{4}\rho^2 R^2(u_0'' + k^2 u_0)' + \cdots,$$

at each end of the tube, where we have used Eq. (33). We will not be able to satisfy this condition for all allowable ρ with $0 \leqslant \rho < \varepsilon$, and so we integrate over each circular end of the tube, and impose

$$0 = u_0'(z) - \tfrac{1}{8}\varepsilon^2 R^2(u_0'' + k^2 u_0)' + \cdots \quad \text{at } z=0 \text{ and at } z=1; \tag{51}$$

this ensures that Eq. (36) is satisfied in an average sense.

The terms in ε^0 from Eqs. (49) and (51) give

$$(R^2 v_0')' + k_0^2 R^2 v_0 = 0 \quad \text{for } 0<z<1, \quad \text{with} \quad v_0'(0) = v_0'(1) = 0. \tag{52}$$

This is Webster's horn equation again, written as a regular Sturm-Liouville problem: it is an eigenvalue problem for k_0^2 and $v_0(z)$; we normalize the solution using

$$\int_0^1 \{v_0(z)R(z)\}^2 dz = 1. \tag{53}$$

The terms in ε^2 give

$$(R^2 v_1')' + k_0^2 R^2 v_1 = -k_2^2 R^2 v_0 + V, \tag{54}$$

where

$$V = \tfrac{1}{8}R^3(R v_0^{\mathrm{iv}} + 4R' v_0''' + 2k_0^2 R v_0'' + 4k_0^2 R' v_0' + k_0^4 R v_0)$$
$$= \tfrac{1}{4}R^2(R'R'' - RR''')v_0' - \tfrac{1}{2}R^3 R'' v_0'', \tag{55}$$

after using Eq. $(52)_1$. Similarly, Eq. (51) gives

$$v_1'(z) = -\tfrac{1}{4}RR'v_0'' = -\tfrac{1}{4}(RR'v_0')' \quad \text{at } z=0 \text{ and at } z=1. \tag{56}$$

Thus v_1 solves a forced version of Webster's horn equation with inhomogeneous boundary conditions. As the homogeneous form of Eq. (54) admits nontrivial solutions, Eq. (54)

will only have solutions if a consistency condition is satisfied. Thus, we multiply Eq. (54) by $v_0(z)$ and Eq. $(52)_1$ by $v_1(z)$, subtract the two, and integrate over $0\leqslant z\leqslant 1$. This gives

$$\left[-\frac{1}{4}R^2 v_0(RR'v_0')'\right]_{z=0}^{1}=-k_2^2+\int_0^1 V v_0\,dz,$$

where we have used Eqs. $(52)_2$, (53), and (56). In order to cancel the left-hand side, write $V=\{V+1/4[R^2(RR'v_0')']'\}-1/4[R^2(RR'v_0')']'$. Then, an integration by parts gives

$$k_2^2=\frac{1}{4}\int_0^1 R^2(RR'v_0')'v_0'\,dz+\int_0^1 W v_0\,dz, \tag{57}$$

where

$$\begin{aligned}W(z)&=V+\tfrac{1}{4}[R^2(RR'v_0')']'\\&=\tfrac{1}{4}R^3R'v_0'''+(RR')^2v_0''+\tfrac{1}{2}(3R^2R'R''+RR'^3)v_0'\end{aligned} \tag{58}$$

and we have used Eq. (55). Integrating the first integral in Eq. (57) by parts, using Eq. $(52)_2$, then gives

$$k_2^2=-\frac{1}{2}\int_0^1 (RR'v_0')^2\,dz+K,$$

where $K=\int_0^1(Wv_0-1/4R^3R'v_0''v_0')dz$. We are going to show that $K=0$.

Noting the first term in Eq. (58), and making use of Eq. $(52)_1$, we have

$$\begin{aligned}v_0'''v_0-v_0''v_0'&=-[k_0^2v_0'+(2R^{-1}R'v_0')']v_0\\&\quad+(k_0^2v_0+2R^{-1}R'v_0')v_0'\\&=2R^{-1}R'v_0'^2-2R^{-2}[(RR''-R'^2)v_0'\\&\quad+RR'v_0'']v_0.\end{aligned}$$

Hence, $K=1/2\int_0^1\{(RR'v_0')^2+[(RR')^2v_0']'v_0\}dz$, and this is seen to vanish after another integration by parts. Thus,

$$k^2=k_0^2-\frac{1}{2}\varepsilon^2\int_0^1 (RR'v_0')^2\,dz+O(\varepsilon^4)\quad\text{as }\varepsilon\to 0. \tag{59}$$

This elegant formula was derived in a different way by Geer and Keller (1983); see their Eq. (102).

B. Wavelength comparable to *a*

In the differential equations, Eqs. (39) and (43), put $k=\kappa/\varepsilon$ and

$$u_0(z)=E(z)w(z)\quad\text{with}\quad E(z)=\exp\left\{\frac{i}{\varepsilon}\int_{z_0}^{z}\Phi(t)\,dt\right\}, \tag{60}$$

where $\Phi(t)$ and $w(z)$ are to be determined, and z_0 is a constant. Suppose further that

$$w(z)=w_0(z)+\varepsilon w_1(z)+\varepsilon^2 w_2(z)+\cdots. \tag{61}$$

Then, we obtain the following approximations:

$$\begin{aligned}&u_0/E=w_0+\varepsilon w_1+O(\varepsilon^2),\quad u_0'/E=\varepsilon^{-1}i\Phi w_0+O(1),\\&u_0''/E=-\varepsilon^{-2}\Phi^2w_0+\varepsilon^{-1}(2i\Phi w_0'+i\Phi'w_0-\Phi^2w_1)\\&\qquad+O(1),\\&u_0'''/E=-\varepsilon^{-3}i\Phi^3w_0+O(\varepsilon^{-2}),\\&u_0^{\mathrm{iv}}/E=\varepsilon^{-4}\Phi^4w_0+\varepsilon^{-3}\Phi^2\{\Phi^2w_1-2i(3\Phi'w_0+2\Phi w_0')\}\\&\qquad+O(\varepsilon^{-2}),\end{aligned}$$

as $\varepsilon\to 0$.

Let us begin with the second-order equation, Eq. (39). We note that $D_m^{(1)}=O(1)$ and $E_m^{(1)}=\varepsilon^{-2}\mathcal{E}_m^{(1)}(\kappa,R)+O(1)$ as $\varepsilon\to 0$. Then, the terms in ε^{-2} give

$$[\Phi(z)]^2=\mathcal{E}_m^{(1)}(\kappa,R). \tag{62}$$

The terms in ε^{-1} give

$$2\Phi w_0'+\{\Phi'+\Phi D_m^{(1)}\}w_0=0,$$

which is a first-order differential equation for w_0. Rearranging gives

$$\frac{(w_0\Phi^{1/2})'}{w_0\Phi^{1/2}}=\frac{w_0'}{w_0}+\frac{\Phi'}{2\Phi}=-\frac{D_m^{(1)}}{2}=-q_m^{(1)}\frac{R'}{R},$$

where $q_m^{(1)}=(2-m^2)/(m+2)$. An integration gives

$$w_0^2\Phi R^{2q}=\text{constant}, \tag{63}$$

where w_0, Φ, and R are all functions of z only, and $q\equiv q_m^{(1)}$.

Next, consider the fourth-order equation, Eq. (43). Substituting as before, we find that

$$\begin{aligned}&B_m^{(2)}=O(1),\quad C_m^{(2)}=\varepsilon^{-2}\mathcal{C}_m(\kappa,R)+O(1),\\&D_m^{(2)}=\varepsilon^{-2}\mathcal{D}_m(\kappa,R)+O(1)\quad\text{and}\\&E_m^{(2)}=\varepsilon^{-4}\mathcal{E}_m^{(2)}(\kappa,R)+O(\varepsilon^{-2})\end{aligned}$$

as $\varepsilon\to 0$. Then, we see that the terms in ε^{-4} give

$$\Phi^4-\Phi^2\mathcal{C}_m(\kappa,R)+\mathcal{E}_m^{(2)}(\kappa,R)=0, \tag{64}$$

which should be compared with Eq. (22). Thus,

$$[\Phi(z)]^2=\mu_m^2(\kappa,R), \tag{65}$$

where μ_m is defined by Eq. (23).

The terms in ε^{-3} give

$$\begin{aligned}0&=\{\Phi^4-\Phi^2\mathcal{C}_m(\kappa,R)+\mathcal{E}_m^{(2)}(\kappa,R)\}w_1\\&\quad+2i\Phi w_0'\{\mathcal{C}_m(\kappa,R)-2\Phi^2\}+iw_0\{\Phi\mathcal{D}_m(\kappa,R)\\&\quad+\Phi'\mathcal{C}_m(\kappa,R)-\Phi^3B_m^{(2)}-6\Phi^2\Phi'\}.\end{aligned}$$

The factor multiplying w_1 vanishes, due to Eq. (64), leaving a first-order differential equation for $w_0(z)$

$$\frac{w_0'}{w_0}+\frac{\Phi^2\mathcal{D}_m(\kappa,R)+\Phi\Phi'\mathcal{C}_m(\kappa,R)-\Phi^4B_m^{(2)}-6\Phi^3\Phi'}{2\Phi^2[\mathcal{C}_m(\kappa,R)-2\Phi^2]}=0.$$

Rearranging this equation gives

$$\frac{(w_0\Phi^{1/2})'}{w_0\Phi^{1/2}}=\frac{4\Phi\Phi'-\mathcal{D}_m(\kappa,R)+\Phi^2B_m^{(2)}}{2[\mathcal{C}_m(\kappa,R)-2\Phi^2]}=-q_m^{(2)}\frac{R'}{R},$$

where

$$q_m^{(2)}=[4(m+2)(m^2-2)-m(m+3)\nu_m]/[(m+4)\chi_m].$$

Here, we have used $\Phi^2=\kappa^2-\nu_m/R^2$, and Eqs. (19), (23), (44), and (45). Hence, an integration gives Eq. (63) again, but with $q\equiv q_m^{(2)}$.

Let us compare our results (obtained by solving ordinary differential equations asymptotically) with those of Ting and Miksis (1983) and Geer and Keller (1983) (obtained by solving an elliptic boundary-value problem asymptotically). First, we obtained approximations for Φ, given by Eqs. (62) and (65), whereas the exact formula is

$$[\Phi(z)]^2=\kappa^2-[j'_{m,\ell}/R(z)]^2. \tag{66}$$

The errors incurred here are exactly the same as those discussed in Sec. IV for a circular tube (constant cross-section): We cannot expect to do any better for tubes of varying cross-section. Second, the analysis of Ting and Miksis (1983) and Geer and Keller (1983) leads to Eq. (63) with $q=1$ (for all m). If we take the minus sign in Eq. (24), we find that $q_0^{(2)}=1$ and $q_1^{(2)}=1.43$; it is not clear why this last number is not closer to 1, given that the corresponding values of Φ are close.

VII. CONCLUDING REMARKS

The method of Sec. V for waves in axisymmetric tubes leads to eigenvalue problems for ordinary differential equations. The simplest (first) approximation leads to a regular Sturm–Liouville problem. This is convenient because efficient software is readily available for solving such problems numerically (Pruess and Fulton, 1993). The next (second) approximation is expected to be more accurate, and, indeed, we have shown this in some asymptotic regimes. The second approximation leads to an eigenvalue problem for a fourth-order differential equation. However, it does not fall into the class of regular fourth-order Sturm-Liouville problems; see, for example, the review by Greenberg and Marletta (2000). In fact, we can say little about the theoretical properties of the simplest equation, namely Eq. (48), which models axisymmetric motions.

Evidently, higher-order approximations could be developed, leading to ordinary differential equations of order $2n$ with $n=3,4,...$; the necessary derivations would be expedited using software for symbolic manipulations. The method of Sec. V could also be extended to other cross-sections, using a scaling ρ that depends on the angle θ as well as on the longitudinal coordinate z.

Another possibility is to abandon power series in favor of Neumann series, which are series of Bessel functions of various orders. This would permit better representation of $u(r,z)$ for fixed z, but at the expense of additional complication.

Finally, the basic power-series method can be extended to various elastodynamic problems, generalizing the work of Boström (2000) on rods and of Boström, Johansson, and Olsson (2001) on plates; for axisymmetric motions in nonuniform anisotropic rods, see (Martin, 2004). Indeed, the fact that the power-series method is relatively insensitive to complications in the governing partial differential equations means that it may be worth developing further.

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