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On functions defined by sums of products of Bessel functions

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Abstract
Various functions, defined as infinite series of products of Bessel functions of the first kind, are studied. Integral representations are obtained, and then used to deduce asymptotic approximations. Although several methods have been investigated (including power series expansions and integral transforms), methods based on Fourier series emerge as the most useful.

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1. Introduction

In a recent investigation into multiple scattering of acoustic waves by random configurations of penetrable circular cylinders [7], we encountered the functions

$$\mathcal{S}_1(x) \equiv \sum_n \mathcal{J}_n^2(x) \qquad \text{and} \qquad \mathcal{S}_2(x;\mu) \equiv \sum_{m,n} \mathcal{J}_n(x)\mathcal{J}_m(x)\mathcal{J}_{n-m}(\mu x). \tag{1}$$

Here, the function $\mathcal{J}_n$ is defined in terms of the Bessel function J_n by

$$\mathcal{J}_n(x) = J_n^2(x) - J_{n-1}(x)J_{n+1}(x), \tag{2}$$

μ is a constant and we have used the shorthand notation

$$\sum_n = \sum_{n=-\infty}^{\infty} \qquad \text{and} \qquad \sum_{m,n} = \sum_m \sum_n .$$

We are especially interested in the asymptotic behaviour of $\mathcal{S}_1(x)$ and $\mathcal{S}_2(x;\mu)$ for large positive values of x. (The behaviour near $x = 0$ is easily obtained; in particular, $\mathcal{S}_1(0) = \mathcal{S}_2(0;\mu) = 1$.) In the application we have in mind, large values of x ($\equiv ka$) correspond to high frequencies and the parameter μ satisfies $\mu \geqslant 2$, with $\mu = 2$ being of special interest.

We shall obtain integral representations for $\mathcal{S}_1$ and $\mathcal{S}_2$, and then we shall show that

$$\mathcal{S}_1(x) \sim \frac{c_1}{x} \qquad \text{and} \quad \mathcal{S}_2(x;\mu) \sim \frac{c_2(\mu)}{x} \quad \text{as} \quad x \to \infty,$$

with explicit expressions for the constants c_1 and c_2. Thus, see (27) for c_1, see (33) for $c_2(\mu)$ and, when $\mu = 2$, see (34):

$$c_2(2) = (4/5)[(2/\pi)\Gamma(3/4)]^4.$$

To see that the determination of the asymptotic behaviour of $\mathcal{S}_1$ and $\mathcal{S}_2$ may not be straightforward, consider the following well-known fact:

$$\mathcal{S}_0(x) \equiv \sum_n J_n^2(x) = 1 \qquad \text{for all} \quad x. \tag{3}$$

Each term in the sum decays as x^{-1} with increasing x, and yet the sum itself does not decay. A related formula is

$$\sum_n J_n(x) J_{n+m}(x) = 0 \qquad \text{for all} \quad x, \qquad \text{where} \quad m = \pm 1, \pm 2, \ldots. \tag{4}$$

Combining this formula with (3) gives

$$\sum_n \mathcal{J}_n(x) = 1 \qquad \text{for all} \quad x. \tag{5}$$

The formulae (3) and (4) are special cases of addition theorems for Bessel functions. For example, it is known that

$$\sum_n J_n(r) J_n(s)\, \mathrm{e}^{-\mathrm{i}n\theta} = J_0(\sqrt{r^2 + s^2 - 2rs\cos\theta}). \tag{6}$$

In particular,

$$\mathcal{S}_0(x;\mu) \equiv \sum_n J_n(x) J_n(\mu x) = J_0(x(\mu - 1)).$$

Thus, $\mathcal{S}_0(x;\mu)$ decays as $x^{-1/2}$ when x increases, provided that $\mu \neq 1$, whereas $\mathcal{S}_0(x;1) = \mathcal{S}_0(x) \equiv 1$, which is (3).

In order to estimate $\mathcal{S}_1, \mathcal{S}_2$ and related infinite series, we need a general method. It is instructive to develop these methods by using them to verify (3); some potential methods (and their drawbacks) are outlined in the following section. We then focus on methods based on Fourier series. These lead to integral representations for $\mathcal{S}_1$ and $\mathcal{S}_2$; we also consider other functions, including

$$\mathcal{S}_3(x) \equiv \sum_n J_n^4(x). \tag{7}$$

The asymptotic approximations follow from the integral representations. We only give the leading term in the asymptotic behaviour (as that is sufficient for our application), but it is likely that full expansions could be derived with more work; such expansions have been obtained previously for $\mathcal{S}_3(x)$, as described in section 3.

2. A short survey of methods for proving (3)

2.1. Use of a differential equation

Perhaps the simplest way to prove (3) is to form a differential equation for $\mathcal{S}_0(x)$. Thus,

$$\mathcal{S}_0'(x) = \sum_n 2J_n(x)J_n'(x) = \sum_n J_n(J_{n-1} - J_{n+1}) = \sum_n J_n J_{n-1} - \sum_m J_{m-1} J_m = 0$$

so that $\mathcal{S}_0(x)$ is a constant; but $\mathcal{S}_0(0) = 1$ and so the result follows. A similar calculation also gives (4). However, we have not succeeded in using this approach for more complicated series, such as $\mathcal{S}_3$.

2.2. Use of power series

Bessel functions are defined by power series, so it is natural to use these. The power series for J_n^2 is known [1, 9.1.14], and this can be summed over n (use the second 'check' on p 822 of [1]) to recover $\mathcal{S}_0 = 1$. In principle, this method could be used for other series. However, power series are not usually convenient when the goal is to estimate functions such as $\mathcal{S}_3(x)$ for large values of x, unless the series obtained can be recognized as a known special function.

2.3. Use of Mellin–Barnes integrals

Mellin transforms can sometimes be used to obtain asymptotic expansions. However, we see immediately that $\mathcal{S}_0(x)$ does not have a Mellin transform. Nevertheless, we can use a Mellin–Barnes integral for J_n^2. Thus, for $n \geqslant 0$, we have the following integral representation (obtained by putting $\mu = \nu = n$ and $s + n = z$ in Watson's formula for $J_\mu(x)J_\nu(x)$ [12, p 436], or by comparison with the power series for $J_n^2(x)$ [1, 9.1.14]),

$$J_n^2(x) = \frac{1}{\sqrt{\pi}}\frac{1}{2\pi \mathrm{i}}\int_{c_n-\mathrm{i}\infty}^{c_n+\mathrm{i}\infty} \frac{\Gamma(n-z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(z+1)\Gamma(n+z+1)} x^{2z}\,\mathrm{d}z, \tag{8}$$

with $-\frac{1}{2} < c_n < n$. (Later, we shall use a Mellin–Barnes integral for J_1; see (32).) To calculate $\mathcal{S}_0$, we sum over n. We want to interchange the order of summation and integration. As [1, 6.1.47]

$$\frac{\Gamma(n-z)}{\Gamma(n+z+1)} \sim \frac{1}{n^{2z+1}} \qquad \text{as} \quad n \to \infty,$$

the interchange will be permissible if $\operatorname{Re} z > 0$. Thus, we fix $c_n = c_1$ (with $0 < c_1 < 1$) and sum over n with $n \geqslant 1$. We have

$$\sum_{n=1}^{\infty} \frac{\Gamma(n-z)}{\Gamma(n+z+1)} = \frac{\Gamma(-z)}{\Gamma(z+1)} \sum_{n=1}^{\infty} \frac{(-z)_n(1)_n}{(z+1)_n n!}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$. The sum is $F(-z, 1; z+1; 1) - 1 = -\frac{1}{2}$, where $F(a, b; c; z)$ is the Gauss hypergeometric function [1, 15.1.1] and we have used a known formula [1, 15.1.20] for $F(a, b; c; 1)$. Hence,

$$\mathcal{S}_0(x) = J_0^2(x) - \frac{1}{\sqrt{\pi}}\frac{1}{2\pi \mathrm{i}}\int_{c_1-\mathrm{i}\infty}^{c_1+\mathrm{i}\infty} \frac{\Gamma(-z)\Gamma\left(z+\frac{1}{2}\right)}{[\Gamma(z+1)]^2} x^{2z}\,\mathrm{d}z. \tag{9}$$

But we know that $J_0^2(x)$ is given by (8); moving the contour from $\operatorname{Re} z = c_0$ to $\operatorname{Re} z = c_1$, we pick up the residue contribution from $z = 0$, and then (9) gives $\mathcal{S}_0(x) = 1$.

It is possible that this method will generalize, but we have not pursued it; for applications to the evaluation of certain integrals of products of Bessel functions, see [5].

2.4. Use of Laplace transforms

Another possibility is to consider the Laplace transform of $\mathcal{S}_0$. It is known that [2]

$$\begin{aligned}\int_0^\infty J_n^2(x)\,\mathrm{e}^{-sx}\,\mathrm{d}x &= \frac{(-1)^n k}{\pi}\int_0^{\pi/2} \frac{\cos 2n\psi\,\mathrm{d}\psi}{(1-k^2\sin^2\psi)^{1/2}} \\ &= \frac{k}{4\pi}\int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta}{[1-k^2\cos^2(\theta/2)]^{1/2}},\end{aligned} \tag{10}$$

where $k^2 = 4/(s^2+4)$ and s is the transform variable. Summing over n gives

$$\int_0^\infty \sum_{n=-N}^{N} J_n^2(x)\,\mathrm{e}^{-sx}\,\mathrm{d}x = \frac{k}{4\pi}\int_{-\pi}^{\pi}\frac{D_N(\theta)\,\mathrm{d}\theta}{[1-k^2\cos^2(\theta/2)]^{1/2}}, \tag{11}$$

where

$$D_N(\theta) = \sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}n\theta} = \frac{\sin(N+1/2)\theta}{\sin(\theta/2)} \tag{12}$$

is the *Dirichlet kernel*; from the theory of Fourier series (see, for example [11, p 317]), we know that D_N has the following filtering property

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} f(t)D_N(t)\,\mathrm{d}t = 2\pi f(0), \tag{13}$$

for any smooth functions f. Hence, letting $N\to\infty$ in (11) gives

$$\int_0^\infty \mathcal{S}_0(x)\,\mathrm{e}^{-sx}\,\mathrm{d}x = \frac{k}{4\pi}\frac{2\pi}{\sqrt{1-k^2}} = \frac{1}{s},$$

which is the Laplace transform of unity.

An obvious limitation with this method is that it needs the Laplace transform to be available. However, the filtering property of the Dirichlet kernel does suggest using Fourier series, and this emerges as our method of choice.

3. Use of Fourier series

If the terms in a series can be expressed as Fourier coefficients (as in (10), for example), then it is trivial to sum the series. Thus, suppose that

$$t_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta;x)\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta, \tag{14}$$

where h is a smooth, 2π-periodic function of θ. Then, $h(\theta;x) = \sum_n t_n(x)\,\mathrm{e}^{-\mathrm{i}n\theta}$ and, in particular,

$$\sum_n t_n(x) = h(0;x). \tag{15}$$

We can also use Parseval's theorem [11, p 128] to sum a related series:

$$\sum_n |t_n(x)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|h(\theta;x)|^2\,\mathrm{d}\theta. \tag{16}$$

More generally, if we have a second function, $g(\theta;x) = \sum_n s_n(x)\,\mathrm{e}^{-\mathrm{i}n\theta}$, then

$$\sum_n \overline{s_n(x)}t_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g(\theta;x)}h(\theta;x)\,\mathrm{d}\theta, \tag{17}$$

where the overbar denotes complex conjugation.

Let us apply these formulae. From [4, 7.7.2 (11)], we have Neumann's formula,

$$J_\nu(x)J_\mu(x) = \frac{2}{\pi}\int_0^{\pi/2} J_{\nu+\mu}(2x\cos\psi)\cos[(\mu-\nu)\psi]\,\mathrm{d}\psi. \tag{18}$$

If we put $\nu = n$, $\mu = -n$ and $\psi = \theta - \pi/2$, we obtain (see [1, 11.4.8] or [4, 7.7.2 (13)])

$$J_n^2(x) = \frac{1}{\pi}\int_0^{\pi} J_0(2x\sin\theta)\cos 2n\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_0(2x\sin[\theta/2])\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta. \tag{19}$$

Comparison with (14) gives $t_n(x) = J_n^2(x)$ and $h(\theta; x) = J_0\big(2x \sin \frac{1}{2}\theta\big)$. Then, (15) gives $\mathcal{S}_0(x) = 1$ whereas (16) gives the integral representation

$$\mathcal{S}_3(x) = \sum_n J_n^4(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_0^2\left(2x\sin\frac{1}{2}\theta\right)\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\pi/2} J_0^2(2x\sin\theta)\,\mathrm{d}\theta. \tag{20}$$

In fact, this is a known formula. It is equation (2b) in a paper by Stoyanov and Farrell [9]. These authors were interested in the integral, but they noted (20). They also obtained the large-x behaviour of the integral's value; using their results, we obtain

$$\mathcal{S}_3(x) = \frac{1}{x\pi^2}(\log x + 5\log 2 + \gamma) + \mathrm{O}(x^{-3/2}) \quad \text{as} \quad x \to \infty,$$

where $\gamma \simeq 0.5772$ is Euler's constant.

3.1. Application to the series $\mathcal{S}_1(x)$

Consider $\mathcal{S}_1(x)$, defined by (1). Using (2), (18) and (19), we obtain

$$\mathcal{J}_n(x) = \frac{1}{\pi}\int_0^{\pi}[J_0(2x\sin\theta) + J_2(2x\sin\theta)]\cos 2n\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta; x)\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta, \tag{21}$$

where

$$g(2\varphi; x) = J_0(2x\sin\varphi) + J_2(2x\sin\varphi) = \frac{J_1(2x\sin\varphi)}{x\sin\varphi}. \tag{22}$$

As $g(0; x) = 1$, applying (15) confirms (5). From (16), we obtain

$$\mathcal{S}_1(x) = \sum_n \mathcal{J}_n^2(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(\theta; x)]^2\,\mathrm{d}\theta, \tag{23}$$

an integral representation for $\mathcal{S}_1(x)$. Explicitly,

$$\mathcal{S}_1(x) = \frac{2}{\pi}\int_0^{\pi/2}[J_0(2x\sin\theta) + J_2(2x\sin\theta)]^2\,\mathrm{d}\theta \tag{24}$$

$$= (2/\pi)[S_{00}(2x) + 2S_{02}(2x) + S_{22}(2x)], \tag{25}$$

where

$$S_{\mu\nu}(\lambda) = \int_0^{\pi/2} J_\mu(\lambda\sin\theta)J_\nu(\lambda\sin\theta)\,\mathrm{d}\theta. \tag{26}$$

The asymptotic behaviour of this integral can be found in the literature: S_{00} is mentioned above, $S_{\nu\nu}$ was studied by Wong [13] and the general case is discussed in [10], [6] and [8]. Specifically,

$$S_{\mu\nu}(\lambda) = \frac{\cos[(\mu-\nu)\pi/2]}{\lambda\pi}(\log\lambda - \psi_{\mu\nu}) + \mathrm{O}(\lambda^{-3/2})$$

as $\lambda \to \infty$, where

$$\psi_{\mu\nu} = \gamma + \psi\left(\frac{1+\mu+\nu}{2}\right) + \frac{1}{2}\psi\left(\frac{1+\mu-\nu}{2}\right) + \frac{1}{2}\psi\left(\frac{1-\mu+\nu}{2}\right)$$

and $\psi(w) = \Gamma'(w)/\Gamma(w)$. In particular, $\psi_{00} = -\gamma - 4\log 2$, $\psi_{22} = -\gamma + \frac{8}{3} - 4\log 2$ and $\psi_{02} = -\gamma + 4 - 4\log 2$. Thus,

$$S_{00}(\lambda) \sim \frac{\log\lambda - \psi_{00}}{\lambda\pi}, \qquad S_{22}(\lambda) \sim \frac{\log\lambda - \psi_{22}}{\lambda\pi}, \qquad S_{02}(\lambda) \sim -\frac{\log\lambda - \psi_{02}}{\lambda\pi},$$

whence

$$S_{00}(\lambda) + 2S_{02}(\lambda) + S_{22}(\lambda) \sim (\lambda\pi)^{-1}(2\psi_{02} - \psi_{00} - \psi_{22}) = (16/3)(\lambda\pi)^{-1}.$$

Thus, (25) gives

$$\mathcal{S}_1(x) \sim \frac{16}{3\pi^2 x} \quad \text{as} \quad x \to \infty. \tag{27}$$

This estimate agrees well with direct numerical evaluation of the infinite series defining $\mathcal{S}_1$.

The fact that the logarithmic terms cancel suggests that the leading behaviour of $\mathcal{S}_1(x)$ could be obtained more directly. Indeed, inspection of (24) suggests that the dominant contribution comes from a neighbourhood of $\theta = 0$, so following Stoyanov and Farrell [9], we have

$$\begin{aligned}\mathcal{S}_1(x) &\simeq \frac{2}{\pi}\int_0^{x^{-1/2}} \left[\frac{J_1(2x\sin\theta)}{x\sin\theta}\right]^2 \mathrm{d}\theta \simeq \frac{2}{\pi}\int_0^{x^{-1/2}} \left[\frac{J_1(2x\theta)}{x\theta}\right]^2 \mathrm{d}\theta \\ &= \frac{4}{x\pi}\int_0^{2x^{1/2}} t^{-2} J_1^2(t)\,\mathrm{d}t \simeq \frac{4}{x\pi}\int_0^{\infty} t^{-2} J_1^2(t)\,\mathrm{d}t.\end{aligned} \tag{28}$$

Note that the last approximation could not have been made in a similar calculation for $\mathcal{S}_3$, because $\int_0^\infty J_0^2(t)\mathrm{d}t$ is divergent: it is this divergence that generates the logarithmic terms. It remains to calculate the value of the infinite integral in (28). We recognize it as a critical case of the Weber–Schafheitlin integrals. Thus, it is known that [12, p 403]

$$\int_0^\infty y^{2s-2} J_1^2(y)\,\mathrm{d}y = \frac{2^{2s-2}\Gamma\left(s+\frac{1}{2}\right)\Gamma(2-2s)}{\left[\Gamma\left(\frac{3}{2}-s\right)\right]^2\Gamma\left(\frac{5}{2}-s\right)} = \frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma(1-s)}{\sqrt{\pi}(3-2s)\left[\Gamma\left(\frac{3}{2}-s\right)\right]^2} \tag{29}$$

for $-\frac{1}{2} < \operatorname{Re} s < 1$; this reduces to $4/(3\pi)$ when $s = 0$, and then (28) agrees with (27). We shall use this method when we estimate $\mathcal{S}_2(x;\mu)$ in the following section.

3.2. Application to the double sum $\mathcal{S}_2(x;\mu)$

The series $\mathcal{S}_2(x;\mu)$, defined by (1), is more complicated than $\mathcal{S}_1$ for two reasons: it is a double sum and it involves Bessel functions with different arguments.

Let us start with the latter complication. From the addition theorem for J_0, (6), we have

$$J_n(x)J_n(\mu x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta;x)\cos n\theta\,\mathrm{d}\theta,$$

with $f(\theta;x) = J_0(x\sqrt{1+\mu^2-2\mu\cos\theta})$. Application of (16) gives

$$\mathcal{S}_3(x;\mu) \equiv \sum_n J_n^2(x)J_n^2(\mu x) = \frac{1}{\pi}\int_0^{\pi} J_0^2(x\sqrt{1+\mu^2-2\mu\cos\theta})\,\mathrm{d}\theta.$$

When $\mu = 1$, this formula reduces to (20). When $\mu \neq 1$, the argument of the square root does not vanish; consequently, it can be shown that $\mathcal{S}_3(x;\mu) = \mathrm{O}(x^{-1})$ as $x \to \infty$ in this case.

More generally, using (21),

$$\mathcal{J}_{n-m}(\mu x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta;\mu x)\,\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_m(\theta;x)\,\mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{d}\theta,$$

where $h_m(\theta;x) = g(\theta;\mu x)\,\mathrm{e}^{-\mathrm{i}m\theta}$ and g is defined by (22). Hence, from (17) (with $h = h_m$ therein, and noting that g is real),

$$\sum_n \mathcal{J}_n(x)\mathcal{J}_{n-m}(\mu x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta;x)g(\theta;\mu x)\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta.$$

Multiplying by $\mathcal{J}_m(x)$, using (21) again, and then summing over m, using (17) again, we obtain

$$\mathcal{S}_2(x;\mu) = \sum_{m,n} \mathcal{J}_m(x)\mathcal{J}_n(x)\mathcal{J}_{n-m}(\mu x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [g(\theta;x)]^2 g(\theta;\mu x)\,\mathrm{d}\theta.$$

This is our integral representation for $\mathcal{S}_2$. Substituting for g from (22), we obtain

$$\mathcal{S}_2(x;\mu) = \frac{2}{\pi}\int_0^{\pi/2}\left[\frac{J_1(2x\sin\theta)}{x\sin\theta}\right]^2 \frac{J_1(2\mu x\sin\theta)}{\mu x\sin\theta}\,\mathrm{d}\theta.$$

To estimate this integral for large values of x, we proceed as at the end of section 3.1. Thus, we have

$$\mathcal{S}_2(x;\mu) \simeq \frac{2}{\pi}\int_0^{x^{-1/2}}\left[\frac{J_1(2x\theta)}{x\theta}\right]^2 \frac{J_1(2\mu x\theta)}{\mu x\theta}\mathrm{d}\theta \simeq \frac{8}{x\mu\pi} I(\mu), \tag{30}$$

say, where

$$I(\mu) = \int_0^\infty t^{-3} J_1^2(t) J_1(\mu t)\,\mathrm{d}t. \tag{31}$$

This integral can be evaluated. We make the following steps. First, we use a Mellin–Barnes integral to represent $J_1(\mu t)$. Then, we integrate with respect to t, using a critical case of the Weber–Schafheitlin integral. The remaining integral is evaluated by residue calculus. The result is proportional to a generalized hypergeometric function, ${}_3F_2$. This function has argument 1 when $\mu = 2$; in this special case, the ${}_3F_2$ can be evaluated in terms of gamma functions. Some details of this calculation follow.

We start with the Mellin–Barnes integral [1, 9.1.26],

$$J_1(\mu t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)(\mu t/2)^{2s+1}}{\Gamma(s+2)}\,\mathrm{d}s, \qquad c = \mathrm{Re}\, s < 0. \tag{32}$$

We substitute in (31), interchange the order of integration and then integrate with respect to t using (29); the result is

$$I(\mu) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(-s)\Gamma\left(s+\frac{1}{2}\right)\Gamma(1-s)(\mu/2)^{2s+1}}{\sqrt{\pi}\Gamma(s+2)(3-2s)\left[\Gamma\left(\frac{3}{2}-s\right)\right]^2}\,\mathrm{d}s, \qquad -\frac{1}{2} < c < 0.$$

We move the integration contour to the left. There are simple poles at $s + \frac{1}{2} = -N$, where $\Gamma\left(s+\frac{1}{2}\right) \simeq (-1)^N/\left[N!\left(s+\frac{1}{2}+N\right)\right]$, $N = 0, 1, 2, \ldots$. Evaluating the residues gives

$$I(\mu) = \sum_{N=0}^{\infty} \frac{\Gamma\left(N+\frac{1}{2}\right)\Gamma\left(N+\frac{3}{2}\right)(\mu/2)^{-2N}}{\sqrt{\pi}\Gamma\left(\frac{3}{2}-N\right)(2N+4)[\Gamma(N+2)]^2}\frac{(-1)^N}{N!}.$$

Use of the reflection formula, $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, with $z = N - \frac{1}{2}$, gives

$$\begin{aligned} I(\mu) &= -\frac{1}{2\pi^{3/2}}\sum_{N=0}^{\infty} \frac{\Gamma\left(N-\frac{1}{2}\right)\Gamma\left(N+\frac{1}{2}\right)\Gamma\left(N+\frac{3}{2}\right)}{\Gamma(N+2)\Gamma(N+3)}\frac{(2/\mu)^{2N}}{N!} \\ &= -\frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\pi^{3/2}\Gamma(2)\Gamma(3)}\,{}_3F_2\left(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2};2,3;4/\mu^2\right); \end{aligned}$$

the factor in front of the ${}_3F_2$ reduces to $\frac{1}{4}$. Thus,

$$\mathcal{S}_2(x;\mu) \sim \frac{2}{x\mu\pi}\,{}_3F_2\left(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2};2,3;4/\mu^2\right) \quad \text{as} \quad x\to\infty. \tag{33}$$

When $\mu = 2$, the generalized hypergeometric function is 'well poised' and can be evaluated using Dixon's theorem [3, 4.4(5)],

$$ {}_3F_2(a, b, c; 1 + a - b, 1 + a - c; 1) \\ = \frac{\Gamma(1 + a/2)\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 - b - c + a/2)}{\Gamma(1 + a)\Gamma(1 - b + a/2)\Gamma(1 - c + a/2)\Gamma(1 + a - b - c)}, $$

with $a = \frac{3}{2}$, $b = \frac{1}{2}$ and $c = -\frac{1}{2}$; the result is

$$ I(2) = \frac{1}{4}\left[\frac{\Gamma(7/4)}{\Gamma(5/2)}\right]^2 \frac{\Gamma(2)\Gamma(3)}{\Gamma(5/4)\Gamma(9/4)} = \frac{16}{5\pi^3}[\Gamma(3/4)]^4. $$

Thus,

$$ \mathcal{S}_2(x; 2) \sim \frac{4}{5x}\left[\frac{2}{\pi}\Gamma(3/4)\right]^4 \quad \text{as} \quad x \to \infty. \tag{34} $$

This estimate (34) agrees well with direct numerical evaluation of the original double sum.

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