Scattering by a Cavity in an Exponentially Graded Half-Space

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1 Introduction

Consider two half-spaces, \( x > 0 \) and \( x < 0 \), welded together along the interface at \( x = 0 \). The left half-space (\( x < 0 \)) is homogeneous. The right half-space is inhomogeneous. If a wave is incident from the left, it will be partly reflected and partly transmitted into the right half-space. We assume that these fields can be calculated.

Suppose now that the right half-space contains a cavity or some other defect (see Fig. 1). How are the basic fields described above modified by the presence of the cavity? In general, it is not easy to answer this question, as the associated mathematical problem is difficult, in general.

In some recent papers, Fang et al. claimed to solve a variety of such problems. All concern “exponential grading,” meaning that the material parameters are proportional to \( e^{\beta x} \) for \( x > 0 \), where \( \beta \) is a given constant. The papers concern antiplane shear waves [1–4], thermal waves [5,6], and shear waves in a piezoelectric material [7]. All of these papers assume that the effect of the interface on the cavity can be found by introducing simple image terms, as if the interface were a mirror or a rigid wall. Unfortunately, this assumption is incorrect.

In this paper, we outline how the problems described above can be solved. We do this in the context of antiplane shear waves with exponential grading and a circular cavity. The main technical part concerns the derivation of suitable multipole potentials; these reveal the complicated image system.

The study of problems involving scatterers near boundaries or interfaces has a long history. For linear surface water waves interacting with a submerged circular cylinder, see the famous paper of Green’s function for the anisotropic Helmholtz equation in a half-space

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \omega^2 \rho u = 0 \]

where \( u(x, y) \) is the antiplane component of displacement and the stress components are given by

\[ \tau_{xx} = \mu(x) \frac{\partial u}{\partial x} \quad \text{and} \quad \tau_{yy} = \mu(x) \frac{\partial u}{\partial y} \]

Thus, in the homogeneous region, where we write \( u_0 \) instead of \( u \), we obtain the two-dimensional Helmholtz equation

\[ \nabla^2 u_0 + 2k^2_0 u_0 = 0 \]

In the inhomogeneous region, we obtain

\[ \nabla^2 u + 2\beta \frac{\partial u}{\partial x} + k^2_0 u = 0 \]

This equation is satisfied by writing

\[ u = u_0 \left( e^{\beta x} \right) \]


An inhomogeneous half-space containing a cavity is bonded to a homogeneous half-space. Waves are incident on the interface and the problem is to calculate the scattered waves. For a circular cavity in an exponentially graded half-space, it is shown how to solve the problem by constructing an appropriate set of multipole functions. These functions are singular on the axis of the cavity, they satisfy the governing differential equation in each half-space, and they satisfy the continuity conditions across the interface between the two half-spaces. Seven recent publications are criticized: They do not take proper account of the interface between the two half-spaces. [DOI: 10.1115/1.3086585]
Fig. 1 The scattering problem. The half-plane on the left of $x = 0$ is homogeneous. The other half-plane is inhomogeneous. The circular cavity has radius $a$. A plane wave is incident from the left.

$$u(x, y) = e^{-i \beta \nu(x, y)}$$

where $w$ satisfies a different two-dimensional Helmholtz equation

$$\nabla^2 w + k^2 w = 0 \quad \text{with} \quad k^2 = k_0^2 - \beta^2$$

For simplicity, we assume that $k_0^2 > \beta^2$ and write $k = \sqrt{k_0^2 - \beta^2}$.

The interface conditions require that the displacements and normal stresses be continuous, so that

$$u_0(0, y) = u(0, y) = w(0, y)$$

$$(\frac{d u_0}{d x})_{x=0} = (\frac{d u}{d x})_{x=0} = (\frac{d w}{d x})_{x=0} - \beta w(0, y)$$

3 Incident Field

Suppose that a plane wave is incident on the interface from the homogeneous side. This wave is given by

$$u_{in}(x, y) = e^{i k_0 (x \cos \alpha_0 y \sin \alpha_0)}$$

where $\alpha_0$ is the angle of incidence, $|\alpha_0| < \pi/2$; $\alpha_0 = 0$ gives normal incidence. There will be a reflected wave $u_{re}$ and a transmitted wave $u_{tr}$.

$$u_{re}(x, y) = R e^{i k_0 (x \cos \alpha_0 y \sin \alpha_0)}, \quad x < 0$$

$$u_{tr}(x, y) = T e^{-i \beta x} e^{i k_0 (x \cos \alpha_0 y \sin \alpha_0)}, \quad x > 0$$

where $R$, $T$, and $\alpha$ are to be found. Writing $u_0 = u_{in} + u_{re}$ and $u = u_{tr}$, Eq. (4) gives

$$1 + R = T \quad \text{and} \quad k_0 \sin \alpha_0 = k \sin \alpha$$

Then, Eq. (5) gives

$$(1 - R) i k_0 \cos \alpha_0 = T (i k_0 \cos \alpha - \beta)$$

Solving for $R$ gives

$$R = \frac{k_0 \cos \alpha_0 - k \cos \alpha - i \beta}{k_0 \cos \alpha_0 + k \cos \alpha + i \beta} = \frac{-i \beta}{k_0 \cos \alpha_0 + k \cos \alpha}$$

and then $T = 1 + R$.

For a simple check, put $\beta = 0$; we obtain $k = k_0$, $\alpha = \alpha_0$, $R = 0$, and $T = 1$, as expected.

4 Scattering by a Buried Cavity

Next, we investigate how the wavefields of Sec. 3 are modified if there is a cavity in the inhomogeneous half-space, $x > 0$. See Fig. 1.

We suppose that the cavity’s cross section is circular, with boundary

$$(x - b)^2 + y^2 = a^2 \quad \text{with} \quad 0 < a < b$$

We also introduce polar coordinates, $(r, \theta)$, so that

$$x = b + r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Thus, the cavity’s boundary is given by $r = a$, and the boundary condition is

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad r = a$$

where $u$ is the total field in the inhomogeneous half-space.

To solve such a scattering problem, we write

$$u = u_0 + u_{tr} + v_0, \quad x < 0$$

$$u = u_0 + v, \quad x > 0, \quad r > a$$

where $v_0$ solves Eq. (2), $v = e^{i \beta x}$, and $w$ solves Eq. (3). Also, $v_0$ must satisfy the Sommerfeld radiation condition and $v$ must decay with $x$.

4.1 Multipole Functions. To represent the scattered field, we introduce functions $\phi_n$ of the form

$$\phi_n = \begin{cases} e^{-i \beta \nu H_n^{(1)}(kr) e^{i \sigma \theta} + \Phi_n}, & x > 0 \\ \Psi_n, & x < 0 \end{cases}$$

where $H_n^{(1)}$ is a Hankel function and $n$ is an arbitrary integer. We require that $\Phi_n$ solves Eq. (3) and $\Psi_n$ solves Eq. (2). In addition, $\Phi_n$ and $\Psi_n$ are to be chosen so that $\phi_n$ satisfies the interface conditions, Eqs. (4) and (5).

The use of polar coordinates is convenient for handling the circular cavity but it is inconvenient when trying to impose the conditions at $x = 0$. Therefore, we convert from polar coordinates to Cartesian coordinates using an integral representation; see the Appendix for details. In particular, if we insert Eq. (8) in Eq. (A4), we obtain the integral representation

$$H_n^{(1)}(kr) e^{i \sigma \theta} = \frac{(-1)^n}{\pi i} \int_{-\infty}^{\infty} e^{i k(x-y) \sin \theta} e^{i k y \cos \theta} e^{-\pi \tau} d\tau$$

for $x < b, |y| < \infty$

Notice that this formula is valid on the interface $x = 0$. The contour of integration in Eq. (10) is also described in the Appendix.

The form of Eq. (10) suggests using a similar integral representation for $\Phi_n$, and so we write

$$\Phi_n(x, y) = \frac{(-1)^n}{\pi i} \int_{-\infty}^{\infty} B(\tau) e^{i k(x-y) \sin \theta} e^{i k y \cos \theta} e^{-\pi \tau} d\tau$$

where $A(\tau)$ is to be found; $\Phi_n$ solves Eq. (3) automatically for any reasonable choice of $A$.

We shall also need a similar integral representation for $\Psi_n(x, y)$ in $x < 0$, where the wavenumber is $k_0$. However, in order to match solutions across the interface at $x = 0$, we shall require the same dependence on $y$ as in Eq. (11). Thus, we consider

$$\Psi_n(x, y) = \frac{(-1)^n}{\pi i} \int_{-\infty}^{\infty} B(\tau) e^{i k(x-y) \sin \theta} e^{i k y \cos \theta} e^{-\pi \tau} d\tau$$

where $B(\tau)$ is to be found,

$$\Delta(\tau) = (k_0^2 \cos^2 \tau - k_0^2)^{1/2} = (k_0^2 \sin^2 \tau - \beta^2)^{1/2}$$

and the square root is taken so that Re $\Delta > 0$ on the contour. Notice that Eq. (2) is satisfied automatically for any reasonable choice of $B$.

We are now ready to enforce the interface conditions. Continuity of $\phi_n$ across $x = 0$ gives $1 + A = B$ whereas continuity of $\partial \phi_n / \partial x$ gives
Hence

\[ A(\tau) = \frac{k \sinh \tau + \Delta + \beta}{k \sinh \tau - \Delta - \beta} \]  

and

\[ B(\tau) = \frac{2k \sinh \tau}{k \sinh \tau - \Delta - \beta} \]  

These formulas complete the construction of the multipole functions \( \phi_n \).

Note that when \( \beta = 0, k = k_0, \Delta = -k \sinh \tau, A = 0, B = 1 \), and \( \Psi_n = H_n^{(1)}(kr)e^{i\mu \theta} \), as expected.

### 4.2 Imposing the Boundary Condition

In the homogeneous half-space, we write

\[ u_0 = u_{in} + u_{ic} + \sum_n c_n \phi_n \]

where \( \Sigma_n \) denotes summation over all integers \( n \). Similarly, in the graded half-space, we write

\[ u(r, \theta) = u_{in} + \sum_n c_n \phi_n \]

Then, by construction, the governing partial differential equations and the interface conditions along \( r = a \), Eq. (9); this gives

\[ \sum_n c_n \left. \frac{\partial \phi_n}{\partial r} \right|_{r=a} = - \left. \frac{\partial u_0}{\partial r} \right|_{r=a} \]  

To proceed, we write both sides of this equation as Fourier series in \( \theta \). For the right-hand side, we have

\[ u_0 = e^{-i\beta_0} T_0 e^{ik \cos \theta} \]

where \( T_0 = T \exp(ik \cos \theta) \) and \( J_n \) is a Bessel function. Also, we have the expansion

\[ e^{-i\beta_0} = e^{-i\beta_0} \sum_s (-1)^s I_s(\beta r) e^{is \theta} \]

where \( I_s \) is a modified Bessel function. Hence,

\[ u_{in}(r, \theta) = e^{-i\beta_0} \sum_m (-1)^m U_m(r) e^{im \theta} \]

where

\[ U_m(r) = T_0 \sum_s (-i)^s I_{m+s}(\beta r) J_s(kr) e^{-isa} \]

In a similar way, we obtain

\[ \Phi_n(r, \theta) = \sum_m (-1)^m f_m^{(n)} J_m(kr) e^{im \theta}, \quad 0 < r < b \]

with

\[ f_m^{(n)} = \frac{(-1)^n}{n+1} \int_{-\pi}^{\pi} \left[ A(\tau) e^{-i\theta} \right] e^{im \theta} d\tau \]

Hence

\[ \Phi_n(r, \theta) = e^{-i\beta_0} \sum_m (-1)^m V_m^n(r) e^{im \theta} \]

where

\[ V_m^n(r) = (-1)^m I_{m+n}(\beta r) H_n^{(1)}(kr), \quad \sum_s f_s^{(n)} I_{m+s}(\beta r) J_s(kr) \]  

Thus, Eq. (15) and orthogonality of \( \{e^{in \theta}\} \) give

\[ \sum_n c_n V_m^n(a) = -U_m^n(a), \quad \text{all } m \]

which is a linear system of algebraic equations for the coefficients \( c_n \).

### 4.3 Far-Field Behavior of \( \Psi_n \)

We should expect cylindrical waves in the homogeneous half-space. These arise from the far-field behavior of \( \Psi_n(x, y) \), for \( x < 0 \). Thus, put

\[ x = -R \cos \Theta, \quad y = R \sin \Theta, \quad |\Theta| < \pi/2 \]

Then, making the substitution \( k \cosh \tau = k_0 \cosh s \) in Eq. (12) gives \( \Delta = -k_0 \sinh s \) and

\[ \Psi_n = \frac{1}{m} \int_{\infty}^{\infty} B_n(k_0 \cosh s; \beta)e^{ik_0 R \sinh(s+\theta)ds}, \quad |\Theta| < \pi/2 \]

where

\[ B_n(\xi; \beta) = \frac{\sqrt{\xi^2 - k_0^2} e^{\xi^2/2} \exp(-b(\xi^2 - k_0^2)^{1/2})}{(\xi^2 - k_0^2)^{1/2} + \beta + (\xi^2 - k_0^2)^{1/2} - k} \]

the square roots being defined to have non-negative real parts.

The formula for \( \Psi_n \), Eq. (18), is convenient for estimating \( \Psi_n \) when \( k_0 R \gg 1 \), as we can use the saddle-point method ([16], Chap. 8). There is one relevant saddle point at \( s = s_0 \) where \( s_0 = \frac{1}{2} \pi + \Theta \). As \( \cos s_0 = -\sin \Theta \) and \( \sinh (s_0 + i\Theta) = i \), the standard argument gives

\[ \Psi_n \sim \frac{1}{\pi} B_n(-k_0 \sin \Theta; \beta)e^{ik_0 R(s - s_0)^2} \]  

where the contour of integration in Eq. (19) passes through the saddle point.

When \( \beta = 0 \), we obtain

\[ B_n(-k_0 \sin \Theta; 0) = \frac{\pi}{\sqrt{2 \pi k_0 R}} e^{ik_0 R \sinh(s_0 + \Theta)ds} \]

Then, Eq. (20) agrees with the known far-field expansion of \( H_n^{(1)}(kr) \times (kr)e^{i\theta} \), when one takes into account that \( \Theta = \pi - \Theta \) and \( r \to R \) + b cos \Theta as \( R \to \infty \).

### 4.4 Near-Field Behavior of \( \Phi_n \)

As the expression for \( \Phi_n \), Eq. (11), is similar to Eq. (10), it is reasonable to ask if \( \Phi_n \) corresponds to a simple image term. To see that it does not, let us define polar coordinates centered at the mirror-image point, \( (x, y) = (-b, 0) \): \( x = -b + r' \cos \theta', y = r' \sin \theta' \). Then, calculations similar to those described in the Appendix show that

\[ H_n^{(1)}(k_r')e^{i\theta'} = \frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} e^{ikr \sin \theta} e^{ikb \sin \theta'} e^{i\theta} d\tau \]

(21)

for \( |\theta'| < \pi/2 \). The integral on the right-hand side of Eq. (21) should be compared with the integral defining \( \Phi_n \), Eq. (11). For them to be equal, the function \( A(\tau) \), defined by Eq. (13), would have to be constant; it is not, and it is not well approximated by a nonzero constant. Thus, it is not justified to replace \( \Phi_n \) with a simple image term: We notice that Fang et al. [2] used image terms similar to those on the left-hand side of Eq. (21), with \( \pi - \theta' \) replaced with \( \theta' \).
5 Discussion

We have outlined how to solve the scattering problem for a cavity buried in a graded half-space; the result is the infinite linear algebraic system, Eq. (17). The system matrix is very complicated: One has to calculate \((\frac{d}{dr}V_\nu''(r))\) at \(r=a\), where \(V_\nu''\) is defined by Eq. (16) as an infinite series of special functions with coefficients given as contour integrals. In principle, the system matrix could be computed but it is unclear whether this is a worthwhile exercise, given the limitations of the underlying model, with both shear modulus and density varying exponentially; see Eq. (1). However, it may be possible to extract asymptotic results from the exact system of equations for small cavities or for cavities that are far from the interface: This remains for future work.

Appendix: Integral Representations

As explained in Sec. 4.1, we need to convert from polar coordinates to Cartesian coordinates in order to apply the interface conditions at \(x=0\). This is done using certain integral formulas. Thus, from Ref. [17] (p. 178, Eq. (2)), we have the integral representation

\[
H_n^{(1)}(kr) = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{i\xi} \left. \frac{1}{\xi} \right|_{\xi=-i\eta} \frac{d\xi}{\xi} \quad (A1)
\]

The integration is along an any contour in the complex \(\xi\)-plane, starting at \(\xi=-\infty\) and ending at \(\xi=\infty\). When \(w = \xi + i\eta\), where \(\xi\) and \(\eta\) are real, \(\left|e^{i\xi} w\right| = e^{i\xi \cos \eta} \cos \eta\). Thus, we can generalize Eq. (A1) to

\[
H_n^{(1)}(kr) = \frac{1}{\pi i} \int_{-\pi}^{\pi} e^{i\xi} \left. \frac{1}{\xi} \right|_{\xi=-i\eta} \frac{d\xi}{\xi} \quad (A2)
\]

where the constants \(\eta_1\) and \(\eta_2\) must satisfy

\[-\frac{1}{2} \pi < \eta_1 < \frac{1}{2} \pi \quad \text{and} \quad \frac{1}{2} \pi < \eta_2 < \frac{1}{2} \pi\]

In other words, we have some flexibility in our choice of contour, flexibility that we shall exploit shortly.

Put \(w = r+i(\theta-\pi)\). Then Eq. (A2) becomes

\[
H_n^{(1)}(kr)e^{i\theta} = \frac{(-1)^n}{\pi i} \int_{-\pi}^{\pi} e^{i\xi} \left. \frac{1}{\xi} \right|_{\xi=-i\eta} \frac{d\xi}{\xi} \quad (A3)
\]

where the constants \(\beta_1\) and \(\beta_2\) must satisfy

\[-\frac{1}{2} \pi < \beta_1 + \theta - \pi < \frac{1}{2} \pi \quad \text{and} \quad \frac{1}{2} \pi < \beta_2 + \theta - \pi < \frac{1}{2} \pi\]

In particular, the choices \(\beta_1=0\) and \(\beta_2=\pi\) show that

\[
H_n^{(1)}(kr)e^{i\theta} = \frac{(-1)^n}{\pi i} \int_{-\pi}^{\pi} e^{i\xi} \left. \frac{1}{\xi} \right|_{\xi=-i\eta} \frac{d\xi}{\xi} \quad (A4)
\]

This is the integral representation that we use in Sec. 4.1.

References