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EXACT SOLUTION OF SOME INTEGRAL EQUATIONS OVER A CIRCULAR DISC

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ABSTRACT. Two-dimensional integral equations of the first kind over a circular disc are considered. The kernels involve the distance between two points on the disc raised to an arbitrary power. A review is given, comparing several published exact solutions for weakly-singular equations: these solutions are complicated, but three of them are shown to be equivalent. Some extensions to hypersingular equations are discussed.

1. Introduction. This paper is concerned with integral equations of the form

$$\int_D \frac{w(x,y)}{R^{2\alpha}}\, dx\, dy = p(x_0,y_0), \quad (x_0,y_0) \in D. \tag{1}$$

Here, $D = \{(x,y) : x^2+y^2 < a^2\}$ is a circular disc of radius a, centered at the origin in the xy-plane, p is a given function and w is to be found. R is the distance between two points in the disc,

$$R = \{(x-x_0)^2 + (y-y_0)^2\}^{1/2},$$

and α is a positive parameter. The kernel $R^{-2\alpha}$ is weakly singular for $0 < \alpha < 1$, and it is hypersingular for $\alpha \geq 1$. (We shall define "hypersingular" later. Note that $\alpha = 1$ does not lead to a "singular" integral equation, as the principal-value integral of R^{-2} does not exist.)

The case $\alpha = 1/2$ is classical: it arises in the problem of the electrified disc [**6, 24, 27**]. This problem requires the determination of a harmonic function in three-dimensional space, $V(x,y,z)$, with the Dirichlet condition, $V = 1$, on the disc and the condition $V \to 0$ at infinity.

More generally, the weakly-singular case ($0 < \alpha < 1$) has been studied by several authors. Complicated formulas for the exact solution of (1)

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are available: one purpose of this paper is to review these formulas, and to confirm that some of them are correct.

The hypersingular case ($\alpha \geq 1$) has received less attention, with one exception: $\alpha = 3/2$. This exceptional case occurs when the Neumann problem for Laplace's equation exterior to a disc is to be solved. The resulting equation can be written in several equivalent forms:

$$\nabla_0^2 \int_D \frac{w(x,y)}{R}\, dx\, dy = p(x_0, y_0), \quad (x_0, y_0) \in D; \tag{2}$$

$$-\!\!\!\!\!\!\int_D \left\{ \frac{\partial w}{\partial x} \frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \frac{\partial w}{\partial y} \frac{\partial}{\partial y}\left(\frac{1}{R}\right) \right\} dx\, dy = p(x_0, y_0), \quad (x_0, y_0) \in D; \tag{3}$$

$$\times\!\!\!\!\!\!\int_D \frac{w(x,y)}{R^3}\, dx\, dy = p(x_0, y_0), \quad (x_0, y_0) \in D; \tag{4}$$

here,

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} \tag{5}$$

is the two-dimensional Laplacian. Usually, these equations are to be solved for w subject to

$$w(x,y) = 0 \quad \text{for } (x,y) \in \partial D, \tag{6}$$

where ∂D is the boundary of D (the edge of the disc).

Equation (2) can be found in Bueckner's article [**7**, p. 287], where it is credited to Panasyuk [**23**, eqn. (VI.21)], who in turn refers to a Russian paper by M.Ya. Leonov published in 1940. Equation (3) was derived by Kossecka [**18**], Bui [**8**] and Guidera and Lardner [**15**]. It involves Cauchy principal-value integrals over D. Equation (4) is a hypersingular integral equation for w; it was first derived by Ioakimidis [**16**]. Further references are given in [**21, 22**]. Notice that the equivalence of (2) and (4) can be taken as *defining* the hypersingular integral in (4).

When considering the integral equation (1), it is natural to introduce polar coordinates, r and θ, defined by $x = r\cos\theta$ and $y = r\sin\theta$, so that

$$D = \{(r,\theta) : 0 \le r < a,\ -\pi \le \theta < \pi\}.$$

Then, putting $w(r\cos\theta, r\sin\theta) = u(r,\theta)$ and $p(r_0\cos\theta_0, r_0\sin\theta_0) = f(r_0,\theta_0)$, (1) becomes

$$\int_{-\pi}^{\pi}\int_0^a \frac{u(r,\theta)}{R^{2\alpha}}\, r\,dr\,d\theta = f(r_0,\theta_0), \quad 0 \le r_0 < a,\ -\pi \le \theta_0 < \pi, \tag{7}$$

where $R = \{r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)\}^{1/2}$. We shall describe various solutions of this integral equation.

We begin in Section 2 by giving the solution of (7) when $f \equiv 1$ and $0 < \alpha < 1$. These results are useful for checking published solutions (supposedly valid when f is more general), they are obtained using methods that generalize, and they reveal the typical behavior near the edge of the disc. Then, three formulas for the exact solution when $0 < \alpha < 1$ are presented in Section 3. These formulas are complicated, and it is not evident that they are equivalent. In fact, they are: this is shown in Section 4 using a Fourier decomposition for f. (Several erroneous formulas are also mentioned.) Finally, some comments on hypersingular equations with $1 < \alpha < 2$ are given in Section 5.

2. The simplest axisymmetric weakly-singular problem. We consider the axisymmetric problem with $0 < \alpha < 1$ and f constant; we choose $f = 1$. Thus, as u does not depend on θ, (7) becomes

$$\int_0^a u(r)\, W(r,r_0;\alpha)\, r\,dr = 1, \quad 0 \le r_0 < a, \tag{8}$$

where

$$W(r,r_0;\alpha) = \int_{-\pi}^{\pi} \frac{d\theta}{R^{2\alpha}}.$$

To solve (8), we can use an integral representation for W; we describe two choices.

2.1 Use of a Copson-type representation. It is known that

$$W(r,r_0;\alpha) = 4\sin\pi\alpha \int_0^{\min(r,r_0)} \frac{t^{2\alpha-1}\,dt}{(r^2-t^2)^\alpha (r_0^2-t^2)^\alpha},$$

for $0 < \alpha < 1$, $r > 0$ and $r_0 > 0$; for a proof, see [**17**, Lemma 1.1] or [**9**, eqn. (1.1.3)]. Substituting for W in (8) and changing the order of integration gives

$$4 \sin \pi\alpha \int_0^{r_0} \frac{t^{2\alpha-1}}{(r_0^2 - t^2)^\alpha} \int_t^a \frac{u(r)\, r\, dr}{(r^2 - t^2)^\alpha}\, dt = 1, \quad 0 \le r_0 < a. \tag{9}$$

This equation can be solved by inverting the Abel-like operators. However, it is simpler to guess: we try

$$u(r) = U(a^2 - r^2)^{\alpha-1}, \tag{10}$$

where the constant U is to be found. With this guess and the substitution $r^2 = (a^2 - t^2)x + t^2$, the inner integral in (9) becomes

$$\frac{U}{2} \int_0^1 x^{-\alpha}(1-x)^{\alpha-1}\, dx = \frac{U\pi}{2 \sin \pi\alpha},$$

which does not depend on t. Then, using the substitution $t^2 = r_0^2 \tau$, (9) becomes

$$U\pi \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}\, d\tau = 1, \quad 0 \le r_0 < a,$$

so that

$$U = \pi^{-2} \sin \pi\alpha. \tag{11}$$

Hence, the (unique) solution of (7) when $f \equiv 1$ is given by

$$u(r) = \frac{\sin \pi\alpha}{\pi^2 (a^2 - r^2)^{1-\alpha}}. \tag{12}$$

In particular, we note that $u(r)$ is unbounded at $r = a$.

2.2 Use of a Bessel-function representation. Fahmy et al. [**13**] proved that

$$W(r, r_0; \alpha) = \frac{\pi\Gamma(1-\alpha)}{2^{2\alpha-2}\Gamma(\alpha)} \int_0^\infty J_0(tr)\, J_0(tr_0)\, t^{2\alpha-1}\, dt, \quad 0 < \alpha < 1.$$

Substituting for W in (8) and changing the order of integration gives
(13)
$$\frac{\pi\Gamma(1-\alpha)}{2^{2\alpha-2}\Gamma(\alpha)}\int_0^\infty t^{2\alpha-1}J_0(tr_0)\int_0^a u(r)\,J_0(tr)\,r\,dr\,dt = 1, \quad 0 \le r_0 < a.$$

Making use of the guess (10), the inner integral becomes

$$Ua^{2\alpha}\int_0^1 J_0(atx)\,(1-x^2)^{\alpha-1}\,x\,dx = 2^{\alpha-1}Ua^\alpha t^{-\alpha}\Gamma(\alpha)\,J_\alpha(at),$$

using [**14**, (6.567.1)]. Thus, (13) becomes

$$\frac{\pi a^\alpha}{2^{\alpha-1}}\,U\,\Gamma(1-\alpha)\int_0^\infty t^{\alpha-1}J_0(tr_0)\,J_\alpha(ta)\,dt = 1, \quad 0 \le r_0 < a. \tag{14}$$

The remaining integral is of Weber-Schafheitlin type; from [**14**, (6.574.1)], its value is $2^{\alpha-1}a^{-\alpha}\Gamma(\alpha)$, since the hypergeometric function satisfies $F(\alpha, 0; 1; r_0^2/a^2) = 1$. Hence, we recover (11) and (12).

3. The general weakly-singular integral equation. Consider the weakly-singular version of (7). We state three formulas for its solution (and mention some erroneous formulas).

3.1 Fabrikant's solution. This solution is given in several papers. It can be written as
(15)
$$u(r_0,\theta_0) = \frac{\sin\pi\alpha}{\pi^2}\left\{\frac{\Phi(a,r_0,\theta_0)}{(a^2-r_0^2)^{1-\alpha}} - \int_{r_0}^a \frac{\partial}{\partial r}\,\Phi(r,r_0,\theta_0)\,\frac{dr}{(r^2-r_0^2)^{1-\alpha}}\right\},$$

where
(16)
$$\Phi(r,r_0,\theta_0) = f(0,\theta_0) + r^{2-2\alpha}\int_0^r \frac{\partial}{\partial\rho}\left\{\left[P\left(\frac{\rho r_0}{r^2}\right)f\right](\rho,\theta_0)\right\}\frac{d\rho}{(r^2-\rho^2)^{1-\alpha}},$$

$$[P(k)f](\rho,\phi) = \int_{-\pi}^{\pi}\mathcal{P}(k,\phi-\phi_0)\,f(\rho,\phi_0)\,d\phi_0 \tag{17}$$

and

$$\mathcal{P}(k,\theta) = \frac{1-k^2}{2\pi(1+k^2-2k\cos\theta)} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} k^{|n|}\,e^{in\theta}, \tag{18}$$

the series converging for $|k| < 1$. P is called the *Poisson operator*. The formula (15) can be obtained from [**12**], correcting earlier work by Fabrikant [**11, 26**]. The formal derivation of (15) was examined carefully in [**20**].

The authorship of [**11, 12, 26**] (and two other papers) was the subject of an investigation [**4**]. It was concluded that Fabrikant was the sole author of these papers, as they "are, in substance, extensive duplications both of an earlier published work of Fabrikant [published in 1971, in Russian; see MR#0285875] which is unreferenced, and of each other, without mutual references" [**4**, Appendix D]. Thus, we are justified in referring to (15) as "Fabrikant's solution."

As $P(k)1 = 1$, we obtain $\Phi = 1$ when $f \equiv 1$, and then (15) reduces to the known solution (12) for this special choice of f.

In his subsequent books, Fabrikant obtained (15) but with Φ replaced by

(19)

$$\Phi_1(r,r_0,\theta_0) = r^{-2\alpha}\int_0^r \frac{\partial}{\partial\rho}\left\{\rho^{2-2\alpha}\left[P\left(\frac{\rho r_0}{r^2}\right)f\right](\rho,\theta_0)\right\}\frac{\rho^{2\alpha}\,d\rho}{(r^2-\rho^2)^{1-\alpha}};$$

see [**9**, eqn. (2.3.10)] or [**10**, eqn. (2.1.7)]. However, when $f \equiv 1$, this gives $\Phi_1 = (1-\alpha)/\alpha$, so that (19) only gives the correct result when $\alpha = 1/2$; notice that this was the only case checked in [**25**].

3.2 Kahane's solution. Kahane [**17**] proved that (7) is uniquely solvable when f is twice continuously differentiable on the closed disc, $f \in C^2(\overline{D})$. (In an earlier unpublished technical report, Ahner and Kahane [**3**] proved similar results for $\alpha = 1/2$.) Kahane's solution is scaled so that $a = 1$; after removing this scaling, his solution is

$$u(r_0,\theta_0) = \frac{\sin\pi\alpha}{\pi^2}\left\{\frac{\mathcal{A}(r_0,\theta_0)}{(a^2-r_0^2)^{1-\alpha}} - \mathcal{B}(r_0,\theta_0)\right\}, \tag{20}$$

where

(21)

$$\begin{aligned}\mathcal{A}(r_0,\theta_0) &= f(0,\theta_0) + a^{2-2\alpha}\int_0^a\int_{-\pi}^{\pi}\mathcal{P}\left(\frac{rr_0}{a^2},\,\theta_0-\theta\right)\frac{\partial f}{\partial r}(r,\theta)\,\frac{d\theta\,dr}{(a^2-r^2)^{1-\alpha}}\\ &\quad - a^{2-2\alpha}\int_0^a\int_{-\pi}^{\pi}\frac{1}{r}\,\mathcal{Q}\left(\frac{rr_0}{a^2},\,\theta_0-\theta\right)\frac{\partial^2 f}{\partial\theta^2}(r,\theta)\,\frac{d\theta\,dr}{(a^2-r^2)^{1-\alpha}},\end{aligned}$$

(22)

$$\begin{aligned}\mathcal{B}(r_0,\theta_0) &= \frac{1}{a^{2\alpha}}\int_{r_0}^a\int_0^a\int_{-\pi}^{\pi}\mathcal{P}\left(\frac{\xi r_0}{at},\,\theta_0-\theta\right)\\ &\qquad\qquad \times\left(\nabla^2 f\right)\big|_{r=\xi t/a}\,\frac{\xi t\,d\theta\,d\xi\,dt}{\{(a^2-\xi^2)(t^2-r_0^2)\}^{1-\alpha}}\\ &= \int_{r_0}^a\int_0^t\int_{-\pi}^{\pi}\mathcal{P}\left(\frac{rr_0}{t^2},\,\theta_0-\theta\right)\nabla^2 f\,\frac{t^{1-2\alpha}\,r\,d\theta\,dr\,dt}{\{(t^2-r^2)(t^2-r_0^2)\}^{1-\alpha}},\\ (\nabla^2 f)(r,\theta) &= \frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2},\end{aligned}$$

$\mathcal{P}$ is defined by (18) and, for $|k|<1$,

$$\mathcal{Q}(k,\theta) = -\frac{1}{2\pi}\log\left|1+k^2-2k\cos\theta\right| = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{k^n}{n}\cos n\theta.$$

When $f\equiv 1$, (20) immediately gives the known solution (12) for this special choice of f.

3.3 Li and Rong's solution. In their Theorem 1, Li and Rong [**19**] obtained

$$u(r_0,\theta_0) = -\frac{(1-\alpha)^2}{\pi^2}\int_D \mathcal{K}_\alpha(r,r_0,\theta-\theta_0)\,f(r,\theta)\,r\,dr\,d\theta \tag{23}$$

with

$$\begin{aligned}&\mathcal{K}_\alpha(r,r_0,\theta-\theta_0)\\ &\quad = -4\sin\pi\alpha\int_{\max(r,r_0)}^a\mathcal{P}\left(\frac{rr_0}{s^2},\,\theta-\theta_0\right)\frac{s^{3-2\alpha}\,ds}{\{(s^2-r^2)(s^2-r_0^2)\}^{2-\alpha}}.\end{aligned} \tag{24}$$

As $0 < \alpha < 1$, we see that the integral in (24) is divergent; it is to be interpreted as a finite-part integral. In their Lemma 1, Li and Rong [**19**] also obtained the (finite-part) integral representation

$$\begin{aligned} \frac{1}{R^{4-2\alpha}} &= -\,4\sin\pi\alpha \\ &\quad\times \int_{\max(r,r_0)}^{\infty} \mathcal{P}\left(\frac{rr_0}{s^2},\, \theta-\theta_0\right) \frac{s^{3-2\alpha}\,ds}{\{(s^2-r^2)(s^2-r_0^2)\}^{2-\alpha}}, \end{aligned} \tag{25}$$

which differs from (24) by the value of the upper limit of integration. Hence, $\mathcal{K}_\alpha$ and $R^{2\alpha-4}$ differ by the value of an ordinary improper integral, giving

$$u(r_0,\theta_0) = v(r_0,\theta_0) + w(r_0,\theta_0), \tag{26}$$

where

$$\begin{aligned} v(r_0,\theta_0) &= -\,\frac{(1-\alpha)^2}{\pi^2}\int_D \frac{f(r,\theta)}{R^{4-2\alpha}}\,r\,dr\,d\theta, \\ w(r_0,\theta_0) &= -\,\frac{(1-\alpha)^2}{\pi^2}\int_D \mathcal{K}_\alpha^\infty(r,r_0,\theta-\theta_0)\,f(r,\theta)\,r\,dr\,d\theta, \end{aligned} \tag{27}$$

with

$$\mathcal{K}_\alpha^\infty(r,r_0,\theta) = 4\sin\pi\alpha\int_a^\infty \mathcal{P}\left(\frac{rr_0}{s^2},\,\theta\right)\frac{s^{3-2\alpha}\,ds}{\{(s^2-r^2)(s^2-r_0^2)\}^{2-\alpha}}.$$

Now, as

$$\nabla^2 R^\beta = \beta^2 R^{\beta-2}, \quad \text{for any } \beta, \tag{28}$$

it is natural to use this fact with $\beta = 2\alpha - 2$ in order to define the hypersingular double integral in (27). Thus, we obtain

$$v(r_0,\theta_0) = -\frac{1}{4\pi^2}\,\nabla_0^2\int_D \frac{f(r,\theta)}{R^{2-2\alpha}}\,r\,dr\,d\theta, \tag{29}$$

where ∇_0^2 is defined by (5). The integral in (29) is weakly singular. As (25) is also valid with α replaced by $\alpha+1$, we obtain

$$\begin{aligned} v(r_0,\theta_0) &= -\,\frac{\sin\pi\alpha}{\pi^2}\,\nabla_0^2 \\ &\quad\times\int_0^a\int_{\max(r,r_0)}^\infty \left[P\left(\frac{rr_0}{s^2}\right)f\right](r,\theta_0)\,\frac{s^{1-2\alpha}\,r\,ds\,dr}{\{(s^2-r^2)(s^2-r_0^2)\}^{1-\alpha}}, \end{aligned} \tag{30}$$

after using (17). Similarly,

$$w(r_0,\theta_0) = -\frac{4}{\pi^2}(1-\alpha)^2 \sin\pi\alpha \times \int_0^a \int_a^\infty \left[P\left(\frac{rr_0}{s^2}\right) f\right](r,\theta_0) \frac{s^{3-2\alpha}\, r\, ds\, dr}{\{(s^2-r^2)(s^2-r_0^2)\}^{2-\alpha}}. \tag{31}$$

Li and Rong's solution, (23), is given as a certain hypersingular integral operator applied to f. Superficially, at least, this is attractive: we can regard the operator in (7) as a pseudodifferential operator of order $2\alpha-2$, and so the inverse should be of order $2-2\alpha$. (These statements can be made more precise, but we choose not to do so.) However, as (23) does not involve any explicit derivatives of f, it does not simplify much when $f\equiv 1$; nevertheless, it does yield the correct solution for this very special case, as shown directly in Appendix A.

4. Fourier decomposition of $f(r,\theta)$. In order to compare the three solutions described in Section 3, it is convenient to assume that $f(r,\theta) = f_m(r)\, e^{im\theta}$, with integer m. Without loss of generality, assume that $m \geq 0$. Let

$$F_m(r) = \frac{1}{r}\frac{d}{dr}\left(r^m f_m(r)\right); \tag{32}$$

assuming that f is in $C^2(\overline{D})$, $F_m(0)$ exists.

4.1 Fabrikant's solution. As $[P(k)f](r,\theta) = k^m f_m(r)\, e^{im\theta}$, (16) reduces to

$$\Phi(r,r_0,\theta_0) = f_0(0)\delta_{0m} + r^{2-2m-2\alpha}\, r_0^m e^{im\theta_0}\, \Psi_m(r)$$

where δ_{ij} is the Kronecker delta and

$$\Psi_m(r) = \int_0^r \frac{\rho\, F_m(\rho)\, d\rho}{(r^2-\rho^2)^{1-\alpha}}. \tag{33}$$

According to (15), we need

$$\frac{\partial}{\partial r}\Phi(r,r_0,\theta_0) = r_0^m e^{im\theta_0}\, r^{1-2m-2\alpha}\left\{2(1-m-\alpha)\Psi_m(r) + r\Psi_m'(r)\right\}. \tag{34}$$

For Ψ_m', we first integrate by parts to give

$$\Psi_m(r) = \frac{r^{2\alpha}}{2\alpha} F_m(0) + \frac{1}{2\alpha} \int_0^r (r^2 - \rho^2)^\alpha F_m'(\rho)\, d\rho. \tag{35}$$

We can then differentiate, giving

$$\Psi_m'(r) = r^{2\alpha-1} F_m(0) + r \int_0^r \frac{F_m'(\rho)\, d\rho}{(r^2-\rho^2)^{1-\alpha}}.$$

Writing $r = r^{-1}(r^2 - \rho^2 + \rho^2)$, we obtain

$$\Psi_m'(r) = r^{2\alpha-1} F_m(0) + \frac{1}{r} \int_0^r (r^2 - \rho^2)^\alpha F_m'(\rho)\, d\rho + \frac{1}{r} \int_0^r \frac{\rho^2 F_m'(\rho)\, d\rho}{(r^2-\rho^2)^{1-\alpha}}.$$

But the first integral also appears in the expression for Ψ_m, (35), whence

$$r\Psi_m'(r) = 2\alpha\Psi_m(r) + \int_0^r \frac{\rho^2 F_m'(\rho)\, d\rho}{(r^2-\rho^2)^{1-\alpha}}. \tag{36}$$

Substituting in (34) gives

$$\begin{aligned} \frac{\partial}{\partial r} \Phi(r, r_0, \theta_0) &= r_0^m e^{im\theta_0}\, r^{1-2m-2\alpha} \\ &\qquad \times \int_0^r \{\rho^2 F_m'(\rho) + (2-2m)\, \rho F_m(\rho)\} \frac{d\rho}{(r^2-\rho^2)^{1-\alpha}} \\ &= r_0^m e^{im\theta_0}\, r^{1-2m-2\alpha} \int_0^r \frac{\rho^{m+1}\, \Delta_m(\rho)}{(r^2-\rho^2)^{1-\alpha}}\, d\rho, \end{aligned}$$

where we have used (33) and put

$$\Delta_m(r) = f_m''(r) + r^{-1} f_m'(r) - (m/r)^2 f_m(r).$$

Thus, we find that Fabrikant's solution (15) reduces to

$$u(r_0, \theta_0) = \frac{\sin \pi\alpha}{\pi^2}\, r_0^m e^{im\theta_0} \left\{ \frac{A_m}{(a^2 - r_0^2)^{1-\alpha}} - B_m(r_0) \right\} \tag{37}$$

where

$$A_m = f_0(0)\delta_{0m} + a^{2-2m-2\alpha} \times \int_0^a \{f_m'(\rho) + (m/\rho)f_m(\rho)\} \frac{\rho^m \, d\rho}{(a^2-\rho^2)^{1-\alpha}}, \tag{38}$$

$$B(r_0) = \int_{r_0}^a \frac{r^{1-2m-2\alpha}}{(r^2-r_0^2)^{1-\alpha}} \int_0^r \frac{\rho^{m+1}\,\Delta_m(\rho)}{(r^2-\rho^2)^{1-\alpha}} \, d\rho \, dr. \tag{39}$$

In particular, when $m = 0$, we have an axisymmetric problem. For this case, we obtain

$$u(r_0) = \frac{\sin \pi\alpha}{\pi^2} \left\{ \frac{A}{(a^2-r_0^2)^{1-\alpha}} - B(r_0) \right\} \tag{40}$$

where

$$A = f_0(0) + a^{2-2\alpha} \int_0^a \frac{f_0'(\rho)\, d\rho}{(a^2-\rho^2)^{1-\alpha}},$$

$$B(r_0) = \int_{r_0}^a \frac{r^{1-2\alpha}}{(r^2-r_0^2)^{1-\alpha}} \int_0^r \frac{\rho\, f_0''(\rho) + f_0'(\rho)}{(r^2-\rho^2)^{1-\alpha}} \, d\rho \, dr.$$

This reduces to (12) when $f_0 \equiv 1$.

Fahmy et al. [**13**] have obtained formulas that are reminiscent of (15) with $f = f_m(r) \cos m\theta$; see [**13**, eqn. (5.8)] or [**1**, eqn. (3.12)]. However, these papers contain many misprints and, moreover, their formulas do not yield the correct result when $f \equiv 1$.

4.2 Kahane's solution. For Kahane's solution, (20), we have

$$\int_{-\pi}^{\pi} \mathcal{P}(k, \theta_0 - \theta)\, e^{im\theta}\, d\theta = k^m\, e^{im\theta_0},$$

$$\int_{-\pi}^{\pi} \mathcal{Q}(k, \theta_0 - \theta)\, e^{im\theta}\, d\theta = \begin{cases} (k^m/m)\, e^{im\theta_0}, & m > 0, \\ 0, & m = 0, \end{cases}$$

and $\nabla^2 f = \Delta_m(r)\, e^{im\theta}$. Thus, (21) and (22) give $\mathcal{A}(r_0, \theta_0) = A_m r_0^m e^{im\theta_0}$ and $\mathcal{B}(r_0, \theta_0) = B_m(r_0)\, r_0^m e^{im\theta_0}$, respectively, in agreement with Fabrikant's solution, (37).

4.3 Li and Rong's solution. Li and Rong's solution is given by (26), (30) and (31). When $f = f_m(r)\, e^{im\theta}$, we obtain

(41)
$$\begin{aligned} v(r_0,\theta_0) &= -\frac{\sin\pi\alpha}{\pi^2}\nabla_0^2 \\ &\quad \times \left\{ r_0^m\, e^{im\theta_0} \int_0^a \int_{\max(r,r_0)}^\infty f_m(r)\, \frac{r^{m+1} s^{1-2m-2\alpha}\, ds\, dr}{\{(s^2-r^2)(s^2-r_0^2)\}^{1-\alpha}} \right\} \\ &= -\pi^{-2}\sin(\pi\alpha)\,\nabla_0^2 \left\{ r_0^m\, e^{im\theta_0} [I_1(r_0) + I_2(r_0)] \right\}, \end{aligned}$$

$$w(r_0,\theta_0) = -\pi^{-2}\sin(\pi\alpha)\, r_0^m\, e^{im\theta_0}\, W(r_0), \tag{42}$$

where

$$\begin{aligned} I_1(r_0) &= \int_{r_0}^a \frac{\Phi_1(s)\, s\, ds}{(s^2-r_0^2)^{1-\alpha}} \\ &= \frac{(a^2-r_0^2)^\alpha}{2\alpha}\,\Phi_1(a) - \int_{r_0}^a \frac{(s^2-r_0^2)^\alpha}{2\alpha}\,\Phi_1'(s)\, ds, \\ I_2(r_0) &= \int_a^\infty \frac{\Phi_2(s)\, s\, ds}{(s^2-r_0^2)^{1-\alpha}}, \end{aligned} \tag{43}$$

$$W(r_0) = 2(\alpha-1)\int_a^\infty \frac{\Phi_3(s)\, s^3\, ds}{(s^2-r_0^2)^{2-\alpha}}, \tag{44}$$

$$\begin{aligned} \Phi_1(s) &= \frac{1}{s^{2m+2\alpha}} \int_0^s \frac{r^{m+1} f_m(r)}{(s^2-r^2)^{1-\alpha}}\, dr, \quad s < a, \\ \Phi_2(s) &= \frac{1}{s^{2m+2\alpha}} \int_0^a \frac{r^{m+1} f_m(r)}{(s^2-r^2)^{1-\alpha}}\, dr, \quad s > a, \\ \Phi_3(s) &= \frac{2(\alpha-1)}{s^{2m+2\alpha}} \int_0^a \frac{r^{m+1} f_m(r)}{(s^2-r^2)^{2-\alpha}}\, dr, \quad s > a. \end{aligned} \tag{45}$$

We have

(46)
$$\begin{aligned} \nabla_0^2 \left\{ r_0^m e^{im\theta_0}\, I_j(r_0) \right\} &= r_0^m e^{im\theta_0} \{ I_j''(r_0) + (2m+1) r_0^{-1} I_j'(r_0) \} \\ &= r_0^m e^{im\theta_0} \left\{ r_0 [r_0^{-1} I_j'(r_0)]' + 2(m+1) r_0^{-1} I_j'(r_0) \right\}. \end{aligned}$$

From (43), we have

$$r_0^{-1} I_1'(r_0) = \int_{r_0}^{a} \frac{\Phi_1'(s)\,ds}{(s^2 - r_0^2)^{1-\alpha}} - \frac{\Phi_1(a)}{(a^2 - r_0^2)^{1-\alpha}}.$$

Similarly,

$$r_0^{-1}\left[r_0^{-1} I_1'(r_0)\right]' = \int_{r_0}^{a} \frac{X_1'(s)\,ds}{(s^2 - r_0^2)^{1-\alpha}} - \frac{X_1(a)}{(a^2 - r_0^2)^{1-\alpha}} + \frac{2(\alpha-1)\Phi_1(a)}{(a^2 - r_0^2)^{2-\alpha}},$$

where $X_1(s) = s^{-1}\Phi_1'(s)$. Hence, multiplication by $r_0^2 = s^2 - (s^2 - r_0^2)$ followed by another integration by parts gives

$$\begin{aligned} r_0\left[r_0^{-1} I_1'(r_0)\right]' = \int_{r_0}^{a} \{s X_1'(s) + 2\alpha X_1(s)\} \frac{s\,ds}{(s^2 - r_0^2)^{1-\alpha}} \\ - \frac{a^2 X_1(a)}{(a^2 - r_0^2)^{1-\alpha}} + 2r_0^2 \frac{(\alpha-1)\Phi_1(a)}{(a^2 - r_0^2)^{2-\alpha}}. \end{aligned}$$

For I_2, the integration limits are constant, so that

$$\begin{aligned} r_0^{-1} I_2'(r_0) &= -2(\alpha-1)\int_a^{\infty} \frac{\Phi_2(s)\,s\,ds}{(s^2 - r_0^2)^{2-\alpha}} \\ &= \int_a^{\infty} \frac{\Phi_2'(s)\,ds}{(s^2 - r_0^2)^{1-\alpha}} + \frac{\Phi_2(a)}{(a^2 - r_0^2)^{1-\alpha}}, \\ r_0^{-1}\left[r_0^{-1} I_2'(r_0)\right]' &= -2(\alpha-1)\int_a^{\infty} \frac{X_2(s)\,s\,ds}{(s^2 - r_0^2)^{2-\alpha}} - \frac{2(\alpha-1)\Phi_2(a)}{(a^2 - r_0^2)^{2-\alpha}}, \end{aligned} \tag{47}$$

where $X_2(s) = s^{-1}\Phi_2'(s)$. (Note that $X_2(a)$ does not exist.) Hence,

$$\begin{aligned} r_0\left[r_0^{-1} I_2'(r_0)\right]' = -2(\alpha-1)\int_a^{\infty} \frac{X_2(s)\,s^3\,ds}{(s^2 - r_0^2)^{2-\alpha}} + 2(\alpha-1) \\ \times \int_a^{\infty} \frac{X_2(s)\,s\,ds}{(s^2 - r_0^2)^{1-\alpha}} - 2r_0^2 \frac{(\alpha-1)\Phi_2(a)}{(a^2 - r_0^2)^{2-\alpha}}. \end{aligned}$$

Let $I \equiv I_1 + I_2$. Noting that $\Phi_1(a) = \Phi_2(a)$, we obtain

$$r_0^{-1} I'(r_0) = \int_{r_0}^{a} \frac{X_1(s)\,s\,ds}{(s^2 - r_0^2)^{1-\alpha}} + \int_a^{\infty} \frac{X_2(s)\,s\,ds}{(s^2 - r_0^2)^{1-\alpha}}$$

and

$$r_0\left[r_0^{-1}I'(r_0)\right]' = \int_{r_0}^{a}\{sX_1'(s)+2\alpha X_1(s)\}\frac{s\,ds}{(s^2-r_0^2)^{1-\alpha}} - \frac{a^2X_1(a)}{(a^2-r_0^2)^{1-\alpha}}$$
$$-2(\alpha-1)\int_a^\infty \frac{X_2(s)\,s^3\,ds}{(s^2-r_0^2)^{2-\alpha}}$$
$$+2(\alpha-1)\int_a^\infty \frac{X_2(s)\,s\,ds}{(s^2-r_0^2)^{1-\alpha}}.$$

Hence, if we write

$$v(r_0,\theta_0) = -\pi^{-2}\sin(\pi\alpha)\,r_0^m e^{im\theta_0}\,V(r_0),$$

(41) and (46) give

(48)
$$V(r_0) = \int_{r_0}^{a}\{sX_1'(s)+2(m+\alpha+1)X_1(s)\}\frac{s\,ds}{(s^2-r_0^2)^{1-\alpha}} - \frac{a^2X_1(a)}{(a^2-r_0^2)^{1-\alpha}}$$
$$-2(\alpha-1)\int_a^\infty \frac{X_2(s)\,s^3\,ds}{(s^2-r_0^2)^{2-\alpha}} + 2(m+\alpha)\int_a^\infty \frac{X_2(s)\,s\,ds}{(s^2-r_0^2)^{1-\alpha}}.$$

Next, consider $W(r_0)$ in (42), defined by (44). We have

$$s^{2m+2\alpha+1}\Phi_3(s) = \left[s^{2m+2\alpha}\Phi_2(s)\right]'$$

so that

$$\Phi_3(s) = X_2(s) + 2(m+\alpha)s^{-2}\Phi_2(s).$$

Hence, using (47), we obtain

$$W(r_0) = 2(\alpha-1)\int_a^\infty \frac{X_2(s)\,s^3\,ds}{(s^2-r_0^2)^{2-\alpha}}$$
$$-2(m+\alpha)\int_a^\infty \frac{X_2(s)\,s\,ds}{(s^2-r_0^2)^{1-\alpha}} - \frac{2(m+\alpha)\Phi_2(a)}{(a^2-r_0^2)^{1-\alpha}}.$$

When this is combined with (48), we obtain

$$V+W = \int_{r_0}^{a}\frac{s^{1-2m-2\alpha}}{(s^2-r_0^2)^{1-\alpha}}\,Y(s)\,ds - \frac{A_m}{(a^2-r_0^2)^{1-\alpha}}, \tag{49}$$

where

$$Y(s) = s^{2m+2\alpha}\{sX_1'(s) + 2(m+\alpha+1)X_1(s)\}$$

and

$$A_m = a^2 X_1(a) + 2(m+\alpha)\Phi_1(a). \tag{50}$$

Now, from (45), we obtain

$$s^{2m+2\alpha}\left\{X_1(s) + 2(m+\alpha)s^{-2}\Phi_1(s)\right\} = s^{2\alpha-2} f_0(0)\delta_{0m} + \Psi_m(s), \tag{51}$$

where $\Psi_m(r)$ is defined by (33). Hence, (50) agrees with (38). Also

$$\begin{aligned} Y(s) &= s^{-1}\left[s^{2m+2\alpha+2}X_1(s)\right]' \\ &= s^{-1}\left[s^{2\alpha} f_0(0)\delta_{0m} + s^2\Psi_m(s) - 2(m+\alpha)s^{2m+2\alpha}\Phi_1(s)\right]' \\ &= -\,2(m+\alpha)s^{2m+2\alpha}\left\{X_1(s) + 2(m+\alpha)s^{-2}\Phi_1(s)\right\} \\ &\quad + 2\alpha s^{2\alpha-2} f_0(0)\delta_{0m} + 2\Psi_m(s) + s\Psi_{m'}(s). \end{aligned}$$

Then, making use of (51) and (36), we obtain

$$Y(s) = 2(1-m)\Psi_m(s) + \int_0^s \frac{\rho^2\, F_m'(\rho)\, d\rho}{(s^2-\rho^2)^{1-\alpha}}.$$

It follows that the integral term in (49) is equal to $B(r_0)$, defined by (39), and so we have complete agreement with Fabrikant's solution, (37).

5. Hypersingular equations. Hypersingular versions of (7), with $1 < \alpha < 2$, have been considered by Li and Rong [**19**]. Making use of (28), we can write (7) as

$$\begin{gathered} \nabla_0^2 \int_{-\pi}^{\pi}\int_0^a \frac{u(r,\theta)}{R^{2\alpha-2}}\, r\, dr\, d\theta = 4(\alpha-1)^2 f(r_0,\theta_0), \\ 0 \le r_0 < a, \quad -\pi \le \theta_0 < \pi, \end{gathered} \tag{52}$$

where $1 < \alpha < 2$. Let us write this equation concisely as

$$\nabla_0^2 \mathcal{L}_{\alpha-1} u = 4(\alpha-1)^2 f \quad \text{on } D, \tag{53}$$

where $\mathcal{L}_{\alpha-1}$ is the weakly-singular integral operator on the left-hand side of (52); explicit formulas for $\mathcal{L}_{\mu}^{-1}$, with $0 < \mu < 1$, were given in Section 3.

Now, (52) says that $\mathcal{L}_{\alpha-1}u$ solves the two-dimensional Poisson equation on D, whence

$$\mathcal{L}_{\alpha-1}u = f_{\mathrm{h}} + f_{\mathrm{p}},$$

where

$$\nabla^2 f_{\mathrm{h}} = 0 \quad \nabla^2 f_{\mathrm{p}} = 4(\alpha-1)^2 f \quad \text{on } D.$$

Thus, f_{p} is a particular solution of the given Poisson equation and f_{h} is a solution of Laplace's equation on D. Once f_{h} and f_{p} have been specified, we deduce that

$$u = \mathcal{L}_{\alpha-1}^{-1} f_{\mathrm{h}} + \mathcal{L}_{\alpha-1}^{-1} f_{\mathrm{p}} = u_{\mathrm{h}} + u_{\mathrm{p}},$$

say. At this point, we notice two facts. First, the solution for u cannot be unique: we must impose side conditions on u. This fact was already known, of course, from the special case with $\alpha = 3/2$: there, the physics of the problem usually dictates that $u = 0$ around the edge of the disc, $r = a$; see (6). Second, the presence of $\nabla^2 f_{\mathrm{h}}$ and $\nabla^2 f_{\mathrm{p}}$ means that Kahane's formula for $\mathcal{L}_{\mu}^{-1}$ is attractive. In particular, $\nabla^2 f_{\mathrm{h}} = 0$ implies that the corresponding $\mathcal{B} = 0$, see (22), and then (20) shows that

$$u_{\mathrm{h}}(r,\theta) = -\,\pi^{-2} \sin(\pi\alpha)\, \mathcal{A}(r,\theta)\, (a^2 - r^2)^{\alpha-2};$$

here, $\mathcal{A}$ is defined by (21) in which f and α are replaced by f_{h} and $\alpha - 1$, respectively. Thus, in general, u_{h} is unbounded at $r = a$. Li and Rong [**19**] gave an expression for u_{p} in their Theorem 2 but did not comment on the inherent non-uniqueness of (52).

Acknowledgments. It is a pleasure for me to contribute this paper in honor of Kendall Atkinson. As a graduate student at the University of Manchester, I learned much from a tattered copy of his 1976 SIAM book. Later, I heard Ken give a lecture at a Durham Symposium [**5**], in which he distilled a wedge problem down to something that could be studied exactly: I like to do the same! Finally, I greatly admire his scholarly work, exemplified by his 1997 CUP book: he is both a fine role model and a fine mathematician!

Appendix

A. *Li and Rong's solution for* $f \equiv 1$. Suppose that $f \equiv 1$ in (30) and (31); we obtain

(A1)
$$\begin{aligned} v(r_0,\theta_0) &= -\frac{\sin\pi\alpha}{\pi^2}\,\nabla_0^2\Big\{\int_{r_0}^{a}\frac{s^{1-2\alpha}}{(s^2-r_0^2)^{1-\alpha}}\int_0^s\frac{r\,dr}{(s^2-r^2)^{1-\alpha}}\,ds \\ &\qquad\qquad + \int_a^\infty\frac{s^{1-2\alpha}}{(s^2-r_0^2)^{1-\alpha}}\int_0^a\frac{r\,dr}{(s^2-r^2)^{1-\alpha}}\,ds\Big\} \\ &= -\frac{\sin\pi\alpha}{2\alpha\pi^2}\,\nabla_0^2\Big\{\int_{r_0}^{a}\frac{s\,ds}{(s^2-r_0^2)^{1-\alpha}} + \int_a^\infty\frac{1-(1-a^2/s^2)^\alpha}{(s^2-r_0^2)^{1-\alpha}}\,s\,ds\Big\} \\ &= -\frac{\sin\pi\alpha}{2\alpha\pi^2}\,\nabla_0^2\Big\{\frac{(a^2-r_0^2)^\alpha}{2\alpha} + \frac{a^{2\alpha}}{2}\int_0^1\frac{x^{-\alpha-1}\{1-(1-x)^\alpha\}}{(1-\zeta x)^{1-\alpha}}\,dx\Big\}, \\ w(r_0,\theta_0) &= -4(1-\alpha)^2\,\frac{\sin\pi\alpha}{\pi^2}\int_a^\infty\frac{s^{3-2\alpha}}{(s^2-r_0^2)^{2-\alpha}}\int_0^a\frac{r\,dr}{(s^2-r^2)^{2-\alpha}}\,ds \\ &= 2(1-\alpha)\,\frac{\sin\pi\alpha}{\pi^2}\int_a^\infty\frac{1-(1-a^2/s^2)^{\alpha-1}}{(s^2-r_0^2)^{2-\alpha}}\,s\,ds \\ &= \frac{\sin\pi\alpha}{\pi^2}\Big\{(a^2-r_0^2)^{\alpha-1}+(\alpha-1)a^{2\alpha-2}\int_0^1\frac{x^{-\alpha}(1-x)^{\alpha-1}}{(1-\zeta x)^{2-\alpha}}\,dx\Big\}, \end{aligned}$$

where $\zeta = r_0^2/a^2$ and we used the substitution $x = a^2/s^2$.

Now, the hypergeometric function, F, satisfies [**2**, 15.3.1]

(A2)
$$\int_0^1\frac{x^{b-1}(1-x)^{c-b-1}}{(1-\zeta x)^a}\,dx = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\,F(a,b;c;\zeta),$$

for $c > b > 0$. Hence,

(A3)
$$w(r_0,\theta_0) = \frac{\sin\pi\alpha}{\pi^2}\,(a^2-r_0^2)^{\alpha-1} + \frac{\alpha-1}{\pi}\,a^{2\alpha-2}\,F(2-\alpha,1-\alpha;1;\zeta).$$

For v, we have

$$\nabla_0^2\left\{(a^2-r_0^2)^\alpha\right\} = -4\alpha\left\{(a^2-r_0^2)^{\alpha-1} + (1-\alpha)r_0^2(a^2-r_0^2)^{\alpha-2}\right\}$$

and

$$\nabla_0^2 F(r_0) = \frac{4}{a^2}\frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial F}{\partial \zeta}\right).$$

Hence (A1) becomes

(A4)

$$v(r_0,\theta_0) = \frac{\sin \pi\alpha}{\alpha\pi^2}\, a^{2\alpha-2}\bigg\{(1-\zeta)^{\alpha-1} + (1-\alpha)\,\zeta(1-\zeta)^{\alpha-2} + (\alpha-1)\,\frac{\partial}{\partial\zeta}\left[\zeta\int_0^1 \frac{x^{-\alpha}\,\{1-(1-x)^\alpha\}}{(1-\zeta x)^{2-\alpha}}\,dx\right]\bigg\}.$$

The remaining integral can be split into two integrals (notice that we could not have done this with (A1)), the first of which is

$$\int_0^1 \frac{x^{-\alpha}}{(1-\zeta x)^{2-\alpha}}\,dx = \frac{(1-\zeta)^{\alpha-1}}{1-\alpha}.$$

When this is substituted in (A4), it is seen to give a contribution that cancels with the first two terms in (A4). Hence,

$$\begin{aligned} v(r_0,\theta_0) &= \frac{\sin\pi\alpha}{\pi^2}\,\frac{(1-\alpha)}{\alpha}\,a^{2\alpha-2}\frac{\partial}{\partial\zeta}\left[\zeta\int_0^1 \frac{x^{-\alpha}(1-x)^\alpha}{(1-\zeta x)^{2-\alpha}}\,dx\right] \\ &= \frac{1-\alpha}{\pi}\,a^{2\alpha-2}\,\frac{\partial}{\partial\zeta}\,[\zeta\, F(2-\alpha, 1-\alpha; 2; \zeta)] \\ &= \frac{1-\alpha}{\pi}\,a^{2\alpha-2}\,F(2-\alpha, 1-\alpha; 1; \zeta), \end{aligned}$$

using (A2) and [**2**, 15.2.4]. When this result is combined with (26) and (A3), we recover (12).

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