Available online at www.sciencedirect.com
SCIENCE DIRECT®
Wave Motion 40 (2004) 387–398
WAVE MOTION
www.elsevier.com/locate/wavemoti

Waves in wood: axisymmetric waves in slender solids of revolution

P.A. Martin*

Department of Mathematical and Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA

Received 15 December 2003; received in revised form 13 January 2004; accepted 19 January 2004

Abstract

Low-frequency axisymmetric waves in slender axisymmetric anisotropic columns are studied. The governing equation of motion is solved using power series with a stretched radial coordinate, leading to a variety of ordinary differential equations in the longitudinal direction. Detailed results for cylindrically orthotropic columns are given.

Keywords: Elastic waves; Frobenius method; Ordinary differential equations

1. Introduction

It is easy to formulate a mathematical problem representing the propagation of elastic waves along a solid waveguide: for time-harmonic waves, one has to solve a three-dimensional elliptic boundary-value problem for the displacement vector $\boldsymbol{u}$. However, this problem is difficult to solve exactly, unless the waveguide is a circular cylinder made from a homogeneous isotropic elastic solid; see, for example [1, Section 6.9] or [2, Section 8.2]. Consequently, many approximate theories have been developed. Here, we are interested in theories that lead to ordinary differential equations that govern the behaviour of some quantity along the waveguide.

To fix ideas, consider an axisymmetric waveguide, defined in cylindrical polar coordinates, $(\tilde{r}, \theta, \tilde{z})$, by

$$0 \leq \tilde{r} < a\tilde{R}(\tilde{z}), \quad 0 \leq \theta < 2\pi, \quad -\ell < \tilde{z} < \ell, \tag{1}$$

where $0 \leq \tilde{R}(\tilde{z}) \leq 1$, so that $2a$ is the maximum diameter of the waveguide, and 2ℓ is its length. (Later, we will use dimensionless versions of $\tilde{r}$ and $\tilde{z}$.) For simplicity, we call such a waveguide a *column* or a *bat* (because a baseball bat gives a good example). In the special case that $\tilde{R}(\tilde{z}) \equiv 1$, the column reduces to a circular cylinder or *rod*.

* Tel.: +1 303 273 3895; fax: +1 303 273 3875.
E-mail address: pamartin@mines.edu.

0165-2125/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.wavemoti.2004.07.001

For low-frequency motions, where the wavelength is long compared to a, we expect to obtain approximate solutions by assuming that $\boldsymbol{u}$ varies little over cross-sections of the column. For isotropic rods, such approximations have a long history; see, for example [3, Chapter 20], the 1960 review by Green [4], [1, Section 6.11] and [2]. However, the literature on columns and on anisotropic rods is sparse. Some asymptotic results for non-uniform isotropic elastic rods can be found in two papers by Rosenfeld and Keller [6,7], whereas anisotropy is included in the analysis of Tromp and Dahlen [8].

Recently, Boström [5] developed a systematic method for axisymmetric motions in isotropic rods. His method begins by seeking solutions of the equation of motion (Navier's equation) in the form

$$u(\tilde{r}, \tilde{z}) = \tilde{r}\, u_1(\tilde{z}) + \tilde{r}^3\, u_3(\tilde{z}) + \cdots,$$

$$w(\tilde{r}, \tilde{z}) = w_0(\tilde{z}) + \tilde{r}^2\, w_2(\tilde{z}) + \cdots,$$

where u and w are the radial and longitudinal components, respectively, of $\boldsymbol{u}$. Substitution then leads to a recursive structure, enabling all the higher unknown functions, $w_2, u_3, w_4, u_5, \ldots$ to be expressed in terms of the two unknown functions, $w_0(\tilde{z})$ and $u_1(\tilde{z})$, and their derivatives. Thus, solutions of Navier's equation are constructed for any choice of w_0 and u_1. These two functions are then obtained by imposing the lateral boundary condition on $\tilde{r} = a$. This condition reduces to power series in a and, for small a, this series may be truncated: doing this leads to ordinary differential equations for w_0 and u_1.

In the present paper, we generalize Boström's approach in two ways. First, we consider columns. To do this, we introduce a scaled radial variable, so that the column is mapped onto a rod. This mapping simplifies the geometry but at the expense of a more complicated system of partial differential equations; however, this complication is handled easily, as we are going to seek solutions in the form of power series in the new radial variable. Second, we consider anisotropic solids. Specifically, we consider materials with *cylindrical anisotropy*. We give results in detail for columns with cylindrical orthotropy, so that isotropy is recovered as a special case. This generalization requires the use of non-integer powers of $\tilde{r}$, as in the method of Frobenius. This method has been used by several authors for wave propagation in rods with cylindrical anisotropy [9–15].

The motivation for this work comes from a study of the vibrations of wooden poles and baseball bats. Wood can be modelled as a cylindrically orthotropic elastic solid. Bats are axisymmetric and slender: they have coaxial circular cross-sections.

2. Governing equations

Consider a wooden axisymmetric column of length 2ℓ. The interior of the bat is specified by (1). It will be convenient to define dimensionless variables, using a typical length scale L; one could choose $L = \ell$, but one may be interested in propagation problems where $\ell = \infty$, in which case one could choose L as a typical wavelength. Thus, we put

$$r = \frac{\tilde{r}}{L}, \quad z = \frac{\tilde{z}}{L}, \quad \tilde{R}(\tilde{z}) = R(z) \quad \text{and} \quad \varepsilon = \frac{a}{L}.$$

In this paper, we are concerned exclusively with axisymmetric motions, so that there is no variation with θ. Then, the governing equation of motion is

$$\frac{\partial}{\partial r}\left(r\tilde{\boldsymbol{t}}_r\right) + \mathsf{K}\tilde{\boldsymbol{t}}_\theta + r\frac{\partial}{\partial z}\tilde{\boldsymbol{t}}_z = \varrho_0 L^2 r \frac{\partial^2 \tilde{\boldsymbol{u}}}{\partial t^2}, \tag{2}$$

where ϱ_0 is the density

$$\tilde{\boldsymbol{t}}_r = \begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \\ \tau_{rz} \end{pmatrix}, \quad \tilde{\boldsymbol{t}}_\theta = \begin{pmatrix} \tau_{\theta r} \\ \tau_{\theta\theta} \\ \tau_{\theta z} \end{pmatrix}, \quad \tilde{\boldsymbol{t}}_z = \begin{pmatrix} \tau_{zr} \\ \tau_{z\theta} \\ \tau_{zz} \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\boldsymbol{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \tag{3}$$

is the displacement vector and τ_{ij} are the stress components. In what follows, we use a generalization of the matrix formulation of Ting [16] for static problems. From [16], we have the following expressions for the traction vectors $\tilde{\boldsymbol{t}}_i(r, z, t)$ in terms of $\tilde{\boldsymbol{u}}(r, z, t)$

$$\tilde{\boldsymbol{t}}_r = \mathsf{Q}\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathsf{RK}\tilde{\boldsymbol{u}} + \mathsf{P}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}}, \qquad \tilde{\boldsymbol{t}}_\theta = \mathsf{R}^{\mathrm{T}}\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathsf{TK}\tilde{\boldsymbol{u}} + \mathsf{S}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}}, \tag{4}$$

$$\tilde{\boldsymbol{t}}_z = \mathsf{P}^{\mathrm{T}}\frac{\partial}{\partial r}\tilde{\boldsymbol{u}} + \frac{1}{r}\mathsf{S}^{\mathrm{T}}\mathsf{K}\tilde{\boldsymbol{u}} + \mathsf{M}\frac{\partial}{\partial z}\tilde{\boldsymbol{u}}. \tag{5}$$

In these expressions

$$\mathsf{Q} = \begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{pmatrix}, \quad \mathsf{R} = \begin{pmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{pmatrix},$$

$$\mathsf{P} = \begin{pmatrix} C_{15} & C_{14} & C_{13} \\ C_{56} & C_{46} & C_{36} \\ C_{55} & C_{45} & C_{35} \end{pmatrix}, \quad \mathsf{M} = \begin{pmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{pmatrix}, \quad \mathsf{S} = \begin{pmatrix} C_{56} & C_{46} & C_{36} \\ C_{25} & C_{24} & C_{23} \\ C_{45} & C_{44} & C_{34} \end{pmatrix}.$$

R^{T} is the transpose of R, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses with $(1, 2, 3) = (r, \theta, z)$. Note that Q, T and M are symmetric matrices.

We look for time-harmonic solutions of (2) in the form

$$\tilde{\boldsymbol{u}}(r, z, t) = \boldsymbol{u}(r, z)\,\mathrm{e}^{-\mathrm{i}\omega t}, \tag{6}$$

with similar expressions for $\tilde{\boldsymbol{t}}_i$, where ω is the radian frequency. We find that $\boldsymbol{u}(r, z)$ solves

$$r\mathsf{Q}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{u}}{\partial r}\right) + r(\mathsf{RK} + \mathsf{KR}^{\mathrm{T}})\frac{\partial \boldsymbol{u}}{\partial r} + r^2(\mathsf{P} + \mathsf{P}^{\mathrm{T}})\frac{\partial^2 \boldsymbol{u}}{\partial r\,\partial z} + r^2\mathsf{M}\frac{\partial^2 \boldsymbol{u}}{\partial z^2} + r(\mathsf{P} + \mathsf{KS} + \mathsf{S}^{\mathrm{T}}\mathsf{K})\frac{\partial \boldsymbol{u}}{\partial z}$$
$$+ \{\varrho_0(\omega L r)^2\mathsf{I} + \mathsf{KTK}\}\boldsymbol{u} = \boldsymbol{0}, \tag{7}$$

where I is the identity. From (4) and (5), we also have

$$\boldsymbol{t}_r = \mathsf{Q}\frac{\partial \boldsymbol{u}}{\partial r} + \frac{1}{r}\mathsf{RK}\boldsymbol{u} + \mathsf{P}\frac{\partial \boldsymbol{u}}{\partial z}, \tag{8}$$

$$\boldsymbol{t}_z = \mathsf{P}^{\mathrm{T}}\frac{\partial \boldsymbol{u}}{\partial r} + \frac{1}{r}\mathsf{S}^{\mathrm{T}}\mathsf{K}\boldsymbol{u} + \mathsf{M}\frac{\partial \boldsymbol{u}}{\partial z}. \tag{9}$$

These equations were studied in [14]. For two-dimensional motions independent of z, we recover the equations studied in [13]. (This paper is mainly concerned with non-axisymmetric motions.) If we also put $\omega = 0$ (static), we obtain the equations solved by Ting [16].

Setting $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$, (7) gives three coupled partial differential equations for the three components of $\boldsymbol{u}$. In general, these equations do not decouple.

The lateral boundary of the bat is free from tractions, whence

$$\boldsymbol{t}_r - \varepsilon R'(z)\,\boldsymbol{t}_z = \boldsymbol{0} \quad \text{on } r = \varepsilon R(z). \tag{10}$$

In this paper, we consider columns of infinite length. For finite-length columns, it is natural to assume that $\tilde{R}(\pm\ell) > 0$, and that the two flat ends of the bat are also free from tractions, whence $\boldsymbol{t}_z = \boldsymbol{0}$ at $\tilde{z} = \pm\ell$. The application of these conditions leads to an eigenvalue problem for the frequencies of free vibration of the column.

3. Reformulation

Let us make a simple change of the independent variables, from (r, z) to (ρ, ζ), so that the new geometry is a circular cylinder. The price of this change is that the new governing partial differential equations are more complicated, but it has the virtue that the lateral boundary condition is to be applied on a coordinate surface.

Thus, define new variables ρ and ζ by

$$\rho = \frac{r}{R(z)} \quad \text{and} \quad \zeta = z, \tag{11}$$

so that the bat is mapped onto the circular pole, given by

$$0 \le \rho < \varepsilon, \quad 0 < \zeta < 1.$$

(Later, we will use z in place of ζ, but it is clearer to distinguish the two variables at this stage.) The chain rule gives

$$\frac{\partial u}{\partial r} = \frac{1}{R}\frac{\partial u}{\partial \rho}, \qquad \frac{\partial^2 u}{\partial r^2} = \frac{1}{R^2}\frac{\partial^2 u}{\partial \rho^2}, \qquad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial \zeta} - \rho\frac{R'}{R}\frac{\partial u}{\partial \rho},$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial \zeta^2} - 2\rho\frac{R'}{R}\frac{\partial^2 u}{\partial \rho\,\partial \zeta} - \rho\frac{\partial u}{\partial \rho}\left(\frac{R'}{R}\right)' + \rho\left(\frac{R'}{R}\right)^2\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right), \qquad \frac{\partial^2 u}{\partial r\,\partial z} = \frac{1}{R}\frac{\partial^2 u}{\partial \rho\,\partial \zeta} - \frac{R'}{R^2}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right).$$

Hence, (7) becomes

$$\begin{aligned}&\{\mathsf{Q} - \rho R'(\mathsf{P} + \mathsf{P}^{\mathrm{T}}) + \rho^2 R'^2\mathsf{M}\}\rho\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \boldsymbol{u}}{\partial \rho}\right) + \{\mathsf{RK} + \mathsf{KR}^{\mathrm{T}} - \rho R'(\mathsf{P} + \mathsf{KS} + \mathsf{S}^{\mathrm{T}}\mathsf{K})\\ &\quad + \rho^2(R'^2 - RR'')\mathsf{M}\}\rho\frac{\partial \boldsymbol{u}}{\partial \rho} + \{\mathsf{P} + \mathsf{P}^{\mathrm{T}} - 2\rho R'\mathsf{M}\}\rho^2 R\frac{\partial^2 \boldsymbol{u}}{\partial \rho\,\partial \zeta} + \rho^2 R^2\mathsf{M}\frac{\partial^2 \boldsymbol{u}}{\partial \zeta^2} + \{\mathsf{P} + \mathsf{KS} + \mathsf{S}^{\mathrm{T}}\mathsf{K}\}\rho R\frac{\partial \boldsymbol{u}}{\partial \zeta}\\ &\quad + \{\varrho_0(\omega L R\rho)^2\mathsf{I} + \mathsf{KTK}\}\boldsymbol{u} = \boldsymbol{0}.\end{aligned} \tag{12}$$

This equation reduces to (7) when $R(z) \equiv 1$. Also, (8) and (9) give

$$\boldsymbol{t}_r = \mathsf{P}\frac{\partial \boldsymbol{u}}{\partial \zeta} + (\mathsf{Q} - \rho R'\mathsf{P})\frac{1}{R}\frac{\partial \boldsymbol{u}}{\partial \rho} + \frac{1}{\rho R}\mathsf{RK}\boldsymbol{u}, \tag{13}$$

$$\boldsymbol{t}_z = \mathsf{M}\frac{\partial \boldsymbol{u}}{\partial \zeta} + (\mathsf{P}^{\mathrm{T}} - \rho R'\mathsf{M})\frac{1}{R}\frac{\partial \boldsymbol{u}}{\partial \rho} + \frac{1}{\rho R}\mathsf{S}^{\mathrm{T}}\mathsf{K}\boldsymbol{u}; \tag{14}$$

these can be used to express the lateral boundary conditions (10) in terms of the new variables.

4. The method of Frobenius

To solve (12), we write

$$\boldsymbol{u}(\rho, \zeta) = \sum_{n=0}^{\infty} \rho^{n+\alpha} \boldsymbol{u}_n(\zeta). \tag{15}$$

Substitution in (12) gives

$$\mathbf{0} = \sum_{n=0}^{\infty} \rho^{n+\alpha} \mathsf{G}_n(\alpha)\, \boldsymbol{u}_n + \sum_{n=1}^{\infty} \rho^{n+\alpha} \mathring{A}_n(\alpha)\{R\boldsymbol{u}'_{n-1} - (n-1+\alpha)R'\boldsymbol{u}_{n-1}\} + \sum_{n=2}^{\infty} \rho^{n+\alpha} \boldsymbol{g}_{n-2}(\alpha), \tag{16}$$

where

$$\begin{aligned} \mathsf{G}_n(\alpha) &= (n+\alpha)^2\mathsf{Q} + (n+\alpha)(\mathsf{RK} + \mathsf{KR}^{\mathrm{T}}) + \mathsf{KTK}, \\ \mathring{A}_n(\alpha) &= (n-1+\alpha)(\mathsf{P} + \mathsf{P}^{\mathrm{T}}) + \mathsf{P} + \mathsf{KS} + \mathsf{S}^{\mathrm{T}}\mathsf{K}, \\ \boldsymbol{g}_{n-2}(\alpha) &= (n-2+\alpha)\{(n-1+\alpha)R'^2 - RR''\}\mathsf{M}\boldsymbol{u}_{n-2} - 2RR'(n-2+\alpha)\mathsf{M}\boldsymbol{u}'_{n-2} \\ &\quad + R^2\mathsf{M}\boldsymbol{u}''_{n-2} + \varrho_0(\omega L R)^2\boldsymbol{u}_{n-2}. \end{aligned} \tag{17}$$

For (16) to be satisfied, we must first have

$$\mathsf{G}_0(\alpha)\, \boldsymbol{u}_0(\zeta) = \mathbf{0}. \tag{18}$$

The terms with $n = 1$ give

$$\mathsf{G}_1(\alpha)\, \boldsymbol{u}_1(\zeta) + \mathring{A}_1(\alpha)\{R\boldsymbol{u}'_0 - \alpha R'\boldsymbol{u}_0\} = \mathbf{0}. \tag{19}$$

Subsequent terms give

$$\mathsf{G}_n(\alpha)\, \boldsymbol{u}_n(\zeta) + \mathring{A}_n(\alpha)\{R\boldsymbol{u}'_{n-1} - (n-1+\alpha)R'\boldsymbol{u}_{n-1}\} + \boldsymbol{g}_{n-2}(\alpha) = \mathbf{0}, \tag{20}$$

for $n \geq 2$. Eq. (18) is the *indicial equation*. It has a non-trivial solution provided that

$$\det \mathsf{G}_0(\alpha) = \det\{\alpha^2\mathsf{Q} + \alpha(\mathsf{RK} + \mathsf{KR}^{\mathrm{T}}) + \mathsf{KTK}\} = 0. \tag{21}$$

This equation determines α; in general, there are six solutions. For each allowable α, (18) then determines the form of (the eigenvector) $\boldsymbol{u}_0$.

Ting [16] has investigated (21) in detail in his study of related static problems; our G_0 is his Γ. He showed that

$$\det \mathsf{G}_0(\alpha) = \alpha^2(\alpha^2 - 1)(\alpha^2|\mathsf{Q}| - |\mathsf{Y}|), \tag{22}$$

where $|\mathsf{Q}| = \det \mathsf{Q}$, $|\mathsf{Y}| = \det \mathsf{Y}$ and

$$\mathsf{Y} = \begin{pmatrix} C_{22} & C_{26} & C_{25} \\ C_{26} & C_{66} & C_{56} \\ C_{25} & C_{56} & C_{55} \end{pmatrix}.$$

Ting [16] has also given a complete discussion of the associated eigenvectors. If we know α and $\boldsymbol{u}_0$, (19) may be used to determine $\boldsymbol{u}_1$, with $\boldsymbol{u}_n$ determined by (20) for $n \geq 2$. This will give solutions of the equation of motion.

To determine $\boldsymbol{u}_0$, we impose the lateral boundary condition. Thus, substituting (15) in (13) and (14), the boundary condition (10) on $\rho = \varepsilon$ gives

$$\sum_{n=0}^{\infty} \varepsilon^{n+\alpha} \boldsymbol{b}_n(\alpha) = \boldsymbol{0}, \tag{23}$$

where

$$\boldsymbol{b}_0(\alpha) = (\alpha\mathsf{Q} + \mathsf{RK})\boldsymbol{u}_0, \tag{24}$$

$$\boldsymbol{b}_1(\alpha) = \{(\alpha + 1)\mathsf{Q} + \mathsf{RK}\}\boldsymbol{u}_1 + R\mathsf{P}\boldsymbol{u}_0' - R'\{\alpha(\mathsf{P} + \mathsf{P}^{\mathrm{T}}) + \mathsf{S}^{\mathrm{T}}\mathsf{K}\}\boldsymbol{u}_0, \tag{25}$$

$$\begin{aligned}\boldsymbol{b}_n(\alpha) = {} & \{(n + \alpha)\mathsf{Q} + \mathsf{RK}\}\boldsymbol{u}_n + R\mathsf{P}\boldsymbol{u}_{n-1}' - R'\{(n - 1 + \alpha)(\mathsf{P} + \mathsf{P}^{\mathrm{T}}) + \mathsf{S}^{\mathrm{T}}\mathsf{K}\}\boldsymbol{u}_{n-1} \\ & + R'\mathsf{M}\{(n - 2 + \alpha)R'\boldsymbol{u}_{n-2} - R\boldsymbol{u}_{n-2}'\}, \quad n \geq 2.\end{aligned} \tag{26}$$

From the calculations above, we know (in principle, at least) how to express $\boldsymbol{u}_n$ in terms of $\boldsymbol{u}_m$ with $0 \leq m < n$; doing this ensures that the governing equation of motion is satisfied. Then, our strategy is to truncate (23) in order to satisfy the lateral boundary condition, approximately. This will lead to ordinary differential equations. In fact, in order to obtain the simplest, non-trivial, frequency-dependent results, we shall truncate (23) at $n = 2$, giving

$$\boldsymbol{b}_0(\alpha) + \varepsilon\boldsymbol{b}_1(\alpha) + \varepsilon^2\boldsymbol{b}_2(\alpha) = \boldsymbol{0}. \tag{27}$$

Evidently, more terms could be included if desired. We could also sum over allowable α in (23); see Section 7.

Let us investigate some of the simplest approximations. First, we are only interested in $\alpha \geq 0$, as we want solutions that are bounded at $\rho = 0$. Second, when there are repeated roots, such as $\alpha = 0$, we do not consider solutions involving $\log \rho$. As various special cases can arise, we choose to consider materials with cylindrical orthotropy. This is a plausible model for wood, and includes isotropic materials as a special case.

5. Cylindrical orthotropy

For materials with cylindrical orthotropy, there are nine non-trivial stiffnesses, namely C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55} and C_{66}. The matrices Q, R, T, P, M and S simplify to

$$\mathsf{Q} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{55} \end{pmatrix}, \quad \mathsf{R} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} C_{66} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}, \quad \mathsf{P} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & 0 \\ C_{55} & 0 & 0 \end{pmatrix},$$

$$\mathsf{M} = \begin{pmatrix} C_{55} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \quad \mathsf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{23} \\ 0 & C_{44} & 0 \end{pmatrix},$$

and the system (7) simplifies accordingly. For example, we find that the torsional component $v(r, z)$ decouples so that (7) then has the exact solution $\boldsymbol{u} = (0, v, 0)^{\mathrm{T}}$ with

$$v(r, z) = J_1(pr)\,\mathrm{e}^{\mathrm{i}\xi z}. \tag{28}$$

Here, $p = [\varrho_0(\omega L)^2 - \xi^2 C_{44}]/C_{66}$ and J_1 is a Bessel function. When $C_{44} = C_{66}$, (28) reduces to a solution found in [17, p. 172].

Note that isotropy is a special case of cylindrical orthotropy. For isotropic materials, $C_{11} = C_{22} = C_{33} = \lambda + 2\mu$, $C_{12} = C_{13} = C_{23} = \lambda$ and $C_{44} = C_{55} = C_{66} = \mu$, where λ and μ are the Lamé moduli. Exact solutions of (7) are well known for isotropic solids; see, for example [2, Section 8.2].

Elementary calculations give $\mathsf{RK} = -\mathsf{KR}^{\mathrm{T}}$,

$$\mathsf{G}_n(\alpha) = \mathrm{diag}\{(n+\alpha)^2 C_{11} - C_{22}, [(n+\alpha)^2 - 1]C_{66}, (n+\alpha)^2 C_{55}\}, \tag{29}$$

$$\alpha\mathsf{Q} + \mathsf{RK} = \mathrm{diag}\{\alpha C_{11} + C_{12}, (\alpha - 1)C_{66}, \alpha C_{55}\},$$

$$\mathsf{P} + \mathsf{P}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & C_{13}+C_{55} \\ 0 & 0 & 0 \\ C_{13}+C_{55} & 0 & 0 \end{pmatrix}, \qquad \mathsf{S}^{\mathrm{T}}\mathsf{K} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{23} & 0 & 0 \end{pmatrix}, \qquad \mathring{A}_n(\alpha) = \begin{pmatrix} 0 & 0 & \gamma_n(\alpha) \\ 0 & 0 & 0 \\ \delta_n(\alpha) & 0 & 0 \end{pmatrix},$$

and $|\mathsf{Y}|/|\mathsf{Q}| = C_{22}/C_{11}$, with

$$\gamma_n(\alpha) = (n - 1 + \alpha)(C_{13} + C_{55}) + C_{13} - C_{23}, \qquad \delta_n(\alpha) = (n - 1 + \alpha)(C_{13} + C_{55}) + C_{23} + C_{55}.$$

6. Some ordinary differential equations

We shall investigate three choices for the parameter α: $\alpha = 0$, $\alpha = 1$ and $\alpha = (C_{22}/C_{11})^{1/2}$. These are the three non-negative solutions of (21). The solutions obtained will be summarized in Section 8.

6.1. *Case 1: $\alpha = 0$*

Writing $\boldsymbol{u}_0 = (u_0, v_0, w_0)^{\mathrm{T}}$, and noting that $\mathsf{G}_0(0) = \mathrm{diag}\{-C_{22}, -C_{66}, 0\}$, (18) (with $\alpha = 0$) gives

$$\boldsymbol{u}_0(\zeta) = (0, 0, w_0(\zeta))^{\mathrm{T}}, \tag{30}$$

for some scalar function $w_0(\zeta)$. Then, (19) gives

$$\mathsf{G}_1(0)\,\boldsymbol{u}_1 = -R\mathring{A}_1(0)\,\boldsymbol{u}_0'. \tag{31}$$

But $\mathsf{G}_1(0) = \mathsf{G}_0(1)$ is singular, as, from (22), $\alpha = 1$ solves (21). (An eigenvector satisfying $\mathsf{G}_1(0)\boldsymbol{a} = \boldsymbol{0}$ is $\boldsymbol{a} = (0, 1, 0)^{\mathrm{T}}$.) Thus, we can only solve (31) if a certain consistency condition is satisfied and, moreover, the solution cannot be unique. If we write (31) explicitly, using $\boldsymbol{u}_1 = (u_1, v_1, w_1)^{\mathrm{T}}$ and (30), we obtain

$$\begin{pmatrix} C_{11} - C_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_{55} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = -Rw_0' \begin{pmatrix} C_{13} - C_{23} \\ 0 \\ 0 \end{pmatrix},$$

whence v_1 is arbitrary, $w_1 = 0$ and

$$(C_{11} - C_{22})u_1 = -R(C_{13} - C_{23})w_0'. \tag{32}$$

This equation determines u_1 when $C_{11} \neq C_{22}$. If $C_{11} = C_{22}$, u_1 is arbitrary but $(C_{13} - C_{23})w_0' = 0$, so that we must then have $C_{13} = C_{23}$ too; this case includes isotropic solids.

Next, we consider (20). As $\mathsf{G}_n(0) = \mathsf{G}_0(n)$, we deduce that $\mathsf{G}_n(0)$ is non-singular for $n \geq 2$ (unless $n^2 = C_{22}/C_{11}$). Hence, (20) can be used to express $\boldsymbol{u}_n(\zeta)$ in terms of $w_0(\zeta)$, $v_1(\zeta)$ and their derivatives. In particular, we have

$$\mathsf{G}_2(0)\,\boldsymbol{u}_2 + \mathring{A}_2(0)\{R\boldsymbol{u}_1' - R'\boldsymbol{u}_1\} + \boldsymbol{g}_0(0) = \boldsymbol{0}, \tag{33}$$

where $\boldsymbol{g}_0(0) = R^2\mathsf{M}\boldsymbol{u}_0'' + \varrho_0(\omega LR)^2\boldsymbol{u}_0$. (Notice that a dependence on the frequency ω enters via $\boldsymbol{g}_0$.) Writing $\boldsymbol{u}_2 = (u_2, v_2, w_2)^{\mathrm{T}}$, we obtain $(4C_{11} - C_{22})u_2 = 0$, $v_2 = 0$ and

$$4C_{55}w_2 + (C_{13} + C_{23} + 2C_{55})(Ru_1' - R'u_1) + R^2C_{33}w_0'' + \varrho_0(\omega LR)^2w_0 = 0. \tag{34}$$

From (30), we have $\mathsf{K}\boldsymbol{u}_0 = \boldsymbol{0}$ so that (24) gives $\boldsymbol{b}_0(0) = \boldsymbol{0}$. Also, (25) and (26) give

$$\begin{aligned} \boldsymbol{b}_1(0) &= [(C_{11} + C_{12})u_1 + C_{13}Rw_0',\ 0,\ 0]^{\mathrm{T}}, \\ \boldsymbol{b}_2(0) &= [0,\ 0,\ 2C_{55}w_2 + C_{55}Ru_1' - (C_{13} + C_{23} + C_{55})R'u_1 - C_{33}RR'w_0']^{\mathrm{T}}, \end{aligned} \tag{35}$$

where we have assumed that $C_{22} \neq 4C_{11}$ so that $u_2 = 0$. Hence, (27) gives

$$(C_{11} + C_{12})u_1 + C_{13}Rw_0' = 0 \tag{36}$$

and

$$2C_{55}w_2 + C_{55}Ru_1' - (C_{13} + C_{23} + C_{55})R'u_1 - C_{33}RR'w_0' = 0. \tag{37}$$

6.1.1. $C_{11} \neq C_{22}$

Suppose first that $C_{11} \neq C_{22}$. Then, (32) and (36) reduce to

$$\{(C_{11} + C_{12})C_{23} - (C_{12} + C_{22})C_{13}\}Rw_0' = 0,$$

so that we can only obtain non-trivial solutions when the stiffnesses satisfy

$$(C_{11} + C_{12})C_{23} = (C_{12} + C_{22})C_{13}. \tag{38}$$

Then, (32), (34) and (37) give

$$\mathcal{A}_0(R^2w_0'' + 2RR'w_0') + \varrho_0(\omega LR)^2w_0 = 0. \tag{39}$$

where $\mathcal{A}_0 = C_{33} + (C_{23}^2 - C_{13}^2)/(C_{11} - C_{22})$.

6.1.2. $C_{11} = C_{22}$

Suppose instead that $C_{11} = C_{22}$ and $C_{13} = C_{23}$. Then, (36) *defines* u_1 if $C_{11} + C_{12} \neq 0$, and (37) gives

$$[(C_{11} + C_{12})C_{33} - 2C_{13}^2](R^2w_0'' + 2RR'w_0') + \varrho_0(C_{11} + C_{12})(\omega LR)^2w_0 = 0. \tag{40}$$

In particular, for isotropic solids, this equation reduces to

$$R^2w_0'' + 2RR'w_0' + \left(\frac{\varrho_0}{E}\right)(\omega LR)^2w_0 = 0, \tag{41}$$

where $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is Young's modulus. Specialising further to cylinders, with $R \equiv 1$, we obtain

$$w_0'' + \left(\frac{\varrho_0}{E}\right)(\omega L)^2w_0 = 0, \tag{42}$$

Eq. (42) is well known: it is used to model the propagation of longitudinal waves in thin cylindrical rods [2, Section 2.1]. The wavespeed $\sqrt{E/\varrho_0}$ is known as the *bar velocity*. Eq. (41) is also known; see [2, Section 2.5.1]. In fact, (41) is an elastodynamic variant of *Webster's horn equation* [18, p. 360].

6.2. Case 2: $\alpha = 1$

Let us make similar calculations for $\alpha = 1$. From (18)–(20) (with $\alpha = 1$), we obtain

$$(C_{11} - C_{22})u_0 = 0, \qquad v_0 \text{ is arbitrary}, \qquad w_0 = 0, \tag{43}$$

$$u_1 = v_1 = 0, \qquad 4C_{55}w_1 + (Ru_0' - R'u_0)\delta_1(1) = 0, \tag{44}$$

$$(9C_{11} - C_{22})u_2 + (Rw_1' - 2R'w_1)\gamma_2(1) + \varrho_0(\omega LR)^2 u_0 + C_{55}\{R^2u_0'' - 2RR'u_0' + (2R'^2 - RR'')u_0\} = 0, \tag{45}$$

$$8C_{66}v_2 + C_{44}\{(2R'^2 - RR'')v_0 - 2RR'v_0' + R^2v_0''\} + \varrho_0(\omega LR)^2 v_0 = 0, \tag{46}$$

and $w_2 = 0$, where we have assumed that $C_{22} \neq 4C_{11}$ in order to deduce that $u_1 = 0$. Similarly, from (24)–(26) (with $\alpha = 1$), we obtain

$$\boldsymbol{b}_0(1) = [(C_{11} + C_{12})u_0,\ 0,\ 0]^{\mathrm{T}}, \tag{47}$$

$$\boldsymbol{b}_1(1) = [0,\ 0,\ 2C_{55}w_1 + RC_{55}u_0' - R'(C_{13} + C_{23} + C_{55})u_0]^{\mathrm{T}}, \tag{48}$$

$$\begin{aligned}\boldsymbol{b}_2(1) = [&(3C_{11} + C_{12})u_2 + RC_{13}w_1' - 2R'(C_{13} + C_{55})w_1 + R'C_{55}(R'u_0 - Ru_0'), 2C_{66}v_2 \\ &+ R'C_{44}(R'v_0 - Rv_0'),\ 0]^{\mathrm{T}}.\end{aligned} \tag{49}$$

We see immediately that only $\boldsymbol{b}_2(1)$ contributes to the torsional (second) component of (27); setting this component to zero, using (46) and (49), gives an equation for the torsional component v_0

$$R^2v_0'' + 2RR'v_0' - (2R'^2 + RR'')v_0 + (\varrho_0/C_{44})(\omega LR)^2 v_0 = 0. \tag{50}$$

In particular, for cylinders (with $R \equiv 1$), we obtain

$$v_0'' + (\varrho_0/C_{44})(\omega L)^2 v_0 = 0.$$

This equation is well known for isotropic rods [2, Section 2.6.1].

6.2.1. $C_{11} \neq C_{22}$

From (43), we obtain $u_0 = 0$ when $C_{11} \neq C_{22}$. It follows that $w_1 = 0$ and $u_2 = 0$, so that only the trivial solution is obtained for this case.

6.2.2. $C_{11} = C_{22}$

In this sub-case, $\alpha = 1$ is a double root of (21). From (43), we see that u_0 is arbitrary, and then w_1 and u_2 are defined by (44) and (45), respectively; explicitly, we obtain

$$32C_{11}C_{55}u_2 = \Lambda\{R^2u_0'' - 2RR'u_0' + (2R'^2 - RR'')u_0\} - 4\varrho_0C_{55}(\omega LR)^2u_0,$$

where

$$\Lambda = \delta_1(1)\,\gamma_2(1) - 4C_{55}^2 = 8C_{13}C_{55} + (C_{13} + C_{23})(3C_{13} - C_{23}).$$

Then, the radial (first) component of (27) gives

$$R^2\mathcal{A}u_0'' - 2RR'\mathcal{B}u_0' + \mathcal{C}u_0 = 0, \tag{51}$$

where $\mathcal{A}$ and $\mathcal{B}$ are known constants, defined by

$$\mathcal{A} = 8C_{13}C_{55}(C_{11}+C_{12}) + \mathcal{D}, \qquad \mathcal{B} = 8C_{55}(C_{12}C_{13} - C_{11}C_{23}) + \mathcal{D},$$
$$\mathcal{D} = (C_{13}+C_{23})\{C_{11}C_{13} - C_{12}C_{23} + 3(C_{12}C_{13} - C_{11}C_{23})\},$$

and $\mathcal{C}$ is a known function

$$\mathcal{C}(z) = 2\mathcal{B}R'^2 - \mathcal{A}RR'' + 32\varepsilon^{-2}C_{11}C_{55}(C_{11}+C_{12}) - 4\varrho_0 C_{55}(3C_{11}+C_{12})(\omega LR)^2.$$

Eq. (51) simplifies for cylinders and for isotropic solids. In the latter case, we obtain

$$(\lambda+2\mu)R^2u_0'' + 6\mu RR'u_0' - \{(\lambda+2\mu)RR'' + 6\mu R'^2\}u_0$$
$$+ \lambda^{-1}\{8\varepsilon^{-2}(\lambda+\mu)(\lambda+2\mu) - \varrho_0(2\lambda+3\mu)(\omega LR)^2\}u_0 = 0$$

and, simplifying further to $R \equiv 1$

$$(\lambda+2\mu)u_0'' + \lambda^{-1}\{8\varepsilon^{-2}(\lambda+\mu)(\lambda+2\mu) - \varrho_0(2\lambda+3\mu)(\omega L)^2\}u_0 = 0.$$

6.3. Case 3: $\alpha = (C_{22}/C_{11})^{1/2}$

Suppose that $\alpha = (C_{22}/C_{11})^{1/2} = \tilde{\alpha}$, say. We assume that $\tilde{\alpha} \neq 1$ (so that isotropy is excluded). Note that $(n+\tilde{\alpha})^2C_{11} - C_{22} = n(n+2\tilde{\alpha})C_{11}$, which simplifies the first entry in $\mathsf{G}_n(\tilde{\alpha})$; see (29). From (18)–(20) (with $\alpha = \tilde{\alpha}$), we obtain $v_0 = w_0 = u_1 = v_1 = v_2 = w_2 = 0$

$$(\tilde{\alpha}+1)^2C_{55}w_1 + (Ru_0' - \tilde{\alpha}R'u_0)\delta_1(\tilde{\alpha}) = 0,$$
$$4(\tilde{\alpha}+1)C_{11}u_2 + \{Rw_1' - (\tilde{\alpha}+1)R'w_1\}\gamma_2(\tilde{\alpha}) + \varrho_0(\omega LR)^2u_0$$
$$+ C_{55}\{R^2u_0'' - 2RR'\tilde{\alpha}u_0' + \tilde{\alpha}[(\tilde{\alpha}+1)R'^2 - RR'']u_0\} = 0. \tag{52}$$

Eliminating w_1 between the last two equations gives

$$4(\tilde{\alpha}+1)^3C_{11}C_{55}u_2 = \tilde{\Lambda}\{R^2u_0'' - 2RR'\tilde{\alpha}u_0' + \tilde{\alpha}[(\tilde{\alpha}+1)R'^2 - RR'']u_0\} - \varrho_0(\tilde{\alpha}+1)^2C_{55}(\omega LR)^2u_0 = 0,$$

where

$$\tilde{\Lambda} = \delta_1(\tilde{\alpha})\gamma_2(\tilde{\alpha}) - (\tilde{\alpha}+1)^2C_{55}^2 = 2(\tilde{\alpha}+1)^2C_{13}C_{55} + (\tilde{\alpha}C_{13}+C_{23})[(\tilde{\alpha}+2)C_{13} - C_{23}].$$

Eqs. (24)–(26) (with $\alpha = \tilde{\alpha}$) give

$$\boldsymbol{b}_0(\tilde{\alpha}) = [(\tilde{\alpha}C_{11}+C_{12})u_0,\ 0,\ 0]^{\mathrm{T}}, \tag{53}$$

$$\boldsymbol{b}_1(\tilde{\alpha}) = [0,\ 0,\ (\tilde{\alpha}+1)C_{55}w_1 + RC_{55}u_0' - R'(\tilde{\alpha}C_{13}+C_{23}+\tilde{\alpha}C_{55})u_0]^{\mathrm{T}}, \tag{54}$$

$$\boldsymbol{b}_2(\tilde{\alpha}) = [\{(\tilde{\alpha}+2)C_{11}+C_{12}\}u_2 + RC_{13}w_1' - R'(\tilde{\alpha}+1)(C_{13}+C_{55})w_1 + R'C_{55}(\tilde{\alpha}R'u_0 - Ru_0'),\ 0,\ 0]^{\mathrm{T}}. \tag{55}$$

Then, the radial component of (27) gives

$$R^2\tilde{\mathcal{A}}u_0'' - 2RR'\tilde{\mathcal{B}}u_0' + \tilde{\mathcal{C}}u_0 = 0, \tag{56}$$

where $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are known constants, defined by

$$\tilde{\mathcal{A}} = 2(\tilde{\alpha}+1)^2 C_{13}C_{55}(\tilde{\alpha}C_{11} + C_{12}) + \tilde{\mathcal{D}},$$

$$\tilde{\mathcal{B}} = 2(\tilde{\alpha}+1)^2 C_{55}(\tilde{\alpha}(\tilde{\alpha}-1)C_{11}C_{13} + \tilde{\alpha}C_{12}C_{13} - C_{11}C_{23}) + \tilde{\alpha}\tilde{\mathcal{D}},$$

$$\tilde{\mathcal{D}} = (\tilde{\alpha}C_{13} + C_{23})\{\tilde{\alpha}^2 C_{11}C_{13} - C_{12}C_{23} + (\tilde{\alpha}+2)(C_{12}C_{13} - C_{11}C_{23})\},$$

and $\tilde{\mathcal{C}}$ is a known function

$$\tilde{\mathcal{C}}(z) = \tilde{\mathcal{C}}_1 R'^2 - \tilde{\alpha}\tilde{\mathcal{A}}RR'' + \tilde{\mathcal{C}}_2,$$

$$\tilde{\mathcal{C}}_1 = 2\tilde{\alpha}(\tilde{\alpha}+1)^2 C_{55}\{\tilde{\alpha}(\tilde{\alpha}-1)C_{11}C_{13} + (\tilde{\alpha}+1)C_{12}C_{13} - 2C_{11}C_{23}\} + \tilde{\alpha}(\tilde{\alpha}+1)\tilde{\mathcal{D}},$$

$$\tilde{\mathcal{C}}_2 = (\tilde{\alpha}+1)^2 C_{55}\{4\varepsilon^{-2}(\tilde{\alpha}+1)C_{11}(\tilde{\alpha}C_{11} + C_{12}) - \varrho_0[(\tilde{\alpha}+2)C_{11} + C_{12}](\omega L R)^2\}.$$

We note that (56) reduces to (51) when $\tilde{\alpha} = 1$. Also, for cylinders ($R \equiv 1$), (56) reduces to

$$u_0'' + (\tilde{\mathcal{C}}_2/\tilde{\mathcal{A}})u_0 = 0; \tag{57}$$

this equation appears to be new.

7. Coupled extensional-radial modes

We saw in Section 6.2 that torsional modes are decoupled; they are given approximately by (50). However, generically (meaning $C_{11} \neq C_{22}$), the extensional ($\alpha = 0$) and radial ($\alpha = \tilde{\alpha}$) solutions are coupled through the boundary conditions. To construct these coupled modes, we first introduce the notation $\langle \boldsymbol{b}\rangle_i$ for the ith component of the vector $\boldsymbol{b}$, $i = 1, 2, 3$. Then, summing (27) over $\alpha = 0$ and $\alpha = \tilde{\alpha}$, we find that the non-trivial terms give

$$\varepsilon^{-2}\langle \boldsymbol{b}_0(\tilde{\alpha})\rangle_1 + \varepsilon^{-1}\langle \boldsymbol{b}_1(0)\rangle_1 + \langle \boldsymbol{b}_2(\tilde{\alpha})\rangle_1 = 0, \tag{58}$$

and

$$\varepsilon^{-1}\langle \boldsymbol{b}_1(\tilde{\alpha})\rangle_3 + \langle \boldsymbol{b}_2(0)\rangle_3 = 0. \tag{59}$$

Eq. (58) yields a modified form of (56)

$$R^2\tilde{\mathcal{A}}u_0'' - 2RR'\tilde{\mathcal{B}}u_0' + \tilde{\mathcal{F}}Rw_0' + \tilde{\mathcal{C}}u_0 = 0, \tag{60}$$

where $\tilde{\mathcal{F}} = 4\varepsilon^{-1}(\tilde{\alpha}+1)^2 C_{55}\{(C_{11} + C_{12})C_{23} - (C_{12} + C_{22})C_{13}\}/(1-\tilde{\alpha})$. Here, we have used (32) and (35). Note that we do not need to assume that (38) is satisfied. Similarly, (59) gives a modified form of (39)

$$\mathcal{A}_0(R^2 w_0'' + 2RR'w_0') + \mathcal{F}_0(Ru_0' + R'u_0) + \varrho_0(\omega L R)^2 w_0 = 0, \tag{61}$$

where $\mathcal{F}_0 = 2\varepsilon^{-1}(1+\tilde{\alpha})^{-1}(\tilde{\alpha}C_{13} + C_{23})$. Here, we have used (52) and (54).

Eqs. (60) and (61) give a system of coupled ordinary differential equations for u_0 and w_0. For cylinders ($R \equiv 1$), we obtain

$$\tilde{\mathcal{A}}u_0'' + \tilde{\mathcal{F}}w_0' + \tilde{\mathcal{C}}_2 u_0 = 0, \qquad \mathcal{A}_0 w_0'' + \mathcal{F}_0 u_0' + \varrho_0(\omega L)^2 w_0 = 0; \tag{62}$$

eliminating u_0 or w_0 yields single fourth-order differential equations.

8. Discussion

In Section 6, we obtained several second-order ordinary differential equations. These yield simple approximations for long waves in slender anisotropic elastic columns. For $\alpha = 0$ and longitudinal motion, equations for w_0 were obtained; see (39) and (40). These equations are of the form

$$R^{-2}(R^2 w_0')' + \left(\frac{\varrho_0}{\mathcal{E}}\right)(\omega L)^2 w_0 = 0, \tag{63}$$

where $\mathcal{E}$ is defined in terms of C_{ij}, and $\mathcal{E}$ reduces to Young's modulus for isotropic solids. Eq. (63) is usually known as Webster's horn equation in the context of sound waves along a tube with slowly varying rigid walls. Longitudinal motions are also the subject of Boström's paper [5]. He shows how higher-order approximations can be obtained, essentially by truncating (23) at a higher value of n; similar calculations could be done here but the algebra is tedious (although it could be expedited using software for symbolic manipulation).

For $\alpha = 1$, we obtained an equation for v_0 (torsional motions), namely (50). For $\alpha = (C_{22}/C_{11})^{1/2}$, we obtained a differential equation for u_0, namely (56). This equation is interesting because it contains the slenderness parameter ε. Similar equations are obtained for analogous acoustic problems.

In Section 7, we discussed the generic situation, in which $C_{11} \neq C_{22}$. Then, we saw that the radial and extensional motions are coupled; they are given approximately by the system of ordinary differential Eqs. (60) and (61).

The next step is to investigate flexural modes. The procedures outlined above will extend to non-axisymmetric modes, but the calculations are more complicated. The specific application that we have in mind is the computation of the "sweet spot" of a baseball bat incorporating the anisotropy of the wood (usually ash); for more information on this problem, see [19–21].

References

[1] J.D. Achenbach, Wave Propagation in Elastic Solids, North-Holland, Amsterdam, 1973.
[2] K.F. Graff, Wave Motion in Elastic Solids, Oxford University Press, Oxford, 1975.
[3] A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, fourth ed., Cambridge University Press, Cambridge, 1927.
[4] W.A. Green, Dispersion relations for elastic waves in bars, in: I.N. Sneddon, R. Hill (Eds.), Progress in Solid Mechanics, vol. I, North-Holland, Amsterdam, 1960 (pp. 223–261).
[5] A. Boström, On wave equations for elastic rods, Z. Angew. Math. Mech. 80 (2000) 245–251.
[6] G. Rosenfeld, J.B. Keller, Wave propagation in elastic rods of arbitrary cross section, J. Acoust. Soc. Am. 55 (1974) 555–561.
[7] G. Rosenfeld, J.B. Keller, Wave propagation in non-uniform elastic rods, J. Acoust. Soc. Am. 57 (1975) 1094–1096.
[8] J. Tromp, F.A. Dahlen, Surface wave propagation in a slowly varying anisotropic waveguide, Geophys. J. Int. 113 (1993) 239–249.
[9] H. Ohnabe, J.L. Nowinski, On the propagation of flexural waves in anisotropic bars, Ingenieur Arch. 40 (1971) 327–338.
[10] F.H. Chou, J.D. Achenbach, Three-dimensional vibrations of orthotropic cylinders, Proc. ASCE J. Eng. Mech. Div. 98 (1972) 813–822.
[11] Š. Markuš, D.J. Mead, Axisymmetric and asymmetric wave motion in orthotropic cylinders, J. Sound Vib. 181 (1995) 127–147.
[12] F.G. Yuan, C.C. Hsieh, Three-dimensional wave propagation in composite cylindrical shells, Compos. Struct. 42 (1998) 153–167.
[13] P.A. Martin, J.R. Berger, Waves in wood: free vibrations of a wooden pole, J. Mech. Phys. Solids 49 (2001) 1155–1178.
[14] P.A. Martin, J.R. Berger, Waves in wood: axisymmetric guided waves along boreholes, Chin. J. Mech. A 19 (2003) 105–111.
[15] A.L. Shuvalov, The Frobenius power series solution for cylindrically anisotropic radially inhomogeneous elastic materials, Q. J. Mech. Appl. Math. 56 (2003) 327–345.
[16] T.C.T. Ting, Pressuring, shearing, torsion and extension of a circular tube or bar of cylindrically anisotropic material, Proc. R. Soc. A 452 (1996) 2397–2421.
[17] M.J.P. Musgrave, Crystal Acoustics, Holden-Day, San Francisco, 1970.
[18] A.D. Pierce, Acoustics, Acoustical Society of America, New York, 1989.
[19] H. Brody, The sweet spot of a baseball bat, Am. J. Phys. 54 (1986) 640–643.
[20] L.L. Van Zandt, The dynamical theory of the baseball bat, Am. J. Phys. 60 (1992) 172–181.
[21] R. Cross, The sweet spot of a baseball bat, Am. J. Phys. 66 (1998) 772–779.