Estimating the dynamic effective mass density of random composites

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(Received 27 January 2010; revised 4 June 2010; accepted 7 June 2010)

The effective mass density of an inhomogeneous medium is discussed. Random configurations of circular cylindrical scatterers are considered, in various physical contexts: fluid cylinders in another fluid, elastic cylinders in a fluid or in another solid, and movable rigid cylinders in a fluid. In each case, time-harmonic waves are scattered, and an expression for the effective wavenumber due to Linton and Martin [J. Acoust. Soc. Am. **117**, 3413–3423 (2005)] is used to derive the effective density in the low frequency limit, correct to second order in the area fraction occupied by the scatterers. Expressions are recovered that agree with either the Ament formula or the effective static mass density, depending upon the physical context.
 [DOI: 10.1121/1.3458849]

PACS number(s): 43.20.Fn, 43.20.Bi, 43.20.Hq [JAT] Pages: 571–577

I. INTRODUCTION

Methods for estimating the effective properties of random composites have been of interest for many decades. In recent years, some of this interest has been directed at the effective mass density, ρ_{eff}. In some circumstances, a simple (static) mixture rule is found. Thus, for a two-phase composite (matrix of density ρ and inclusions of density ρ_0), the static estimate is $\rho_{\text{eff}}^{\text{st}}=(1-\phi)\rho+\phi\rho_0$, where ϕ is the fraction of space (area density or volume density) occupied by the inclusions. Evidently, $\rho_{\text{eff}}^{\text{st}}$ could be determined by weighing samples.

However, one can also consider dynamic problems, followed by taking a low-frequency limit. This is appropriate if one wants to estimate effective wavespeeds. In some circumstances, this "quasistatic" limit gives a different estimate, namely,

$$\frac{\rho_{\text{eff}}}{\rho}=\frac{\rho+\rho_0-\phi(\rho-\rho_0)}{\rho+\rho_0+\phi(\rho-\rho_0)}. \tag{1}$$

The three-dimensional version of this formula is due to Ament.[1] Wavespeed estimates based on Eq. (1) have been shown to agree with experiments.[2] One purpose of this paper is to provide independent analytical verification of Eq. (1).

When is the static effective density inappropriate? If the matrix is a fluid, inertia is important, and so one might not expect to obtain the static effective density. On the other hand, if the matrix is solid, there is a well-defined static limit, and so one may expect to recover $\rho_{\text{eff}}^{\text{st}}$. Indeed, this is what has been found in the literature, as reviewed in Sec. II.

In this paper, a variety of two-dimensional problems is considered, and the low-frequency limit is examined. The prototype problem consists of the Helmholtz equation, $(\nabla^2+k^2)u=0$, outside circular cylinders. Inside each cylinder, there is another Helmholtz equation, $(\nabla^2+k_0^2)u_0=0$, with various transmission conditions connecting u and u_0 across the circular boundaries. For simplicity, it is assumed that each circle has radius a.

To analyze multiple scattering by random configurations of small scatterers, a formula for the effective wavenumber, K, obtained recently by Linton and Martin[3] is used. Their formula [see Eq. (21) in Sec. IV] uses the Lax quasicrystalline approximation (QCA), it compares favorably with experiments,[4] and it has been confirmed by an independent method that is valid for weak scattering.[5] The Linton-Martin formula is accurate to second order in ϕ. It requires the solution of a scattering problem for one scatterer; this scalar (transmission) problem is discussed briefly in Sec. III.

The Linton-Martin formula is applied to four problems. The first (Sec. V) concerns fluid cylinders in a fluid matrix. Agreement with the Ament formula, Eq. (1), is found, correct to second order in ϕ. Then, in Sec. VI, antiplane motions (SH waves) in a solid-solid composite are considered: the static estimate for ρ_{eff} is obtained. In Sec. VII, elastic cylinders in a fluid matrix are considered, whereas movable rigid cylinders in a fluid are considered in Sec. VIII. In both of these cases, agreement with the Ament formula, Eq. (1), is found, correct to second order in ϕ.

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The methods used herein can be extended to three-dimensional problems, with random configurations of spheres. The appropriate estimate for the effective wavenumber is the Lloyd-Berry formula.[6,7] The basic scattering problems for one sphere can be solved by standard methods. For one elastic sphere in a fluid, see Faran's paper.[8] For one movable rigid sphere in a fluid, see the paper by Hickling and Wang.[9]

II. REVIEW

Consider acoustic scattering in two dimensions. In the exterior, there is a homogeneous compressible fluid with density ρ. The scatterers are homogeneous with density ρ_0. Scattering by random configurations of scatterers is of interest.

The main focus of this paper is on formulas for the effective density of the random medium, ρ_{eff}. There are many such formulas. The simplest is the static estimate,

$$\rho_{\text{eff}}^{\text{st}} = (1-\phi)\rho + \phi\rho_0 = \rho + \phi\Delta, \quad \text{where} \quad \Delta = \rho_0 - \rho \tag{2}$$

is the density difference and ϕ is the area fraction occupied by the cylinders. In three dimensions (many identical spherical scatterers), ϕ is the volume fraction occupied by the spheres. Although $0<\phi<1$, ϕ is usually regarded as being small.

Ament[1] obtained another estimate in 1953, incorporating dynamic and viscous effects. In the absence of viscosity, his estimate, $\rho_{\text{eff}}^{\text{A}}$, becomes [see Eq. (9) in Ref. 1 or Eq. (1.5) in Ref. 10]

$$\rho_{\text{eff}}^{\text{A}} = \rho_{\text{eff}}^{\text{st}} - \frac{2\Delta^2\phi(1-\phi)}{2\Delta(1-\phi)+3\rho}. \tag{3}$$

This formula is for three-dimensional problems. Substituting for $\rho_{\text{eff}}^{\text{st}}$ and Δ, Eq. (3) reduces (exactly) to

$$\frac{\rho_{\text{eff}}^{\text{A}}}{\rho} = \frac{1-\phi Q_3}{1+2\phi Q_3} \quad \text{with} \quad Q_3 = \frac{\rho-\rho_0}{\rho+2\rho_0}; \tag{4}$$

the subscript 3 denotes three dimensions. Also, if Eq. (4) is approximated for small $\phi|Q_3|$,

$$\rho_{\text{eff}}^{\text{A}} \simeq \rho(1-3\phi Q_3 + 6\phi^2 Q_3^2). \tag{5}$$

In 1961, Waterman and Truell [Eq. (3.35) in Ref. 11] obtained the estimate

$$\rho_{\text{eff}}^{\text{WT}} = \rho(1-3\phi Q_3), \tag{6}$$

using a Foldy-type method, including effects of multiple scattering. Thus, they obtained the linear term in Eq. (5) (but did not cite Ament's paper[1]). Later, Fikioris and Waterman [Eq. (4.15) in Ref. 12] obtained precisely Eq. (4).

In 1974, Kuster and Toksöz [Eq. (25) in Ref. 13] used a different argument in which multiple scattering was ignored. Their estimate, $\rho_{\text{eff}}^{\text{KT}}$, is defined by

$$\frac{\rho_{\text{eff}}^{\text{KT}}-\rho}{\rho+2\rho_{\text{eff}}^{\text{KT}}} = \phi\frac{\rho_0-\rho}{\rho+2\rho_0}, \tag{7}$$

which gives $\rho_{\text{eff}}^{\text{KT}} = \rho_{\text{eff}}^{\text{A}}$, exactly, as they noted. Kuster and Toksöz considered elastic spheres in a fluid, and they "emphasized that $[\rho_{\text{eff}}^{\text{KT}}]$ is an effective inertial density" (see p. 593 of Ref. 13).

Berryman [Eq. (32) of Ref. 14] also considered elastic spheres in a compressible fluid. He used a self-consistent method and he ignored multiple scattering. His estimate, $\rho_{\text{eff}}^{\text{B}}$, is given by

$$\frac{1}{3\rho_{\text{eff}}^{\text{B}}} = \frac{1-\phi}{\rho_{\text{eff}}^{\text{B}}+2\rho} + \frac{\phi}{\rho_{\text{eff}}^{\text{B}}+2\rho_0}.$$

This equation can be rewritten as

$$\frac{\rho_{\text{eff}}^{\text{B}}-\rho}{3\rho_{\text{eff}}^{\text{B}}} = \phi\frac{\rho_0-\rho}{\rho_{\text{eff}}^{\text{B}}+2\rho_0}, \tag{8}$$

which may be compared with Eq. (7). Equation (8) is a quadratic equation for $\rho_{\text{eff}}^{\text{B}}$. For small ϕ, it gives

$$\rho_{\text{eff}}^{\text{B}} = \rho\left(1-3\phi Q_3 + 6\phi^2 Q_3^2\frac{3\rho_0}{\rho+2\rho_0}\right),$$

which differs from Eq. (5) in the ϕ^2 term.

More recently, Sheng and his colleagues[2,15,16] have given a new derivation of the two-dimensional version of Eq. (7), which is

$$\frac{\rho_{\text{eff}}-\rho}{\rho_{\text{eff}}+\rho} = \phi\frac{\rho_0-\rho}{\rho_0+\rho}. \tag{9}$$

They derive Eq. (9) by starting from a consideration of waves in a periodic square arrangement of cylinders. They also mistakenly refer to Eqs. (7) and (9) as defining the "Berryman effective mass density:" Berryman's formula is different. The same mis-attribution of Eq. (9) has also been made by Torrent and Sánchez-Dehesa;[17] these authors have also given a generalization of Eq. (9) covering non-random distributions of cylinders.

Solving Eq. (9) gives

$$\frac{\rho_{\text{eff}}}{\rho} = \frac{1-\phi Q_2}{1+\phi Q_2} \quad \text{with} \quad Q_2 = \frac{\rho-\rho_0}{\rho+\rho_0}. \tag{10}$$

This can be seen as the two-dimensional version of Ament's formula, Eq. (4). For small $\phi|Q_2|$, Eq. (10) gives

$$\rho_{\text{eff}} \sim \rho(1-2\phi Q_2 + 2\phi^2 Q_2^2). \tag{11}$$

This formula defines what is called the *small-ϕ Ament estimate* below.

III. SCATTERING BY ONE CYLINDER

For one circular scatterer, the exact solution can be constructed by separation of variables. Thus, consider one circle of radius a, centered at the origin. Then $(\nabla^2+k^2)u=0$ for $r>a$ and $(\nabla^2+k_0^2)u_0=0$ for $r<a$. The interface conditions are

$$u = u_0 \quad \text{and} \quad \frac{1}{\rho}\frac{\partial u}{\partial r} = \frac{1}{\rho_0}\frac{\partial u_0}{\partial r} \quad \text{on} \quad r=a. \tag{12}$$

These are appropriate for a fluid cylinder surrounded by a different fluid; u is the pressure. Outside the cylinder, $u=u_{\text{in}}+u_{\text{sc}}$ where $u_{\text{in}}=e^{ikx}$ and u_{sc} satisfies the Sommerfeld radiation condition. Then[3]

$$u_{\mathrm{in}}(r,\theta)=\sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in\theta}, \tag{13}$$

$$u_{\mathrm{sc}}(r,\theta)=\sum_{n=-\infty}^{\infty} A_n Z_n H_n(kr)e^{in\theta},$$

$$u_0(r,\theta)=\sum_{n=-\infty}^{\infty} B_n J_n(k_0 r)e^{in\theta}, \tag{14}$$

where J_n is a Bessel function, $H_n = H_n^{(1)}$ is a Hankel function and

$$Z_n=\frac{(\rho_0/\rho)J_n'(ka)J_n(k_0a)-(k_0/k)J_n'(k_0a)J_n(ka)}{(\rho_0/\rho)H_n'(ka)J_n(k_0a)-(k_0/k)J_n'(k_0a)H_n(ka)}. \tag{15}$$

Also, $Z_n = Z_{-n}$. The interface conditions, Eq. (12), yield equations for A_n and B_n; in particular, $A_n = -i^n$. Note that, from Eq. (14), $A_n Z_n$ could have been replaced by a single quantity. However, Z_n is retained for two reasons. First, it was used in Ref. 3. Second, Z_n characterizes scattering by a single cylinder, for any incident field, and so it arises naturally when scattering by many identical cylinders is considered.

The far-field pattern, $f(\theta)$, is defined by

$$u_{\mathrm{sc}} \sim \sqrt{2/(\pi kr)}\,f(\theta)\exp(ikr - i\pi/4) \quad \text{as} \quad r\to\infty.$$

Hence, Eq. (14) gives

$$f(\theta)=-\sum_{n=-\infty}^{\infty} Z_n e^{in\theta}. \tag{16}$$

A. Behavior of Z_n for small ka

The behavior of Z_n [defined by Eq. (15)] is studied for small ka, in various situations, starting with $n=0$:

$$Z_0=\frac{(\rho_0/\rho)J_1(ka)J_0(k_0a)-(k_0/k)J_1(k_0a)J_0(ka)}{(\rho_0/\rho)H_1(ka)J_0(k_0a)-(k_0/k)J_1(k_0a)H_0(ka)}.$$

For $\rho_0 = 0$,

$$Z_0=\frac{J_0(ka)}{H_0(ka)} \simeq \frac{1}{H_0(ka)}; \tag{17}$$

this corresponds to a soft scatterer (Dirichlet condition).

For $\rho_0 \neq 0$, one uses $J_1(x)\sim x/2$ and $H_1(x)\sim 2/(\pi i x)$ as $x\to 0$, giving

$$Z_0 \sim \frac{\pi i}{4}(ka)^2\left(1-\frac{\rho k_0^2}{\rho_0 k^2}\right). \tag{18}$$

For $n>0$, one can use the approximations

$$J_n(x)\sim \frac{x^n}{2^n n!}, \quad J_n'(x)\sim \frac{x^{n-1}}{2^n(n-1)!},$$

$$H_n(x)\sim \frac{2^n(n-1)!}{\pi i x^n}, \quad H_n'(x)\sim -\frac{2^n n!}{\pi i x^{n+1}},$$

as $x\to 0$. Then

$$Z_n \sim \frac{\pi i (ka/2)^{2n}}{n!(n-1)!}\,\frac{\rho-\rho_0}{\rho+\rho_0}, \tag{19}$$

with no dependence on k_0 at leading order. In particular, when $\rho_0 = 0$, comparison with Eq. (17) shows that Z_0 is dominant in this case.

When $\rho_0 \neq 0$ and $\rho_0 \neq \rho$, Z_0 and Z_1 are both $O((ka)^2)$ as $ka\to 0$: in general, small cylinders behave as a combination of a source and a dipole. This is the generic situation. It includes sound-hard scatterers (Neumann condition) by setting $\rho = 0$.

Exceptionally, when $\rho = \rho_0 \neq 0$, one must use

$$J_n(x)\sim \frac{x^n}{2^n n!}\left(1-\frac{(x/2)^2}{n+1}\right),$$

$$J_n'(x)\sim \frac{x^{n-1}}{2^n(n-1)!}\left(1-\frac{(n+2)x^2}{4n(n+1)}\right),$$

as $x\to 0$, giving

$$Z_n \sim \frac{\pi i}{2}\,\frac{(ka/2)^{2n+2}}{n!(n+1)!}\left(1-\frac{k_0^2}{k^2}\right).$$

Comparison with Eq. (18) shows that Z_0 is dominant when the density is constant, $\rho = \rho_0$. (These results are consistent with a paper by Exner and Šeba,[18] who consider only the two special cases $\rho = \rho_0$ and $\rho_0 = 0$.)

As is well known, the sound-soft case, $\rho_0 = 0$, is atypical: the scattering is isotropic. This suggests that care is needed with the commutativity of the two limits $\rho_0 \to 0$ and $ka\to 0$. For investigations in this direction, see Refs. 19 and 20.

IV. SCATTERING BY RANDOM ARRANGEMENTS OF CIRCULAR CYLINDERS

For scattering by random arrangements of scatterers, one can use Foldy-type theories.[21] If n_0 is the number of scatterers per unit area, the effective wavenumber, K, is given by

$$K^2 = k^2 - 4in_0 f(0). \tag{20}$$

This formula is linear in n_0 and it is derived using Foldy's "closure assumption."

Linton and Martin[3] obtained a second-order correction to Eq. (20):

$$K^2 = k^2 - 4in_0 f(0) + \frac{8n_0^2}{\pi k^2}\int_0^{\pi}\cot(\theta/2)\frac{d}{d\theta}[f(\theta)]^2 d\theta. \tag{21}$$

This formula is derived using the Lax QCA as the closure assumption. It was derived in the limit $b\to 0$, where b is the "hole radius" in the hole correction used to ensure that circles do not overlap during the averaging process. Therefore, as $b \geq 2a$, it is natural to approximate the far-field pattern $f(\theta)$ assuming that ka is small. Assuming that $\rho_0 \neq 0$ and $\rho_0 \neq \rho$, Z_0 and $Z_1 = Z_{-1}$ are retained (see Sec. III A), giving

$$f(\theta) \simeq -Z_0 - 2Z_1\cos\theta \tag{22}$$

with Z_0 given by Eq. (18),

$$Z_1 = \frac{\pi i}{4}(ka)^2 Q_2 \quad \text{and} \quad Q_2 = \frac{\rho - \rho_0}{\rho + \rho_0}. \tag{23}$$

As $[f(\theta)]^2 = Z_0^2 + 4Z_0Z_1 \cos\theta + 4Z_1^2 \cos^2\theta$, the integral term in Eq. (21) is

$$\int_0^\pi \cot(\theta/2)\frac{d}{d\theta}[f(\theta)]^2 d\theta$$
$$= -8Z_1 \int_0^\pi \cos^2(\theta/2)(Z_0 + 2Z_1\cos\theta)d\theta$$
$$-16Z_1\int_0^{\pi/2}(Z_0 - 2Z_1 + 4Z_1\cos^2\xi)\cos^2\xi d\xi$$
$$= -4\pi Z_1(Z_0 - 2Z_1) - 12\pi Z_1^2$$
$$= -4\pi Z_1(Z_0 + Z_1).$$

Hence, Eq. (21) becomes

$$K^2 = k^2 + 4in_0(Z_0 + 2Z_1) - 32n_0^2k^{-2}Z_1(Z_0 + Z_1). \tag{24}$$

This is the formula that is used below to estimate the effective mass density.

It is worth recalling the estimate for K obtained by Waterman and Truell,[11] K_{WT}, given by

$$K_{\mathrm{WT}}^2 = k^2 - 4in_0 f(0) + (2n_0/k)^2\{[f(\pi)]^2 - [f(0)]^2\}. \tag{25}$$

It is known that the term in n_0^2 is incorrect[6] but, nevertheless, Eq. (25) has been used to estimate effective properties. Using Eq. (22) gives

$$K_{\mathrm{WT}}^2 = k^2 - 4in_0(Z_0 + 2Z_1) - 16n_0^2k^{-2}Z_0Z_1. \tag{26}$$

Evidently, Eqs. (24) and (26) agree when $2(Z_0+Z_1)=Z_0$, that is, when $f(0)=0$ according to the approximation Eq. (22).

V. EFFECTIVE MASS DENSITY IN FLUID-FLUID SYSTEMS

Define the area fraction occupied by the cylinders, ϕ:

$$\phi = n_0\pi a^2.$$

Then, as already noted, a (static) effective density could be defined by Eq. (2). However, other definitions are possible, especially in a dynamic context, and it is these that are of interest here. Expressions for ρ_{eff} will be extracted from the small-ka approximations to the effective wavenumber, K, and these will be compared with Eq. (11). Notice that, in this section, sound waves in a compressible fluid are considered. (SH-waves in an elastic composite will be considered in Sec. VI) Thus, $k=\omega/c$ and $c^2=M/\rho$, where ω is the frequency, c is the wavespeed and M is the bulk modulus. Then, using obvious notation,

$$K^2 = \omega^2\rho_{\mathrm{eff}}/M_{\mathrm{eff}}, \quad k^2 = \omega^2\rho/M, \quad k_0^2 = \omega^2\rho_0/M_0,$$

and, from Eq. (18), $Z_0=(\pi i/4)(ka)^2(1-M/M_0)$. Evidently, Z_0 involves the bulk moduli only whereas Z_1 [defined by Eq. (23)] involves the densities only.

Start with the Foldy estimate,

$$(K/k)^2 = 1 + 4in_0k^{-2}(Z_0 + 2Z_1) + O(n_0^2) = (1 + 4in_0k^{-2}Z_0) \times(1 + 8in_0k^{-2}Z_1) + O(n_0^2). \tag{27}$$

As $(K/k)^2=(\rho_{\mathrm{eff}}/\rho)(M/M_{\mathrm{eff}})$, this suggests that

$$\frac{1}{M_{\mathrm{eff}}} = \frac{1}{M}(1 + 4in_0k^{-2}Z_0) = \frac{1-\phi}{M} + \frac{\phi}{M_0} \tag{28}$$

and

$$\rho_{\mathrm{eff}} = \rho(1 + 8in_0k^{-2}Z_1) = \rho(1 - 2\phi Q_2). \tag{29}$$

Thus, Eq. (28) gives M_{eff} as the harmonic average of M and M_0, whereas Eq. (29) agrees with the small-ϕ Ament estimate, Eq. (11), to first order in ϕ.

It is noteworthy that Waterman and Truell[11] used the same argument (but in three dimensions) and obtained the estimate Eq. (6). They regarded their estimate as "not so readily interpretable" as Eq. (28), and went on to consider the limit $\rho_0 \to \rho$.

Next, consider the Linton-Martin small-ka estimate for K, Eq. (24). It gives

$$(K/k)^2 = (1 + 4in_0k^{-2}Z_0)(1 + 8in_0k^{-2}Z_1 - 32n_0^2k^{-4}Z_1^2), \tag{30}$$

correct to order n_0^2. Note that the first factor on the right-hand side is exactly the same as in Eq. (27), so that the estimate for M_{eff}, Eq. (28), is unchanged. The second factor gives a refined estimate for ρ_{eff}; it agrees precisely with the small-ϕ Ament estimate, Eq. (11).

Aristégui and Angel [Eq. (21) in Ref. 22] used the Waterman-Truell estimate, Eq. (25), to obtain an estimate, $\rho_{\mathrm{eff}}^{\mathrm{AA}}$, for the effective density,

$$\rho_{\mathrm{eff}}^{\mathrm{AA}}/\rho = 1 - 2in_0k^{-2}[f(0) - f(\pi)]. \tag{31}$$

It is noted that Eq. (31) is linear in $\phi=n_0\pi a^2$; it gives $\rho_{\mathrm{eff}}^{\mathrm{AA}} \sim 1-2\phi Q_2$ as $ka\to 0$. Aristégui and Angel [Eq. (18) in Ref. 23] have obtained a similar estimate for SH waves in a random composite: this problem is discussed in Sec. VI.

One of the advantages of using formulas such as those above, involving the far-field pattern f, is that they often transpire to be useful well beyond the regime of small ka under which they were derived. However they are, of course, limited to the regime of small ϕ. An alternative approach in classical multiple scattering methods is to use the assumption of small ka earlier on in the analysis and retain the leading order terms in ka: classically this was done for two-dimensional elasticity problems by Bose and Mal.[24,25] This approach gives an effective wavenumber which is physically viable (meaning that predictions of effective properties reside inside strict variational bounds) for all ϕ. Using such an approach leads to M_{eff} as in Eq. (28) and ρ_{eff} as in Eq. (10).

VI. SH WAVES IN A SOLID-SOLID COMPOSITE

Here, anti-plane (SH) motions in a solid composite are considered. The problem of scattering by one cylinder is almost the same as for the fluid-fluid problem discussed in Sec. III. Thus, $(\nabla^2+k^2)u=0$ for $r>a$ and $(\nabla^2+k_0^2)u_0=0$ for $r<a$, where u and u_0 are the out-of-plane displacement components. The shear wavenumbers, k and k_0, are given by

$$k^2=\omega^2\rho/\mu \quad \text{and} \quad k_0^2=\omega^2\rho_0/\mu_0,$$

where μ and μ_0 are the shear moduli. The interface conditions differ from Eq. (12); they are

$$u=u_0 \quad \text{and} \quad \mu\frac{\partial u}{\partial r}=\mu_0\frac{\partial u_0}{\partial r} \quad \text{on} \quad r=a. \tag{32}$$

Outside the cylinder, $u=u_{\rm in}+u_{\rm sc}$ (as before) where $u_{\rm in}=e^{ikx}$ and $u_{\rm sc}$ satisfies the Sommerfeld radiation condition.

Comparing Eq. (32) with Eq. (12) shows that all the results of Sec. III can be reused: replace ρ_0/ρ by μ/μ_0 therein. Thus, for small ka, Eqs. (18) and (23) give

$$Z_0 \sim (\pi i/4)(ka)^2(1-\rho_0/\rho) \quad \text{and} \quad Z_1 \sim (\pi i/4)(ka)^2\mathcal{Q}$$

with $\mathcal{Q}=(\mu_0-\mu)/(\mu_0+\mu)$, in agreement with Bose and Mal [Eq. (16) in Ref. 24]. Then, proceeding as in Sec. V, one defines an effective shear modulus $\mu_{\rm eff}$ by $K^2=\omega^2\rho_{\rm eff}/\mu_{\rm eff}$. Working correct to order n_0^2, the Linton-Martin estimate, Eq. (30), gives

$$\rho_{\rm eff}=\rho(1+4in_0k^{-2}Z_0)=(1-\phi)\rho+\phi\rho_0 \tag{33}$$

and

$$\mu/\mu_{\rm eff}=1+8in_0k^{-2}Z_1-32n_0^2k^{-4}Z_1^2=1-2\phi\mathcal{Q}+2\phi^2\mathcal{Q}^2. \tag{34}$$

Thus, one obtains the static estimate for $\rho_{\rm eff}$, Eq. (2), whereas Eq. (34) gives

$$\mu_{\rm eff}/\mu \sim 1+2\phi\mathcal{Q}+2\phi^2\mathcal{Q}^2$$

which agrees with the formula of Bose and Mal [Eq. (39) in Ref. 24], namely,

$$\frac{\mu_{\rm eff}}{\mu}=\frac{1+\phi\mathcal{Q}}{1-\phi\mathcal{Q}}, \tag{35}$$

correct to second order in $\phi\mathcal{Q}$. The expression Eq. (35) was derived in Ref. 24 using the assumption of $ka \ll 1$ from the outset, as described at the end of Sec. V. Also, it is noted that Eq. (35) is the well-known static result of Hashin and Rosen [Eq. (71) in Ref. 26]. Yang and Mal[27] have noted that it cannot be obtained from the Waterman-Truell estimate, $K_{\rm WT}$, defined by Eq. (25).

The static effective density in Eq. (33) is recovered by employing asymptotic homogenization for elasticity problems, e.g., the SH problem solved by Parnell and Abrahams.[28] In this case, because the arrangement of scatterers was periodic, the effective shear modulus was anisotropic in general. Reciprocity between the SH and acoustics problems, as described above, therefore gives a static effective bulk modulus and an anisotropic effective density in the case of acoustics. This latter point is particularly pertinent to note, considering recent comments regarding possible applications in metamaterials.[29] The static effective density is also recovered by application of a recently proposed scheme of homogenization applied to the SH problem: the so-called *integral equation method* proposed by Parnell and Abrahams.[30] This method also recovers Eq. (35) at leading order, with successive lattice corrections for materials where scatterers are positioned on a periodic lattice. Finally, as should be expected, for the in-plane two-dimensional elasticity problem (the so-called P/SV problem) the static effective density $\rho_{\rm eff}$, as in Eq. (33), is recovered.[25,31]

VII. FLUID-SOLID PROBLEM

Consider an elastic cylinder in a compressible inviscid fluid. The transmission conditions at $r=a$ are

$$\frac{1}{\rho}\frac{\partial u}{\partial r}=\omega^2\hat{\mathbf{r}}\cdot\mathbf{u}_0 \quad \text{and} \quad -u\hat{\mathbf{r}}=T_0\mathbf{u}_0,$$

where ρ is the fluid density, $\hat{\mathbf{r}}$ is a unit vector in the radial direction, $\mathbf{u}_0$ is the elastic displacement in the cylinder and T_0 is the traction operator. With $\mathbf{u}_0=(u_r^0,u_\theta^0)$, these conditions become

$$\frac{1}{\rho}\frac{\partial u}{\partial r}=\omega^2u_r^0, \quad \tau_{rr}^0=-u \quad \text{and} \quad \tau_{r\theta}^0=0 \quad \text{on} \quad r=a.$$

The problem of plane-wave scattering by an elastic cylinder in a fluid can be solved by standard methods.[8] For a convenient resolution, see p. 46 in Ref. 32, where the total pressure field in the fluid is given by

$$u=\sum_{n=0}^{\infty}\epsilon_n i^n\{J_n(kr)-Z_nH_n(kr)\}\cos n\theta, \tag{36}$$

where $\epsilon_0=1$, $\epsilon_n=2$ for $n\geq 1$, $Z_n=b_n/D_n$,

$$b_n=\begin{vmatrix}(\rho/\rho_0)x_{\rm T}^2J_n(x) & a_{12}^n & a_{13}^n\\ xJ_n'(x) & a_{22}^n & a_{23}^n\\ 0 & a_{32}^n & a_{33}^n\end{vmatrix},$$

$$D_n=\begin{vmatrix}(\rho/\rho_0)x_{\rm T}^2H_n(x) & a_{12}^n & a_{13}^n\\ xH_n'(x) & a_{22}^n & a_{23}^n\\ 0 & a_{32}^n & a_{33}^n\end{vmatrix},$$

$$a_{12}^n=-2x_{\rm L}J_n'(x_{\rm L})+(2n^2-x_{\rm T}^2)J_n(x_{\rm L}),$$

$$a_{22}^n=-x_{\rm L}J_n'(x_{\rm L}), \quad a_{32}^n=2n[x_{\rm L}J_n'(x_{\rm L})-J_n(x_{\rm L})],$$

$$a_{13}^n=-2n[x_{\rm T}J_n'(x_{\rm T})-J_n(x_{\rm T})],$$

$$a_{23}^n=nJ_n(x_{\rm T}), \quad a_{33}^n=2x_{\rm T}J_n'(x_{\rm T})-(2n^2-x_{\rm T}^2)J_n(x_{\rm T}),$$

$x=ka$, $x_{\rm T}=k_{\rm T}a$, $x_{\rm L}=k_{\rm L}a$, $k_{\rm T}$ is the shear wavenumber, $k_{\rm L}$ is the compressional wave-number and ρ_0 is the solid density. Note that $Z_n=Z_{-n}$, so that Eq. (16) remains valid.

For $n\geq 1$, all a_{ij}^n are $O(x^n)$ as $x\to 0$. Hence, a rough calculation gives

$$Z_n \sim \frac{J_n'(x)}{H_n'(x)} \sim -\frac{\pi i(x/2)^{2n}}{n!(n-1)!}.$$

This is a correct estimate for $n\geq 2$ but there is some cancellation for $n=1$. Thus,

$$\begin{vmatrix}a_{22}^1 & a_{23}^1\\ a_{32}^1 & a_{33}^1\end{vmatrix} \sim \frac{1}{8}x_{\rm L}x_{\rm T}(x_{\rm L}^2-x_{\rm T}^2)$$

and

$$\begin{vmatrix} a_{12}^1 & a_{13}^1 \\ a_{32}^1 & a_{33}^1 \end{vmatrix} \sim \frac{1}{8} x_{\mathrm{L}} x_{\mathrm{T}}^3 (x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2),$$

which are both smaller than first expected. Hence, a more careful calculation gives

$$Z_1 \sim \frac{\dfrac{\rho x_{\mathrm{T}}^2 x}{2\rho_0} \begin{vmatrix} a_{22}^1 & a_{23}^1 \\ a_{32}^1 & a_{33}^1 \end{vmatrix} - \dfrac{x}{2} \begin{vmatrix} a_{12}^1 & a_{13}^1 \\ a_{32}^1 & a_{33}^1 \end{vmatrix}}{\dfrac{2\rho x_{\mathrm{T}}^2}{\rho_0 \pi i x} \begin{vmatrix} a_{22}^1 & a_{23}^1 \\ a_{32}^1 & a_{33}^1 \end{vmatrix} + \dfrac{2}{\pi i x} \begin{vmatrix} a_{12}^1 & a_{13}^1 \\ a_{32}^1 & a_{33}^1 \end{vmatrix}} \sim \frac{\pi i}{4} x^2 Q_2,$$

exactly as in Eq. (23).

For $n=0$, $a_{13}^0=a_{23}^0=a_{32}^0=0$, and so $Z_0=\widetilde{b}_0/\widetilde{D}_0$, where

$$\widetilde{b}_0 = \begin{vmatrix} (\rho/\rho_0) x_{\mathrm{T}}^2 J_0 & a_{12}^0 \\ -xJ_1(x) & a_{22}^0 \end{vmatrix}, \quad \widetilde{D}_0 = \begin{vmatrix} (\rho/\rho_0) x_{\mathrm{T}}^2 H_0 & a_{12}^0 \\ -xH_1(x) & a_{22}^0 \end{vmatrix},$$

$a_{12}^0=2x_{\mathrm{L}}J_1(x_{\mathrm{L}})-x_{\mathrm{T}}^2 J_0(x_{\mathrm{L}})$ and $a_{22}^0=x_{\mathrm{L}}J_1(x_{\mathrm{L}})$. Then, $a_{12}^0 \sim x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2$, $a_{22}^0 \sim x_{\mathrm{L}}^2/2$,

$$\widetilde{b}_0 \sim \begin{vmatrix} (\rho/\rho_0) x_{\mathrm{T}}^2 & x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2 \\ -x^2/2 & x_{\mathrm{L}}^2/2 \end{vmatrix},$$

$$\widetilde{D}_0 \sim \begin{vmatrix} (\rho/\rho_0) x_{\mathrm{T}}^2 H_0(x) & x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2 \\ 2i/\pi & x_{\mathrm{L}}^2/2 \end{vmatrix} \sim \frac{2}{\pi i}(x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2),$$

$$Z_0 \sim \frac{\pi i}{4} x^2 \left(1 + \frac{(\rho/\rho_0) x_{\mathrm{T}}^2 x_{\mathrm{L}}^2}{x^2 (x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2)} \right).$$

To simplify, $x^2=(ka)^2=(\omega a)^2\rho/M$, $x_{\mathrm{T}}^2=(k_{\mathrm{T}}a)^2=(\omega a)^2\rho_0/\mu_0$ and $x_{\mathrm{L}}^2=(k_{\mathrm{L}}a)^2=(\omega a)^2\rho_0/(\lambda_0+2\mu_0)$. Hence

$$\frac{(\rho/\rho_0) x_{\mathrm{T}}^2 x_{\mathrm{L}}^2}{x^2 (x_{\mathrm{L}}^2 - x_{\mathrm{T}}^2)} = -\frac{M}{\lambda_0 + \mu_0}.$$

In summary, the dominant terms are Z_0 and $Z_1=Z_{-1}$, where

$$Z_0 = \frac{\pi i}{4} x^2 \left(1 - \frac{M}{\lambda_0 + \mu_0} \right) \quad \text{and} \quad Z_1 = \frac{\pi i}{4} x^2 Q_2,$$

with Q_2 given by Eq. (23). Using these estimates in Eq. (30) gives

$$\frac{1}{M_{\mathrm{eff}}} = \frac{1-\phi}{M} + \frac{\phi}{\lambda_0 + \mu_0} \tag{37}$$

for M_{eff} and the small-ϕ Ament estimate, Eq. (11), for ρ_{eff}. Notice that, in Eq. (37), the quantity $\lambda_0+\mu_0$ is the plane-strain bulk modulus for the solid.

VIII. MOVABLE RIGID CYLINDERS IN A FLUID

Consider a rigid cylinder in a compressible inviscid fluid. The cylinder is free to move, so that its equation of motion is needed. The corresponding scattering problem has been solved by Zhuk.[33]

As before, let u be the total pressure field in the fluid, and suppose that it is expanded as Eq. (36). The resultant force per unit length on the cylinder due to the incident wave $u_{\mathrm{in}}=e^{ikx}$ is in the x-direction, and it is given by

$$F_x = -\int_0^{2\pi} u(a,\theta)\cos\theta\, a\, d\theta = -2\pi i a\{J_1(ka) - Z_1 H_1(ka)\}.$$

The cylinder oscillates in the x-direction with velocity U. Thus, Newton's law gives $F_x=-i\omega U(\pi a^2\rho_0)$, and so

$$U = 2(\omega a \rho_0)^{-1}\{J_1(ka) - Z_1 H_1(ka)\}.$$

As the fluid velocity is $(i\omega\rho)^{-1}\mathrm{grad}\, u$,

$$\frac{1}{i\omega\rho}\frac{\partial u}{\partial r} = U\cos\theta \quad \text{on} \quad r=a.$$

Applying this boundary condition, using Eq. (36), determines Z_n. For $n \neq 1$, $Z_n=J_n'(ka)/H_n'(ka)$, which is the same as would be obtained for a sound-hard cylinder; in particular $Z_0 \sim (\pi i/4)(ka)^2$ as $ka \to 0$. For $n=1$,

$$Z_1 = \frac{(\rho_0/\rho) ka J_1'(ka) - J_1(ka)}{(\rho_0/\rho) ka H_1'(ka) - H_1(ka)}.$$

For small ka, one finds that Z_1 behaves precisely as in Eq. (23), implying that the effective properties for movable rigid cylinders in a fluid are exactly the same as for the fluid-fluid problem discussed in Sec. V.

IX. CONCLUSION

According to Ament,[1] the dynamic (low-frequency limit) effective mass density of a random composite, ρ_{eff}, is given by Eq. (1). This formula gives $\rho_{\mathrm{eff}} \to \rho$ as $\phi \to 0$ and $\rho_{\mathrm{eff}} \to \rho_0$ as $\phi \to 1$, both of which are physically correct limits. In this paper, the small-ϕ behavior has been explored in more detail, retaining terms correct to second order in ϕ. It was found that, to this order, Ament's formula agrees precisely with the predictions of an independent multiple-scattering theory,[3] for several (two-dimensional) physical systems. These systems have in common that the exterior medium ($\phi=0$) is a compressible fluid. For SH waves in a solid, however, the familiar static estimate, Eq. (2), was recovered.

The detailed agreement between the two theories is gratifying, given that the precise conditions for validity of the Ament formula or the Linton–Martin multiple-scattering formalism are unknown, at present. However, the same degree of agreement would not have been found if Ament's formula had been replaced with Berryman's formula, Eq. (8), for example, or if the Linton–Martin theory had been replaced with the Waterman–Truell theory,[11] for example. Consequently, analogous three-dimensional investigations are warranted.

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