Multiple scattering by random configurations of circular cylinders: Second-order corrections for the effective wavenumber

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A formula for the effective wavenumber in a dilute random array of identical scatterers in two dimensions is derived, based on Lax's quasicrystalline approximation. This formula replaces a widely-used expression due to Twersky, which is shown to be based on an inappropriate choice of pair-correlation function. [DOI: 10.1121/1.1904270]

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I. INTRODUCTION

Multiple scattering by random arrangements of scatterers is a topic with an extensive literature. See, for example, the recent book by Tsang *et al.* (2001). The modern era dates from the work of Foldy (1945), Lax (1951, 1952), Waterman and Truell (1961), and Twersky (1962). Major applications include wave propagation through suspensions [see, for example McClements *et al.* (1990), Povey (1997), Dukhin and Goetz (2001), and Spelt *et al.* (2001)] and through elastic composites [see Mal and Knopoff (1967), Kim *et al.* (1995), and Kanaun (2000)]. In this paper, we are mainly interested in two-dimensional problems, motivated by the calculation of sound propagation through forests [Embleton (1966); Price *et al.* (1988)]. An important paper on acoustic scattering by arrays of circular cylinders is that of Bose and Mal (1973); see Sec. IV below. For subsequent work, see Varadan *et al.* (1978), Yang and Mal (1994), Bose (1996), Kanaun and Levin (2003), and Kim (2003). For analogous plane-strain elastodynamics, see Varadan *et al.* (1986), Yang and Mal (1994), Bussink *et al.* (1995), and Verbis *et al.* (2001).

A typical problem is the following. The region $x<0$ is filled with a homogeneous compressible fluid of density ρ and sound-speed c. The region $x>0$ contains the same fluid and many scatterers; to fix ideas, we suppose that the scatterers are identical circles (parallel circular cylinders). Then, a time-harmonic plane wave with wavenumber $k=\omega/c$ (ω is the angular frequency) is incident on the scatterers: what is the reflected wave field? This field may be computed exactly for any given configuration (ensemble) of N circles, but the cost increases as N increases. If the computation can be done, it may be repeated for other configurations, and then the average reflected field could be computed (this is the Monte Carlo approach). Instead of doing this, one can try to do some ensemble averaging in order to calculate the average (coherent) field. One result of this is a formula for the *effective wavenumber K*. This can then be used to replace the "random medium" occupying $x>0$ by a homogeneous effective medium.

Foldy (1945) began by considering isotropic point scatterers; this is an appropriate model for small sound-soft scatterers. He obtained the formula

$$K^2=k^2-4\mathrm{i}gn_0, \tag{1}$$

where n_0 is the number of circles per unit area and g is the scattering coefficient for an individual scatterer. [In fact, Foldy considered scattering in three dimensions; the two-dimensional formula, Eq. (1), can be found as Eq. (3.20) in Twersky (1962) and Eq. (26) in Aristégui and Angel (2002), for example.] The formula (1) assumes that the scatterers are independent and that n_0 is small. We are interested in calculating the correction to Eq. (1) (a term proportional to n_0^2), and this will require saying more about the distribution of the scatterers; specifically, we shall use pair correlations. Thus, our goal is a formula of the form

$$K^2=k^2+\delta_1 n_0+\delta_2 n_0^2, \tag{2}$$

with computable expressions for δ_1 and δ_2. Moreover, we do not only want to restrict our formula to sound-soft scatterers.

There is some controversy over the proper value for δ_2. In order to state one such formula, we introduce the *far-field pattern f* for scattering by one circular cylinder. Thus, we have $u_{\mathrm{in}}=\exp[\mathrm{i}kr\cos(\theta-\theta_{\mathrm{in}})]$ for the incident plane wave, where (r,θ) are plane polar coordinates ($x=r\cos\theta$, $y=r\sin\theta$) and θ_{in} is the angle of incidence. The scattered waves satisfy

$$u_{\mathrm{sc}}\sim\sqrt{2/(\pi kr)}\,f(\theta-\theta_{\mathrm{in}})\exp(\mathrm{i}kr-\mathrm{i}\pi/4)\quad\text{as}\quad r\to\infty. \tag{3}$$

Then, Twersky (1962) has given the following formula:

$$K^2=k^2-4\mathrm{i}n_0f(0)+(2n_0/k)^2\sec^2\theta_{\mathrm{in}}\{[f(\pi-2\theta_{\mathrm{in}})]^2-[f(0)]^2\}. \tag{4}$$

This formula involves θ_{in}, so that it gives a different effective wavenumber for different incident fields.

The three-dimensional version of Eq. (4) is older. For a random collection of identical spheres, it is

$$K^2 = k^2 - 4\pi \mathrm{i}(\hat{n}_0/k)f(0) + \delta_2 \hat{n}_0^2$$

with [see Twersky (1962)]

$$\delta_2 = (4\pi^2/k^4)\sec^2\theta_{\mathrm{in}}\{[f(\pi - 2\theta_{\mathrm{in}})]^2 - [f(0)]^2\}, \tag{5}$$

where the far-field pattern is now defined by $u_{\mathrm{sc}} \sim (\mathrm{i}kr)^{-1}\mathrm{e}^{\mathrm{i}kr}f(\vartheta)$, r and ϑ are spherical polar coordinates, and $\hat{n}_0$ is the number of spheres per unit volume. The same formula but with $\theta_{\mathrm{in}}=0$ (normal incidence) was given by Waterman and Truell (1961). However, it was shown by Lloyd and Berry (1967) that Eq. (5) is incorrect; they obtained

$$\delta_2 = \frac{4\pi^2}{k^4}\left\{ -[f(\pi)]^2 + [f(0)]^2 + \int_0^\pi \frac{1}{\sin(\vartheta/2)}\frac{\mathrm{d}}{\mathrm{d}\vartheta}[f(\vartheta)]^2 \mathrm{d}\vartheta \right\} \tag{6}$$

(with no dependence on θ_{in}). Lloyd and Berry (1967) used methods (and language) coming from nuclear physics. Thus, in their approach, which they ''call the 'resummation method,' a point source of waves is considered to be situated in an infinite medium. The scattering series is then written out completely, giving what Lax has called the 'expanded' representation. In this expanded representation the ensemble average may be taken exactly [but then] the coherent wave does not exist; the series must be resummed in order to obtain any result at all.'' One purpose of the present paper is to demonstrate that a proper analysis of the semi-infinite two-dimensional model problem (with arbitrary angle of incidence) leads to a formula that is reminiscent of the (three-dimensional) Lloyd–Berry formula; specifically, instead of Eq. (4), we obtain

$$K^2 = k^2 - 4\mathrm{i}n_0 f(0) + \frac{8n_0^2}{\pi k^2}\int_0^\pi \cot(\theta/2)\frac{\mathrm{d}}{\mathrm{d}\theta}[f(\theta)]^2 \mathrm{d}\theta. \tag{7}$$

Our analysis does not involve ''resumming'' series or divergent integrals. It builds on a conventional approach, in the spirit of the papers by Fikioris and Waterman (1964) and by Bose and Mal (1973).

The paper is organized, as follows. Some elementary probability theory is recalled in Sec. II. In particular, the pair-correlation function is introduced; this leads to the notion of ''hole correction''—individual cylinders must not be allowed to overlap during the averaging process. In Sec. III, we derive the integral equations of Foldy (isotropic scatterers, no hole correction) and of Lax (isotropic scatterers, hole correction included). Foldy's integral equation can be solved exactly whereas we have been unable to solve Lax's integral equation. Nevertheless, we have developed a rigorous method for extracting an expression for K from these integral equations without actually solving the integral equations themselves. Then, we use the same method in Sec. IV but without the restriction to isotropic scatterers. We start by following Bose and Mal (1973), and use an exact (deterministic) theory for scattering by N circles followed by ensemble averaging. We give a clear derivation of a certain homogeneous infinite linear system of algebraic equations, obtained previously by Bose and Mal (1973) for the case of normal incidence; the system does not depend on θ_{in} and the existence of a nontrivial solution determines K. We solve the system for small n_0, and obtain Eq. (7). We also show that Eq. (4) is obtained if the hole correction is not done correctly. Concluding remarks are given in Sec. V.

II. SOME PROBABILITY THEORY

In this section, we give a very brief summary of the probability theory needed. For more information, see Foldy (1945), Lax (1951), Aristégui and Angel (2002) or Chap. 14 of Ishimaru (1978).

Suppose we have N scatterers located at the points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$; denote the configuration of points by $\Lambda_N = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$. Then, the ensemble (or configurational) average of any quantity $F(\mathbf{r}|\Lambda_N)$ is defined by

$$\langle F(\mathbf{r})\rangle = \int \cdots \int p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)F(\mathbf{r}|\Lambda_N)\mathrm{d}V_1 \cdots \mathrm{d}V_N, \tag{8}$$

where the integration is over N copies of the volume B_N containing N scatterers. Here, $p(\mathbf{r}_1, \ldots, \mathbf{r}_N)\mathrm{d}V_1\mathrm{d}V_2 \cdots \mathrm{d}V_N$ is the probability of finding the scatterers in a configuration in which the first scatterer is in the volume element $\mathrm{d}V_1$ about $\mathbf{r}_1$, the second scatterer is in the volume element $\mathrm{d}V_2$ about $\mathbf{r}_2$, and so on, up to $\mathbf{r}_N$. The joint probability distribution $p(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is normalized so that $\langle 1\rangle = 1$. Similarly, the average of $F(\mathbf{r}|\Lambda_N)$ over all configurations for which the first scatterer is fixed at $\mathbf{r}_1$ is given by

$$\langle F(\mathbf{r})\rangle_1 = \int \cdots \int p(\mathbf{r}_2, \ldots, \mathbf{r}_N|\mathbf{r}_1)F(\mathbf{r}|\Lambda_N)\mathrm{d}V_2 \cdots \mathrm{d}V_N, \tag{9}$$

where $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = p(\mathbf{r}_1)p(\mathbf{r}_2, \ldots, \mathbf{r}_N|\mathbf{r}_1)$ defines the conditional probability $p(\mathbf{r}_2, \ldots, \mathbf{r}_N|\mathbf{r}_1)$. If two scatterers are fixed, say the first and the second, we can define

$$\langle F(\mathbf{r})\rangle_{12} = \int \cdots \int p(\mathbf{r}_3, \ldots, \mathbf{r}_N|\mathbf{r}_1, \mathbf{r}_2)F(\mathbf{r}|\Lambda_N)\mathrm{d}V_3 \cdots \mathrm{d}V_N, \tag{10}$$

where $p(\mathbf{r}_2, \ldots, \mathbf{r}_N|\mathbf{r}_1) = p(\mathbf{r}_2|\mathbf{r}_1)p(\mathbf{r}_3, \ldots, \mathbf{r}_N|\mathbf{r}_1, \mathbf{r}_2)$.

Now, as each of the N scatterers is equally likely to occupy $\mathrm{d}V_1$, the density of scatterers at $\mathbf{r}_1$ is $Np(\mathbf{r}_1) = n_0$, the (constant) number of scatterers per unit volume. Thus

$$p(\mathbf{r}) = n_0/N = |B_N|^{-1}, \tag{11}$$

where $|B_N|$ is the volume of B_N. Also, as $p(\mathbf{r}_1, \mathbf{r}_2) = p(\mathbf{r}_1)p(\mathbf{r}_2|\mathbf{r}_1)$, we obtain

$$\int\int p(\mathbf{r}_2|\mathbf{r}_1)\mathrm{d}V_1\mathrm{d}V_2 = \frac{N}{n_0} = |B_N|. \tag{12}$$

We have to specify $p(\mathbf{r}_2|\mathbf{r}_1)$, consistent with Eq. (12). Also, we want to ensure that scatterers do not overlap. For circular cylinders of radius a, a simple choice is $p(\mathbf{r}_2|\mathbf{r}_1) = p_0 H(R_{12} - b)$ with $b \geqslant 2a$, where $H(x)$ is the Heaviside unit function, $R_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and p_0 is a constant determined by Eq. (12). Thus,

$$p_0 = \{|B_N| - \pi b^2\}^{-1} \simeq n_0/N, \tag{13}$$

assuming that $b^2 n_0/N \ll 1$. [The equality in Eq. (13) assumes that the "hole" at $\mathbf{r}_1$ of radius b does not cut the boundary of B_N. Evidently, taking this possibility into account would not change the approximation $p_0 \simeq n_0/N$.] Hence, the simplest sensible choice for the pair-correlation function is

$$p(\mathbf{r}_2|\mathbf{r}_1)=\begin{cases}0, & R_{12}<b,\\ n_0/N, & R_{12}>b.\end{cases} \tag{14}$$

This simple choice will be used for most of our analysis. More generally, we could use

$$p(\mathbf{r}_2|\mathbf{r}_1)=\begin{cases}0, & R_{12}<b,\\ (n_0/N)[1+\chi(R_{12};n_0)], & R_{12}>b,\end{cases} \tag{15}$$

where the function χ is to be chosen, subject to some constraints. The effect of using Eq. (15) instead of Eq. (14) is calculated in Sec. IV D. One could also consider functions χ that depend on $\mathbf{r}_1-\mathbf{r}_2$ (instead of just $|\mathbf{r}_1-\mathbf{r}_2|$); such possibilities are discussed in Twersky (1978) and Siqueira *et al.* (1995).

III. FOLDY–LAX THEORY: ISOTROPIC SCATTERERS

Foldy's theory begins with a simplified deterministic model for scattering by N identical scatterers, each of which is supposed to scatter isotropically. Thus, the total field is assumed to be given by the incident field plus a point source at each scattering center, $\mathbf{r}_j$:

$$u(\mathbf{r}|\Lambda_N)=u_{\mathrm{in}}(\mathbf{r})+g\sum_{j=1}^{N} u_{\mathrm{ex}}(\mathbf{r}_j;\mathbf{r}_j|\Lambda_N)H_0(k|\mathbf{r}-\mathbf{r}_j|). \tag{16}$$

Here, $H_n(w)\equiv H_n^{(1)}(w)$ is a Hankel function, g is the (assumed known) scattering coefficient, and the exciting field u_{ex} is given by

$$u_{\mathrm{ex}}(\mathbf{r};\mathbf{r}_n|\Lambda_N)=u_{\mathrm{in}}(\mathbf{r})+g\sum_{\substack{j=1\\ j\neq n}}^{N} u_{\mathrm{ex}}(\mathbf{r}_j;\mathbf{r}_j|\Lambda_N)H_0(k|\mathbf{r}-\mathbf{r}_j|). \tag{17}$$

The second term in Eq. (17) is the field near the cylinder at $\mathbf{r}_n$ due to scattering by all the other cylinders. The N numbers $u_{\mathrm{ex}}(\mathbf{r}_j;\mathbf{r}_j|\Lambda_N)$ $(j=1,2,...,N)$ required in Eq. (16) are to be determined by solving the linear system obtained by evaluating Eq. (17) at $\mathbf{r}=\mathbf{r}_n$; direct numerical solutions of this system have been given by Fikioris (1966) and by Groenenboom and Snieder (1995).

Let us try to compute the ensemble average of u, using Eqs. (16) and (8). The result is

$$\langle u(\mathbf{r})\rangle=u_{\mathrm{in}}(\mathbf{r})+gn_0\int_{B_N}\langle u_{\mathrm{ex}}(\mathbf{r}_1)\rangle_1 H_0(k|\mathbf{r}-\mathbf{r}_1|)\mathrm{d}V_1, \tag{18}$$

where we have used Eqs. (9) and (11), and the indistinguishability of the scatterers. For $\langle u_{\mathrm{ex}}(\mathbf{r}_1)\rangle_1$ [which is given explicitly by Eq. (9) in which $u_{\mathrm{ex}}(\mathbf{r}_1;\mathbf{r}_1|\Lambda_N)$ is substituted for $F(\mathbf{r}|\Lambda_N)$], we obtain

$$\langle u_{\mathrm{ex}}(\mathbf{r})\rangle_1=u_{\mathrm{in}}(\mathbf{r})+g(N-1)\int_{B_N}p(\mathbf{r}_2|\mathbf{r}_1)\times\langle u_{\mathrm{ex}}(\mathbf{r}_2)\rangle_{12}H_0(k|\mathbf{r}-\mathbf{r}_2|)\mathrm{d}V_2, \tag{19}$$

where we have used Eqs. (10) and (17). Equations (18) and (19) are the first two in a hierarchy, involving more and more complicated information on the statistics of the scatterer distribution. In practice, the hierarchy is broken using an additional assumption. At the lowest level, we have Foldy's assumption,

$$\langle u_{\mathrm{ex}}(\mathbf{r})\rangle_1\simeq\langle u(\mathbf{r})\rangle, \tag{20}$$

at least in the neighborhood of $\mathbf{r}_1$. When this is used in Eq. (18), we obtain

$$\langle u(\mathbf{r})\rangle=u_{\mathrm{in}}(\mathbf{r})+gn_0\int_{B_N}\langle u(\mathbf{r}_1)\rangle H_0(k|\mathbf{r}-\mathbf{r}_1|)\mathrm{d}V_1,\quad \mathbf{r}\in B_N. \tag{21}$$

We call this *Foldy's integral equation* for $\langle u\rangle$. The integral on the right-hand side is an acoustic volume potential. Hence, an application of (∇^2+k^2) to Eq. (21) eliminates the incident field and shows that $(\nabla^2+K^2)\langle u\rangle=0$ in B_N, where K^2 is given by Foldy's formula, Eq. (1).

At the next level, we have the Lax (1952) quasicrystalline assumption (QCA),

$$\langle u_{\mathrm{ex}}(\mathbf{r})\rangle_{12}\simeq\langle u_{\mathrm{ex}}(\mathbf{r})\rangle_2. \tag{22}$$

When this is used in Eq. (19) evaluated at $\mathbf{r}=\mathbf{r}_1$, we obtain

$$v(\mathbf{r})=u_{\mathrm{in}}(\mathbf{r})+g(N-1)\int_{B_N}p(\mathbf{r}_1|\mathbf{r})v(\mathbf{r}_1)H_0(k|\mathbf{r}-\mathbf{r}_1|)\mathrm{d}V_1,\quad \mathbf{r}\in B_N, \tag{23}$$

where $v(\mathbf{r})=\langle u_{\mathrm{ex}}(\mathbf{r})\rangle_1$. We call this *Lax's integral equation*.

In what follows, we let $N\to\infty$ so that $B_N\to B_\infty$, a semi-infinite region, $x>0$.

A. Foldy's integral equation: Exact treatment

Consider a plane wave at oblique incidence, so that

$$u_{\mathrm{in}}=\mathrm{e}^{\mathrm{i}(\alpha x+\beta y)}\quad\text{with}\quad \alpha=k\cos\theta_{\mathrm{in}}\quad\text{and}\quad \beta=k\sin\theta_{\mathrm{in}}. \tag{24}$$

For a semi-infinite domain B_∞, Foldy's integral equation, Eq. (21), becomes

$$\langle u(x,y)\rangle=\mathrm{e}^{\mathrm{i}(\alpha x+\beta y)}+gn_0\int_0^\infty\int_{-\infty}^{\infty}\langle u(x_1,y+Y)\rangle\times H_0(k\rho_1)\mathrm{d}Y\mathrm{d}x_1,\quad x>0,\ -\infty<y<\infty,$$

where $\rho_1=\sqrt{(x-x_1)^2+Y^2}$. This equation can be solved exactly. Thus, writing

$$\langle u(x,y)\rangle=U(x)\mathrm{e}^{\mathrm{i}\mu y},\quad x>0,\quad -\infty<y<\infty, \tag{25}$$

we obtain

$$U(x)=e^{i\alpha x}e^{i(\beta-\mu)y}+gn_0\int_0^\infty\int_{-\infty}^\infty U(x_1)\times H_0(k\rho_1)e^{i\mu Y}dY\,dx_1, \quad x>0,\ -\infty<y<\infty. \tag{26}$$

Hence, for a solution in the form Eq. (25), we must have $\mu=\beta=k\sin\theta_{\text{in}}$.

Now,

$$\int_{-\infty}^\infty H_0(k\rho_1)e^{i\beta Y}dY=\frac{2}{\alpha}e^{i\alpha|x-x_1|}, \tag{27}$$

where $\alpha=\sqrt{k^2-\beta^2}=k\cos\theta_{\text{in}}$. Thus, we see that U solves

$$U(x)=e^{i\alpha x}+\frac{2gn_0}{\alpha}\int_0^\infty U(x_1)e^{i\alpha|x-x_1|}dx_1, \quad x>0. \tag{28}$$

Now, put $U(x)=U_0e^{i\lambda x}$, so that Eq. (25) gives

$$\langle u(x,y)\rangle=U_0e^{i(\lambda x+\beta y)}, \quad x>0, \quad -\infty<y<\infty, \tag{29}$$

and Eq. (28) gives

$$U_0e^{i\lambda x}-e^{i\alpha x}=\frac{2gn_0U_0}{i\alpha}\left(\frac{2\alpha e^{i\lambda x}}{\lambda^2-\alpha^2}-\frac{e^{i\alpha x}}{\lambda-\alpha}\right),$$

where we have assumed that $\text{Im}\,\lambda>0$. If we compare the coefficients of $e^{i\lambda x}$, we see that U_0 cancels, leaving

$$\lambda^2-\alpha^2=-4ign_0, \tag{30}$$

which determines λ. Then, the coefficients of $e^{i\alpha x}$ give $U_0=2\alpha/(\lambda+\alpha)$. A similar method can be used to find $\langle u\rangle$ when B_∞ is a slab of finite thickness, $0<x<h$, say; see Aristégui and Angel (2002).

From Eq. (29), it is natural to write

$$\lambda=K\cos\varphi \quad \text{and} \quad \beta=K\sin\varphi=k\sin\theta_{\text{in}}. \tag{31}$$

These define the effective wave number K; the last equality is recognized as Snell's law, even though K and φ are complex, with $\text{Im}\,K>0$. Hence, we see that

$$\lambda^2-\alpha^2=K^2-k^2, \tag{32}$$

and so Eq. (30) reduces to Foldy's formula, Eq. (1).

B. Foldy's integral equation: Approximate treatment

We have seen that Foldy's integral equation can be solved exactly, and that the solution process has two parts: first find λ (and hence the effective wavenumber) and then find U_0. In fact, λ can be found without finding the complete solution; the reason for pursuing this is that we cannot usually find exact solutions. Thus, consider Eq. (28), and suppose that

$$U(x)=U_0e^{i\lambda x} \quad \text{for } x>\ell,$$

where U_0, λ, and ℓ are unknown. To proceed, we need say nothing about the solution U in the "boundary layer" $0<x<\ell$. Now, evaluate the integral equation for $x>\ell$; we find that

$$U_0e^{i\lambda x}-e^{i\alpha x}=\frac{2gn_0}{\alpha}e^{i\alpha x}\int_0^\ell U(t)e^{-i\alpha t}dt+\frac{2gn_0}{\alpha}\times\int_\ell^\infty U(t)e^{i\alpha|x-t|}dt=\mathcal{A}e^{i\lambda x}+\mathcal{B}e^{i\alpha x} \quad \text{for } x>\ell,$$

where $\mathcal{A}=-4ign_0U_0/(\lambda^2-\alpha^2)$ and

$$\mathcal{B}=\frac{2gn_0}{\alpha}\int_0^\ell U(t)e^{-i\alpha t}dt+\frac{2ign_0U_0}{\alpha(\lambda-\alpha)}e^{i(\lambda-\alpha)\ell}.$$

Then, setting $U_0=\mathcal{A}$ gives Eq. (30) again, without knowing the solution U everywhere. This basic method will be used again below.

C. Lax's integral equation

Using the approximation $p(\mathbf{r}_1|\mathbf{r})=(n_0/N)H(R_1-b)$ in Lax's integral equation, Eq. (23) gives

$$v(\mathbf{r})=u_{\text{in}}(\mathbf{r})+gn_0\frac{N-1}{N}\int_{B_N^b(\mathbf{r})}v(\mathbf{r}_1)H_0(kR_1)d\mathbf{r}_1, \quad \mathbf{r}\in B_N, \tag{33}$$

where $B_N^b(\mathbf{r})=\{\mathbf{r}_1\in B_N:R_1=|\mathbf{r}-\mathbf{r}_1|>b\}$, which is B_N with a (possibly incomplete) disk excluded.

Let $N\to\infty$ and take an incident plane wave, Eq. (24), giving

$$v(x,y)=e^{i(\alpha x+\beta y)}+gn_0\int_{x_1>0,\rho_1>b}v(x_1,y+Y)\times H_0(k\rho_1)dY\,dx_1, \quad x>0,\ -\infty<y<\infty.$$

As in Sec. III A, we write

$$v(x,y)=V(x)e^{i\beta y}, \quad x>0, \quad -\infty<y<\infty, \tag{34}$$

giving

$$V(x)=e^{i\alpha x}+gn_0\int_{x_1>0,\rho_1>b}V(x_1)H_0(k\rho_1)\times e^{i\beta Y}dY\,dx_1, \quad x>0. \tag{35}$$

Then, using Eq. (27), we see that V solves

$$V(x)=e^{i\alpha x}+gn_0\int_0^\infty V(x_1)L(x-x_1)dx_1, \quad x>0, \tag{36}$$

where the kernel, $L(x-x_1)$, is given by

$$L(X)=\frac{2}{\alpha}e^{i\alpha|X|}-2\int_0^{c(X)}H_0(k\sqrt{X^2+Y^2})e^{i\beta Y}dY \tag{37}$$

with $c(X)=\sqrt{b^2-X^2}H(b-|X|)$; here, we have written the integral over Y in Eq. (35) as an integral over all Y minus an integral through the disk, if necessary.

We have been unable to solve Eq. (36) exactly (even though it is an integral equation of Wiener–Hopf-type). However, the approximate method described in Sec. III B can be used. Thus, let us suppose that

$$V(x)=V_0 e^{i\lambda x} \quad \text{for } x>\ell, \tag{38}$$

where V_0, λ, and ℓ are unknown. Then, consider Eq. (36) for $x>\ell+b$, so that the interval $|x-x_1|<b$ is entirely within the range $x_1>\ell$. Making use of Eq. (37), Eq. (36) gives

$$\frac{V_0 e^{i\lambda x}-e^{i\alpha x}}{g n_0} = \frac{2}{\alpha} e^{i\alpha x}\int_0^{\ell} V(t) e^{-i\alpha t} dt + \frac{2}{\alpha}\int_{\ell}^{\infty} V(t) e^{i\alpha|x-t|} dt - 2\int_{x-b}^{x+b} V(t)\int_0^{c(x-t)} H_0\left(k\sqrt{(x-t)^2+Y^2}\right) e^{i\beta Y} dY dt \tag{39}$$

for $x>\ell+b$. Equation (38) can be used in the second and third integrals. The second integral is elementary, and has the value

$$\frac{2iV_0}{\alpha(\lambda-\alpha)} e^{i(\lambda-\alpha)\ell} e^{i\alpha x} - \frac{4iV_0}{\lambda^2-\alpha^2} e^{i\lambda x}.$$

The third integral becomes

$$-2V_0\int_{-b}^{b} e^{i\lambda(x+\xi)}\int_0^{\sqrt{b^2-\xi^2}} H_0\left(k\sqrt{\xi^2+Y^2}\right) e^{i\beta Y} dY d\xi$$

$$= -V_0 e^{i\lambda x}\int_0^{2\pi}\int_0^{b} e^{iKr\cos(\theta-\varphi)} H_0(kr) r dr d\theta$$

$$= -2\pi V_0 e^{i\lambda x}\int_0^{b} J_0(Kr) H_0(kr) r dr$$

$$= V_0 e^{i\lambda x}\left\{\frac{4i}{K^2-k^2} - \frac{2\pi\mathcal{N}_0(Kb)}{K^2-k^2}\right\},$$

where $\mathcal{N}_0(Kb)=KbH_0(kb)J_1(Kb)-kbH_1(kb)J_0(Kb)$. Using these results in Eq. (39), noting Eq. (32), we obtain

$$V_0 e^{i\lambda x}-e^{i\alpha x}=\mathcal{A}e^{i\lambda x}+\mathcal{B}e^{i\alpha x} \quad \text{for } x>\ell+b,$$

where

$$\mathcal{A}=\frac{2\pi g n_0 V_0}{k^2-K^2}\mathcal{N}_0(Kb),$$

$$\mathcal{B}=\frac{2gn_0}{\alpha}\int_0^{\ell} V(t) e^{-i\alpha t} dt + \frac{2ign_0V_0}{\alpha(\lambda-\alpha)} e^{i(\lambda-\alpha)\ell}.$$

For a solution, we must have $\mathcal{A}=V_0$, and so

$$K^2=k^2-2\pi g n_0\mathcal{N}_0(Kb), \tag{40}$$

which is a nonlinear equation for *K*. Notice that this equation does not depend on the angle of incidence, θ_{in}.

We have $\mathcal{N}_0(Kb)\to 2i/\pi$ as $b\to 0$ so that, in this limit, we recover Foldy's formula for the effective wavenumber, Eq. (1).

Let us solve Eq. (40) for small n_0. (Alternatively, we could use the dimensionless area fraction $\pi a^2 n_0$.) Begin by writing

$$K^2=k^2+\delta_1 n_0+\delta_2 n_0^2+\cdots, \tag{41}$$

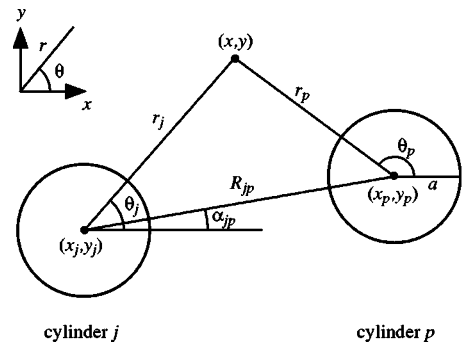

FIG. 1. A view of two typical cylinders.

where δ_1 and δ_2 are to be found; for δ_1, we expect to obtain the result given by Eq. (1). It follows from Eq. (41) that $K=k+\frac{1}{2}\delta_1 n_0/k+O(n_0^2)$ and then

$$\mathcal{N}_0(Kb)=\mathcal{N}_0(kb)+(Kb-kb)\mathcal{N}_0'(kb)+\cdots$$

$$=2i/\pi+\tfrac{1}{2}b^2\delta_1 d_0(kb) n_0+O(n_0^2),$$

where $d_0(x)=J_0(x)H_0(x)+J_1(x)H_1(x)$. When this approximation for $\mathcal{N}_0(Kb)$ is used in Eq. (40), we obtain

$$K^2=k^2-4ign_0-\pi b^2 g\delta_1 d_0(kb) n_0^2.$$

Comparison of this formula with Eq. (41) gives $\delta_1=-4ig$ (as expected) and $\delta_2=4\pi i(gb)^2 d_0(kb)$, so that we obtain the approximation

$$K^2=k^2-4ign_0+4\pi i(gbn_0)^2 d_0(kb). \tag{42}$$

Note that the second-order term in Eq. (42) vanishes in the limit $kb\to 0$.

IV. FINITE-SIZE EFFECTS

The theory described above relies on the assumption of isotropy. Here, we use a more complete theory. We start with an exact theory (due to Záviška) for acoustic scattering by *N* identical circular cylinders of radius *a*; for details and references, see p. 173 of Linton and McIver (2001). The cylinders can be soft, hard or penetrable. Then (in Sec. IV B), we form averaged equations, and we invoke the QCA. This leads to an infinite homogeneous system of linear algebraic equations from which the effective wave number, *K*, is to be determined; the equations are independent of the angle of incidence. An approximate solution for *K* is found in Sec. IV C, correct to $O(n_0^2)$. In Sec. IV D, it is shown that this approximation does not depend on the choice of the function $\chi(r;n_0)$, appearing in the pair-correlation function, Eq. (15). In Sec. IV E, it is shown how Twersky's formula for *K* can be derived, using an unreasonable choice for the pair-correlation function.

A. A finite array of identical circular cylinders: Exact theory

We use polar coordinates (r,θ) centered at the origin and (r_j,θ_j), centered at $\mathbf{r}_j=(x_j,y_j)$, the center of the *j*th cylinder. The various parameters relating to the relative positions of the cylinders are shown in Fig. 1.

Exterior to the cylinders the pressure field is *u*, where

$$\nabla^2 u + k^2 u = 0. \tag{43}$$

In the interior of cylinder j, the field is u_j, where

$$\nabla^2 u_j + \kappa^2 u_j = 0. \tag{44}$$

A plane wave, given by Eq. (24), is incident on the cylinders. A phase factor for each cylinder, I_j, is defined by

$$I_j = \mathrm{e}^{\mathrm{i}(\alpha x_j + \beta y_j)} \tag{45}$$

and then we can write

$$u_{\mathrm{in}} = I_j \mathrm{e}^{\mathrm{i}kr_j \cos(\theta_j - \theta_{\mathrm{in}})} = I_j \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\pi/2 - \theta_j + \theta_{\mathrm{in}})} J_n(kr_j). \tag{46}$$

We seek a solution to Eqs. (43) and (44) in the form

$$u = u_{\mathrm{in}} + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} A_n^j Z_n H_n(kr_j) \mathrm{e}^{\mathrm{i}n\theta_j}, \tag{47}$$

$$u_j = \sum_{n=-\infty}^{\infty} B_n^j J_n(\kappa r_j) \mathrm{e}^{\mathrm{i}n\theta_j}, \tag{48}$$

for some set of unknown complex coefficients A_n^j and B_n^j. The factor

$$Z_n = \frac{q J_n'(ka) J_n(\kappa a) - J_n(ka) J_n'(\kappa a)}{q H_n'(ka) J_n(\kappa a) - H_n(ka) J_n'(\kappa a)} = Z_{-n} \tag{49}$$

has been introduced for later convenience. Here $\kappa = \omega/\tilde{c}$ and $q = \tilde{\rho}\tilde{c}/(\rho c)$, where $\tilde{\rho}$ and $\tilde{c}$ are the density and sound speed, respectively, inside the cylinders. Note that we recover the sound-soft results in the limit $q \to 0$, whereas the limit $q \to \infty$ gives the sound-hard results. The boundary conditions on the cylinders are

$$u = u_s, \quad \frac{1}{\rho}\frac{\partial u}{\partial r_s} = \frac{1}{\tilde{\rho}}\frac{\partial u_s}{\partial r_s} \quad \text{on } r_s = a, \quad s = 1,\ldots,N. \tag{50}$$

Using Graf's addition theorem for Bessel functions, it can be shown that provided $r_s < R_{js}$ for all j, we can write the field exterior to cylinder s as

$$\begin{aligned} u(r_s,\theta_s) &= \sum_{n=-\infty}^{\infty} \left(I_s J_n(kr_s) \mathrm{e}^{\mathrm{i}n(\pi/2 - \theta_s + \theta_{\mathrm{in}})} + A_n^s Z_n H_n(kr_s) \mathrm{e}^{\mathrm{i}n\theta_s} \right) \\ &\quad + \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} A_n^j Z_n \sum_{m=-\infty}^{\infty} J_m(kr_s) H_{n-m}(kR_{js}) \\ &\quad \times \mathrm{e}^{\mathrm{i}m\theta_s} \mathrm{e}^{\mathrm{i}(n-m)\alpha_{js}}. \end{aligned} \tag{51}$$

The geometrical restriction implies that this expression is only valid if the point (r_s,θ_s) is closer to the center of cylinder s than the centers of any of the other cylinders. This is certainly true on the surface of cylinder s and so Eq. (51) can be used to apply the body boundary conditions which leads, after using the orthogonality of the functions $\exp(\mathrm{i}m\theta_s)$, $m \in \mathbb{Z}$, and eliminating the coefficients B_n^j, to the system of equations

$$\begin{aligned} A_m^s &+ \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} A_n^j Z_n \mathrm{e}^{\mathrm{i}(n-m)\alpha_{js}} H_{n-m}(kR_{js}) \\ &= -I_s \mathrm{e}^{\mathrm{i}m(\pi/2 - \theta_{\mathrm{in}})}, \quad s = 1,2,\ldots,N, \quad m \in \mathbb{Z}. \end{aligned} \tag{52}$$

Note that the quantities q, κ, and a only enter the equations through the terms Z_n.

For a single cylinder the solution is immediate: $A_m^1 = -\mathrm{i}^m I_1 \exp(-\mathrm{i}m\theta_{\mathrm{in}})$ and then the far-field pattern, defined by Eq. (3), is given by

$$f(\theta) = -\sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n\theta}. \tag{53}$$

B. Arrays of circular cylinders: Averaged equations

The above analysis applies to a specific configuration of scatterers. Now we follow Bose and Mal (1973) and take ensemble averages. Specifically, setting $s=1$ in Eq. (52) and then taking the conditional average, using Eq. (14), we get

$$\begin{aligned} \langle A_m^1 \rangle_1 &+ n_0 \frac{N-1}{N} \sum_{n=-\infty}^{\infty} Z_n \int_{B_N: R_{12} > b} H_{n-m}(kR_{21}) \\ &\times \mathrm{e}^{\mathrm{i}(n-m)\alpha_{21}} \langle A_n^2 \rangle_{12} \mathrm{d}V_2 \\ &= -I_1 \mathrm{e}^{\mathrm{i}m(\pi/2 - \theta_{\mathrm{in}})}, \quad m \in \mathbb{Z}. \end{aligned} \tag{54}$$

Now we let $N \to \infty$ so that B_N becomes the half-space $x > 0$, and invoke Lax's QCA, Eq. (22). This implies that

$$\langle A_m^2 \rangle_{12} = \langle A_m^2 \rangle_2. \tag{55}$$

We seek a solution to Eq. (54) in the form

$$\langle A_m^s \rangle_s = \mathrm{i}^m \mathrm{e}^{\mathrm{i}\beta y_s} \Phi_m(x_s) \tag{56}$$

so that

$$\begin{aligned} \Phi_m(x_1) &+ n_0 \sum_{n=-\infty}^{\infty} Z_n (-\mathrm{i})^{n-m} \int_{x_2 > 0, R_{12} > b} \psi_{n-m}(x_{21}, y_{21}) \\ &\times \mathrm{e}^{\mathrm{i}\beta y_{21}} \Phi_n(x_2) \mathrm{d}x_2 \mathrm{d}y_2 \\ &= -\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{in}}} \mathrm{e}^{\mathrm{i}\alpha x_1}, \quad m \in \mathbb{Z}, \end{aligned} \tag{57}$$

where we have written $x_{21} = x_2 - x_1$ and $y_{21} = y_2 - y_1$, used $\alpha_{21} = \alpha_{12} - \pi$, and defined $\psi_n(X,Y) = H_n(kR)\mathrm{e}^{\mathrm{i}n\Theta}$ with $X = R\cos\Theta$ and $Y = R\sin\Theta$.

Proceeding as before, suppose that for sufficiently large x (say $x > \ell$) we can write

$$\Phi_m(x) = F_m \mathrm{e}^{-\mathrm{i}m\varphi} \mathrm{e}^{\mathrm{i}\lambda x}, \tag{58}$$

where λ and φ are defined by Eq. (31). We assume that $\mathrm{Im}\,\lambda > 0$ so that $\Phi_m \to 0$ as $x \to \infty$. Then if $x_1 > \ell + b$, Eq. (57) becomes

$$\begin{aligned} F_m \mathrm{e}^{-\mathrm{i}m\varphi} \mathrm{e}^{\mathrm{i}\lambda x_1} &+ n_0 \sum_{n=-\infty}^{\infty} Z_n (-\mathrm{i})^{n-m} \\ &\times \left\{ \int_0^{\ell} \Phi_n(x_2) L_{n-m}(x_{21}) \mathrm{d}x_2 + F_n \mathrm{e}^{-\mathrm{i}n\varphi} \mathrm{e}^{\mathrm{i}\lambda x_1} M_{n-m} \right\} \\ &= -\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{in}}} \mathrm{e}^{\mathrm{i}\alpha x_1}, \quad m \in \mathbb{Z}, \end{aligned} \tag{59}$$

where

$$L_n(X)=\int_{-\infty}^{\infty}\psi_n(X,Y)e^{i\beta Y}dY, \tag{60}$$

$$M_n=\int_{x_2>\ell, R_{12}>b}\psi_n(x_{21},y_{21})\Psi(x_{21},y_{21})dx_2dy_2, \tag{61}$$

$$\Psi(X,Y)=e^{i(\lambda X+\beta Y)}=e^{iKR\cos(\Theta-\varphi)}, \tag{62}$$

and we have used Eq. (31). Next, we shall evaluate L_n and M_n; note that we have $x_2<\ell<x_1$ in Eq. (59) so that $x_{21}<0$.

Consider the integral $L_n(X)$ for $X<0$. From Eq. (27), we have

$$L_0(X)=(2/\alpha)e^{-i\alpha X} \quad \text{and } L_0'=-i\alpha L_0. \tag{63}$$

For L_n with $n>0$, we use the fact that

$$L_n(X)=\int_{-\infty}^{\infty}-\frac{1}{k}e^{i\beta Y}\left(\frac{\partial}{\partial X}+i\frac{\partial}{\partial Y}\right)\psi_{n-1}(X,Y)dY. \tag{64}$$

Then, as the $\partial/\partial X$ can be taken outside the integral and the $\partial/\partial Y$ can be removed using an integration by parts, we have $kL_n=-L_{n-1}'-\beta L_{n-1}$, which expresses L_n in terms of L_{n-1} and L_{n-1}'. It follows from Eq. (63) that

$$L_n=\frac{2(i\alpha-\beta)^n}{\alpha k^n}e^{-i\alpha X}=\frac{2i^n}{\alpha}e^{in\theta_{in}}e^{-i\alpha X}. \tag{65}$$

This formula also holds for $n<0$. Hence, for $x_1>x_2$,

$$L_n(x_2-x_1)=(2/\alpha)i^n e^{in\theta_{in}}e^{i\alpha(x_1-x_2)}. \tag{66}$$

The double integral M_n can be evaluated using Green's theorem as follows. We have $\psi_n\nabla^2\Psi-\Psi\nabla^2\psi_n=(k^2-K^2)\psi_n\Psi$. It follows that

$$M_m=\frac{1}{k^2-K^2}\int_{\partial B}\left[\psi_m\frac{\partial\Psi}{\partial n}-\Psi\frac{\partial\psi_m}{\partial n}\right]ds_2,$$

where ∂B consists of two parts, the line $x_2=\ell$ and the circle $R_{12}=b$. Now, on $x_2=\ell$, $\partial/\partial n=-\partial/\partial x_2$ and so we have

$$\begin{aligned}
-\int_{x_2=\ell}&\left[\psi_n\frac{\partial\Psi}{\partial x_2}-\Psi\frac{\partial\psi_n}{\partial x_2}\right]dy_2 \\
&=e^{i\lambda(\ell-x_1)}\int_{-\infty}^{\infty}e^{i\beta y_{21}}\left[-i\lambda\psi_n+\cos\alpha_{12}\frac{\partial\psi_n}{\partial R_{12}}\right.\\
&\quad\left.-\frac{\sin\alpha_{12}}{R_{12}}\frac{\partial\psi_n}{\partial\alpha_{12}}\right]_{x_2=\ell}dy_2 \\
&=e^{i\lambda(\ell-x_1)}\int_{-\infty}^{\infty}e^{i\beta y_{21}}\\
&\quad\times\left[-i\lambda\psi_n+\frac{k}{2}(\psi_{n-1}-\psi_{n+1})\right]_{x_2=\ell}dy_2 \\
&=\frac{2}{\alpha}e^{i(\alpha-\lambda)(x_1-\ell)}i^{n-1}e^{in\theta_{in}}(\lambda+\alpha),
\end{aligned} \tag{67}$$

using $2H_n'(x)=H_{n-1}(x)-H_{n+1}(x)$, $(2n/x)H_n(x)=H_{n-1}(x)+H_{n+1}(x)$, and Eq. (66) thrice.

The contribution from the circle $R_{12}=b$ is

$$\begin{aligned}
-\int_0^{2\pi}&\left[\psi_n\frac{\partial}{\partial R}(e^{iKR\cos(\Theta-\varphi)})-e^{iKR\cos(\Theta-\varphi)}\frac{\partial\psi_n}{\partial R}\right]_{R=b}b\,d\Theta \\
&=-b\int_0^{2\pi}e^{iKb\cos(\Theta-\varphi)}e^{in\Theta}\\
&\quad\times[iKH_n(kb)\cos(\Theta-\varphi)-kH_n'(kb)]d\Theta \\
&=-be^{in\varphi}\int_0^{2\pi}e^{in\theta}\sum_{q=-\infty}^{\infty}i^qJ_q(Kb)e^{-iq\theta}\\
&\quad\times\left[\frac{iK}{2}H_n(kb)(e^{i\theta}+e^{-i\theta})-kH_n'(kb)\right]d\theta \\
&=-2\pi b i^n e^{in\varphi}[KH_n(kb)J_n'(Kb)-kH_n'(kb)J_n(Kb)].
\end{aligned} \tag{68}$$

Thus, the system (59) can be written as

$$\begin{aligned}
&\mathcal{A}_m e^{-im\varphi}e^{i\lambda x}+\mathcal{B}e^{-im\theta_{in}}e^{i\alpha x}\\
&\quad=-e^{-im\theta_{in}}e^{i\alpha x}, \quad x>\ell+b, \quad m\in\mathbb{Z},
\end{aligned} \tag{69}$$

where

$$\mathcal{A}_m=F_m+\frac{2n_0\pi}{k^2-K^2}\sum_{n=-\infty}^{\infty}F_nZ_n\mathcal{N}_{n-m}(Kb),$$

$$\begin{aligned}
\mathcal{B}=&\frac{2n_0}{\alpha}\sum_{n=-\infty}^{\infty}Z_ne^{in\theta_{in}}\\
&\times\left\{\int_0^{\ell}\Phi_n(t)e^{-i\alpha t}dt+\frac{iF_ne^{-in\varphi}}{\lambda-\alpha}e^{i(\lambda-\alpha)\ell}\right\},
\end{aligned}$$

and

$$\mathcal{N}_n(Kb)=kbH_n'(kb)J_n(Kb)-KbH_n(kb)J_n'(Kb). \tag{70}$$

In particular, note that $\mathcal{N}_0$ appeared in Sec. III C during our analysis of Lax's integral equation.

From Eq. (69), we immediately obtain $\mathcal{B}=-1$ and $\mathcal{A}_m=0$ for all m; the second of these, namely

$$F_m+\frac{2n_0\pi}{k^2-K^2}\sum_{n=-\infty}^{\infty}F_nZ_n\mathcal{N}_{n-m}(Kb)=0, \quad m\in\mathbb{Z}, \tag{71}$$

is of most interest to us. It is an infinite homogeneous system of linear algebraic equations for F_m, $m\in\mathbb{Z}$. The existence of a nontrivial solution to Eq. (71) determines K. Notice that Eq. (71) does not depend on θ_{in}, so that the effective wavenumber cannot depend on θ_{in}.

Equation (71) is the same as Eq. (33) in Bose and Mal (1973) [with the choice Eq. (14)]; these authors began by considering normal incidence, $\theta_{in}=0$. However, the *derivation* of Eq. (71) given here has some advantages over that given by Bose and Mal (1973). First, we do not invoke "the so-called 'extinction theorem'" of Lax; this is described in Sec. VI of Lax (1952). Roughly speaking, this "theorem" asserts that one may simply delete the incident field when calculating the effective wavenumber, in the limit $N\to\infty$. Along with this come some divergent integrals; for example, the integrals in the unnumbered equation between Eqs. (32)

and (33) of Bose and Mal (1973) are divergent, because e^{iKx} is exponentially large as $x \to -\infty$. In fact, we can say that our analysis *proves* Lax's theorem in our particular case.

Second, when dealing with a half-space containing scatterers, we know from the work of Lloyd and Berry (1967) that the boundary of the half-space can cause difficulties. Here, we give a proper treatment of this boundary. In particular, we do not assume that all fields are proportional to $e^{i\lambda x}$ *everywhere* inside the half-space, $x>0$, but only in $x>\ell$, away from the boundary: the width of the boundary layer, ℓ, is not specified, and need not be specified if one only wants to calculate K.

A more recent analysis was given by Siqueira and Sarabandi (1996). They allow noncircular and nonidentical cylinders (using a T-matrix formulation) but they do assume that the effective field is proportional to $e^{i\lambda x}$ for all $x>0$.

C. Approximate determination of K for small n_0

The only approximation made in the derivation of Eq. (71) is the QCA, which is expected to be valid for small values of the scatterer concentration $(n_0 a^2 \ll 1)$. We now assume (as in Sec. III C) that n_0/k^2 is also small and write $K^2 = k^2 + \delta_1 n_0 + \delta_2 n_0^2 + \cdots$. We then have

$$\mathcal{N}_n(Kb) = 2i/\pi + \tfrac{1}{2} b^2 \delta_1 d_n(kb) n_0 + O(n_0^2), \tag{72}$$

where

$$d_n(x) = J_n'(x) H_n'(x) + [1-(n/x)^2] J_n(x) H_n(x) \tag{73}$$

and so

$$\frac{\mathcal{N}_n(Kb)}{k^2-K^2} = -\frac{2i}{\pi\delta_1 n_0} - \frac{b^2 d_n(kb)}{2} + \frac{2i\delta_2}{\pi\delta_1^2} + O(n_0). \tag{74}$$

If Eq. (74) is substituted in Eq. (71) and $O(n_0^2)$ terms neglected we get

$$F_m = \frac{4i}{\delta_1}\sum_{n=-\infty}^{\infty} Z_n F_n + n_0 \sum_{n=-\infty}^{\infty} Z_n F_n \times\left(\pi b^2 d_{n-m}(kb) - \frac{4i\delta_2}{\delta_1^2}\right), \quad m\in\mathbb{Z}. \tag{75}$$

At leading order this gives

$$F_m = \frac{4i}{\delta_1}\sum_{n=-\infty}^{\infty} Z_n F_n, \quad m\in\mathbb{Z}, \tag{76}$$

which implies that all the F_m are equal. If we write $F_m = F$, Eq. (76) becomes

$$\delta_1 = 4i\sum_{s=-\infty}^{\infty} Z_s = -4i f(0), \tag{77}$$

where f is the far-field pattern, given by Eq. (53).

Returning to Eq. (75), we now put $F_m = F + n_0 q_m$, and then the $O(n_0)$ terms give

$$q_m = -\frac{1}{f(0)}\sum_{n=-\infty}^{\infty} Z_n q_n + \pi b^2 F \sum_{n=-\infty}^{\infty} Z_n d_{n-m}(kb) - \frac{iF\delta_2}{4f(0)}, \quad m\in\mathbb{Z}. \tag{78}$$

It follows that $q_m - \pi b^2 F \sum_{n=-\infty}^{\infty} Z_n d_{n-m}$ must be independent of m, call it Q:

$$\begin{aligned} Q &= -\frac{1}{f(0)}\sum_{n=-\infty}^{\infty} Z_n q_n - \frac{iF\delta_2}{4f(0)} \\ &= -\frac{1}{f(0)}\sum_{n=-\infty}^{\infty} Z_n\left(Q + F\pi b^2 \sum_{s=-\infty}^{\infty} Z_s d_{s-n}(kb)\right) - \frac{iF\delta_2}{4f(0)}. \end{aligned} \tag{79}$$

Hence

$$\delta_2 = 4\pi i b^2 \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} Z_n Z_s d_{s-n}(kb) \tag{80}$$

and so we obtain the approximation

$$K^2 = k^2 - 4i n_0 f(0) + 4\pi i b^2 n_0^2 \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} Z_n Z_s d_{s-n}(kb) + \cdots. \tag{81}$$

For isotropic point scatterers, we have $|Z_0| \gg |Z_n|$ for all $n\neq 0$ and $g = -Z_0$, so that Eq. (81) reduces to Eq. (42) in this limit.

So far we have not made any assumptions about the size of ka or kb (though clearly $kb \geq 2ka$). Now we will assume that kb is small. In the limit $x\to 0$, we have $x^2 d_n(x) \sim 2i|n|/\pi$. Hence as $kb\to 0$,

$$\delta_2 \sim -\frac{8}{k^2}\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} |s-n| Z_n Z_s. \tag{82}$$

Now

$$\begin{aligned} [f(\theta)]^2 &= \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} Z_n Z_s e^{i(n+s)\theta} \\ &= \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} Z_n Z_s \cos(n-s)\theta \end{aligned} \tag{83}$$

since $Z_n = Z_{-n}$. Thus

$$\frac{d}{d\theta}[f(\theta)]^2 = -\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} (n-s) Z_n Z_s \sin(n-s)\theta. \tag{84}$$

Also

$$\int_0^{\pi} \cot\tfrac{1}{2}\theta \sin m\theta \, d\theta = \pi\,\mathrm{sgn}(m), \tag{85}$$

see Eq. 3.612(7) in Gradshteyn and Ryzhik (2000). Thus, setting $kb=0$ gives

$$K^2=k^2-4\mathrm{i}n_0 f(0)+\frac{8n_0^2}{\pi k^2}\int_0^\pi \cot(\theta/2)\frac{\mathrm{d}}{\mathrm{d}\theta}[f(\theta)]^2\mathrm{d}\theta. \tag{86}$$

The integral appearing here is convergent because $f'(0)=0$.

D. Effect of pair-correlation function choice

Here, we consider the effect of using a more complicated pair-correlation function, defined by Eq. (15) in terms of the function $\chi(r;n_0)$. This function must decay rapidly to zero as $r\to\infty$ and, in addition, $\chi(r;n_0)\to 0$ as $n_0\to 0$ for any fixed r. For example, Bose and Mal (1973) suggest using $\chi(r;n_0)=\mathrm{e}^{-r/L(n_0)}$, where the correlation length $L(n_0)\to 0$ as $n_0\to 0$. Other authors have supposed that $\chi(r;n_0)=0$ for $r>b'>b$, where the radius b' may be taken as $2b$; see, for example, p. 1072 of Bose (1996) or Eq. (27) in Twersky (1978).

Proceeding as in Sec. IV B, we obtain Eq. (57) with an additional factor of $[1+\chi(R_{12};n_0)]$ in the integrand. Evaluating this equation for $x_1>\ell+b'$, assuming that $\chi(r;n_0)=0$ for $r>b'$, we obtain Eq. (59) with M_{n-m} replaced by M'_{n-m}, where

$$\begin{aligned}M'_n&=M_n+\int_{b<R_{12}<b'}\psi_n(x_{21},y_{21})\\&\quad\times\Psi(x_{21},y_{21})\chi(R_{12};n_0)\mathrm{d}x_2\mathrm{d}y_2\\&=M_n+2\pi\mathrm{i}^n\mathrm{e}^{\mathrm{i}n\varphi}W_n,\end{aligned}$$

$$W_n=\int_b^{b'}H_n(kR)J_n(KR)\chi(R;n_0)R\mathrm{d}R,$$

and M_n is defined by Eq. (61). Hence, we obtain a modified form of Eq. (71), namely

$$F_m+2n_0\pi\sum_{n=-\infty}^{\infty}F_nZ_n\left\{\frac{\mathcal{N}_{n-m}(Kb)}{k^2-K^2}+W_{n-m}\right\}=0,\quad m\in\mathbb{Z}, \tag{87}$$

from which K is to be determined. This homogeneous system for F_n is Eq. (33) in Bose and Mal (1973) and it is a special case of Eq. (24) in Siqueira and Sarabandi (1996). Moreover, the fact that $W_n=o(1)$ as $n_0\to 0$ means that the approximations for K obtained in Sec. IV C, namely Eqs. (81) and (86), are unchanged by the presence of χ.

E. Reproducing Twersky's formula

It is implicit in the work of Twersky (1962) (and others) that the complications arising when a scatterer center is closer to the boundary $x=0$ than its radius are ignored. It was pointed out by Lloyd and Berry (1967) that, since all scatterers are treated equally, ignoring the boundary-layer effects is equivalent to using a pair-correlation function with the following property: if one scatterer is at (x_1,y_1), then no other scatterer [with center (x_2,y_2)] can occupy the infinite strip $x_1-a<x_2<x_1+a$. Thus, instead of Eq. (14), the choice

$$Np(\mathbf{r}_2|\mathbf{r}_1)=\begin{cases}0, & |x_{21}|<a,\\ n_0, & |x_{21}|>a,\end{cases} \tag{88}$$

was made. We shall show that use of Eq. (88) leads to Twersky's formula, Eq. (4).

Setting $s=1$ and taking the conditional average of Eq. (52) in the usual way, and looking for a solution in the form of Eq. (56) now leads to

$$\begin{aligned}&\Phi_m(x_1)+n_0\sum_{n=-\infty}^{\infty}Z_n(-\mathrm{i})^{n-m}\left(\int_0^{x_1-a}+\int_{x_1+a}^{\infty}\right)\\&\quad\times L_{n-m}(x_{21})\Phi_n(x_2)\mathrm{d}x_2=-\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{in}}}\mathrm{e}^{\mathrm{i}\alpha x_1},\quad m\in\mathbb{Z},\end{aligned} \tag{89}$$

where $L_n(X)$ is defined by Eq. (60).

Suppose that for $x>\ell$ we can write [cf. Eq. (58)]

$$\Phi_m(x)=F_m\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{in}}}\mathrm{e}^{\mathrm{i}\lambda x}, \tag{90}$$

where $\mathrm{Im}\,\lambda>0$. Then if $x_1>\ell+a$, Eq. (89) becomes

$$\begin{aligned}&F_m\mathrm{e}^{\mathrm{i}\lambda x_1}+n_0\sum_{n=-\infty}^{\infty}Z_n(-\mathrm{i})^{n-m}\mathrm{e}^{\mathrm{i}m\theta_{\mathrm{in}}}\int_0^{\ell}\Phi_n(x_2)L_{n-m}(x_{21})\\&\quad\times\mathrm{d}x_2+n_0\mathrm{e}^{\mathrm{i}\lambda x_1}\sum_{n=-\infty}^{\infty}F_nZ_n\mathrm{e}^{-\mathrm{i}(n-m)(\pi/2+\theta_{\mathrm{in}})}\\&\quad\times\left(\int_{\ell}^{x_1-a}+\int_{x_1+a}^{\infty}\right)L_{n-m}(x_{21})\mathrm{e}^{\mathrm{i}\lambda x_{21}}\mathrm{d}x_2=-\mathrm{e}^{\mathrm{i}\alpha x_1},\\&\quad m\in\mathbb{Z}.\end{aligned} \tag{91}$$

We have already evaluated $L_n(x)$ for $x<0$, see Eq. (66). Now, we also need its value for $x>0$; we have

$$\alpha L_n(x)=\begin{cases}2(-\mathrm{i})^n\mathrm{e}^{-\mathrm{i}n\theta_{\mathrm{in}}}\mathrm{e}^{\mathrm{i}\alpha x} & x>0\\ 2\mathrm{i}^n\mathrm{e}^{\mathrm{i}n\theta_{\mathrm{in}}}\mathrm{e}^{-\mathrm{i}\alpha x} & x<0.\end{cases} \tag{92}$$

Using these in Eq. (91) gives

$$\tilde{\mathcal{A}}_m\mathrm{e}^{\mathrm{i}\lambda x}+\tilde{\mathcal{B}}\mathrm{e}^{\mathrm{i}\alpha x}=-\mathrm{e}^{\mathrm{i}\alpha x},\quad x>\ell+a,\quad m\in\mathbb{Z},$$

where

$$\begin{aligned}\tilde{\mathcal{A}}_m&=F_m-\frac{2\mathrm{i}n_0}{\alpha}\sum_{n=-\infty}^{\infty}F_nZ_n\\&\quad\times\left\{\frac{\mathrm{e}^{-\mathrm{i}(\lambda-\alpha)a}}{\lambda-\alpha}-\frac{\mathrm{e}^{\mathrm{i}(\lambda+\alpha)a}}{\lambda+\alpha}\mathrm{e}^{\mathrm{i}(n-m)\theta_{\mathrm{T}}}\right\},\end{aligned}$$

$$\tilde{\mathcal{B}}=\frac{2n_0}{\alpha}\sum_{n=-\infty}^{\infty}Z_n\left\{\mathrm{e}^{\mathrm{i}n\theta_{\mathrm{in}}}\int_0^{\ell}\Phi_n(t)\mathrm{e}^{-\mathrm{i}\alpha t}\mathrm{d}t+\frac{\mathrm{i}F_n\mathrm{e}^{\mathrm{i}(\lambda-\alpha)\ell}}{\lambda-\alpha}\right\}$$

and $\theta_{\mathrm{T}}=\pi-2\theta_{\mathrm{in}}$. Thus, λ is to be found from $\tilde{\mathcal{A}}_m=0$ for all $m\in\mathbb{Z}$.

As before, we write $K^2-k^2=\lambda^2-\alpha^2=\delta_1n_0+\delta_2n_0^2+\cdots$. Hence,

$$\begin{aligned}&\frac{\mathrm{e}^{-\mathrm{i}(\lambda-\alpha)a}}{\lambda-\alpha}-\frac{\mathrm{e}^{\mathrm{i}(\lambda+\alpha)a}}{\lambda+\alpha}\mathrm{e}^{\mathrm{i}(n-m)\theta_{\mathrm{T}}}\\&\quad=\frac{2\alpha}{\delta_1n_0}+\frac{1}{2\alpha}\{1-2\mathrm{i}\alpha a-\mathrm{e}^{2\mathrm{i}\alpha a}\mathrm{e}^{\mathrm{i}(n-m)\theta_{\mathrm{T}}}\}-\frac{2\alpha\delta_2}{\delta_1^2}\\&\qquad+O(n_0).\end{aligned}$$

Substituting in $\widetilde{\mathcal{A}}_m = 0$ and neglecting terms that are $O(n_0^2)$, we obtain

$$F_m - \sum_{n=-\infty}^{\infty} F_n Z_n \left\{ \frac{4\mathrm{i}}{\delta_1} + \frac{\mathrm{i}n_0}{\alpha^2}\{1 - 2\mathrm{i}\alpha a - \mathrm{e}^{2\mathrm{i}\alpha a}\mathrm{e}^{\mathrm{i}(n-m)\theta_\mathrm{T}}\} - \frac{4\mathrm{i}n_0\delta_2}{\delta_1^2} \right\} = 0$$

for $m \in \mathbb{Z}$. Proceeding as in Sec. IV C, we obtain Eq. (77), as before. Then, the $O(n_0)$ terms give

$$\delta_2 = \frac{\delta_1^2}{4\alpha^2}(1 - 2\mathrm{i}\alpha a) + \frac{4}{\alpha^2}\mathrm{e}^{2\mathrm{i}\alpha a} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s \mathrm{e}^{\mathrm{i}(s-n)\theta_\mathrm{T}}$$

$$= \frac{4}{\alpha^2}\{\mathrm{e}^{2\mathrm{i}\alpha a}[f(\theta_T)]^2 - (1 - 2\mathrm{i}\alpha a)[f(0)]^2\}.$$

Hence, if we let $\alpha a \to 0$ in this formula, we recover Twersky's (erroneous) formula, Eq. (4).

V. CONCLUDING REMARKS

We have derived a two-dimensional version of the three-dimensional Lloyd–Berry formula for the effective wave number in a dilute random configuration of scatterers, using methods that differ from those used by Lloyd and Berry (1967). Much remains to be done in order to validate the new formula. Specifically, it should be possible to compare its predictions with those obtained from full numerical simulations (using Monte Carlo methods) and from experiments. Some comparisons between Monte Carlo results and solutions of the infinite system (87), for various choices of χ, have been reported by Siqueira and Sarabandi (1996). They used lossy cylinders, and found good agreement for low area fractions, with little dependence on χ.

Price *et al.* (1988) have compared the predictions of Twersky's formula, Eq. (4), with experimental results obtained from sound propagation through three forests; they found "poor" agreement, but perhaps this could be attributed to errors in the formula and the crude approximation of an actual forest by a random array of sound-hard circular cylinders. For a recent review of the quantification of attenuation effects due to trees, see (Attenborough, 2002). Several other experimental studies, in the context of fiber-reinforced materials, are cited in the paper by Verbis *et al.* (2001).

In three dimensions, there is an extensive literature on comparisons between experiments, direct numerical simulations, and various theories, including the Lloyd–Berry formula; see, for example, Sec. 4.3.12 of Povey (1997), Hipp *et al.* (1999), and references therein. For low volume fractions and properly modelled spheres, the agreement is generally good: according to Povey (1997), p. 133, there is "a sizable body of evidence in support of the acoustic multiple scattering theory."

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