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Multiple scattering of flexural waves by random configurations of inclusions in thin plates

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ABSTRACT

Flexural waves are scattered by inclusions in a thin plate. For a single inclusion of arbitrary shape, reciprocity relations are obtained connecting coefficients in circular multipole expansions. Then, a formula for the effective wavenumber in a random arrangement of identical circular inclusions is derived, using the Lax quasi-crystalline approximation.

1. Introduction

The simplest model for wave propagation in a thin plate is based on Kirchhoff theory, which gives a partial differential equation, $(\nabla^4 - k^4)w = 0$, for the out-of-plane component of displacement, w. (Notation, formulation, and boundary conditions will be described fully in Section 2.) More refined models are available, but we limit ourselves here to the Kirchhoff theory; extensions to other models seem feasible.

Flexural waves in the plate will be scattered by inclusions, including the cases of holes (cavities) and defects, for example. We are interested in describing the scattering by many small inclusions, arranged randomly.

Early work on scattering by one inclusion is discussed in the book by Pao and Mow [23, Section III.6]. Since then, several authors have analysed scattering by one circular inclusion. For example, Norris and Vemula [22] gave results for the limiting cases of a rigid inclusion and a cavity. Chou et al. [4] and Squire and Dixon [25] used similar methods but in the context of surfactant effects and ice dynamics, respectively. Lee and Chen have given numerical results for scattering by two circular inclusions [13,14] and by three circular cavities [15]. Evans and Porter [8] have discussed scattering by N point scatterers, located arbitrarily in an infinite plate. Movchan et al. [21] have studied waves in a plate perforated by an infinite square array of identical circular holes. Matus and Emets [19] have developed a T-matrix method for scattering by one non-circular inclusion in a plate.

The classical (deterministic) multiple scattering problem, with many identical circular inclusions in the plate, can be tackled by combining separated solutions with addition theorems. We give such an exact treatment in Section 3.1: it serves as a starting point for problems in which the inclusions are located randomly.

There is previous work on flexural waves in plates with various kinds of randomness. We mention Beran's paper [1] on random density fluctuations and Weaver's paper [27] on plates with randomly attached sprung masses. Dixon and Squire [7] have used several approaches, including a self-consistent approach in which the solution of a certain scattering problem, with one inclusion embedded in the (unknown) effective medium [25], is employed.

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The literature on multiple scattering of acoustic waves by many inclusions (in both deterministic and random configurations) is extensive. For a review, see [17]. The literature on acoustic scattering by random arrangements of identical circular scatterers is reviewed in the paper by Linton and Martin [16]. That paper starts with a classical, exact multiple scattering theory (built using separated solutions of the Helmholtz equation and appropriate addition theorems), as used earlier by, for example, Fikioris and Waterman [9] and by Bose and Mal [3], and ends with an estimate for the effective acoustic wavenumber, K, in the random medium,

$$(Ka)^2 = (ka)^2 - \frac{4i}{\pi}\phi f(0) + \frac{8\phi^2}{\pi^3 (ka)^2}\int_0^\pi \cot(\theta/2)\frac{d}{d\theta}[f(\theta)]^2 d\theta + O(\phi^3). \tag{1}$$

In this formula, $f(\theta)$ is the far-field pattern for scattering of a plane wave (with wavenumber k) by a single inclusion of radius a, and ϕ is the fractional area occupied by the scatterers. Formula (1) has been shown to compare well with experiments [6]. It was derived using the Lax quasi-crystalline approximation (QCA) [12], a heuristic closure assumption that has been shown to give good theoretical results in certain situations [18,28]. See Parnell and Abrahams [24] for a comparison of Eq. (1) with other multiple scattering theories in the low-frequency (homogenization) limit. The generalization of Eq. (1) to plane-strain elastodynamics was given recently by Conoir and Norris [5].

In this paper, we develop a thin plate theory analogous to the acoustic theory given by Linton and Martin [16]. We derive the following formula for the effective flexural wavenumber K (in (58)) below:

$$(Ka)^2 = (ka)^2 - \frac{4i}{\pi}\phi f(0) + \frac{8\phi^2}{\pi^3 (ka)^2}\int_0^\pi \cot(\theta/2)\frac{d}{d\theta}[f(\theta)]^2 d\theta + \frac{8\phi^2}{\pi^2 (ka)^2}[g(0)]^2 + O(\phi^3). \tag{2}$$

Note that this is identical to Eq. (1) to $O(\phi^2)$ except for the additional term involving a second far-field pattern $g(\theta)$: whereas $f(\theta)$ is generated by a plane wave, e^{ikx}, $g(\theta)$ is generated by an incident field, e^{-ky}, a field that has no physical interest in itself. As an example, we calculate effective properties of a plate perforated by many small circular holes.

We also obtain reciprocity relations connecting coefficients in multipole expansions of the field scattered by a single inclusion. These relations (see Eq. (A.6) below) hold for inclusions of all shapes and composition, and so they are of independent interest. We use one of them in the derivation of our formula for K. Also, as pointed out by Conoir and Norris [5], when the inclusions are such that $f(\theta) \neq f(-\theta)$, $[f(\theta)]^2$ in Eq. (2) should be replaced by $f(\theta)f(-\theta)$.

2. Flexural wave scattering by a single inclusion in a thin plate

We use a Cartesian coordinate system (x, y, z) where the xy plane is the plane of the thin plate and z is perpendicular to this. Under the assumptions of the Kirchhoff thin plate theory, the amplitude of flexural waves $W(x,y,t)$ normal to the plane of the plate is described by

$$D\nabla^2\nabla^2 W + \rho h \partial^2 W/\partial t^2 = q \tag{3}$$

where q is the external forcing, ρ is the density, h is the thickness of the plate, the bending rigidity is given by $D = Eh^3/[12(1-\nu^2)]$, E is Young's modulus and ν is Poisson's ratio.

Assuming time harmonic behaviour, $W(x,y,t) = \mathrm{Re}\{w(x,y)e^{-i\omega t}\}$, and taking $q = 0$, from Eq. (3) we have

$$(\nabla^2\nabla^2 - k^4)w = (\nabla^2 + k^2)(\nabla^2 - k^2)w = 0 \quad \text{with } k^4 = \rho h \omega^2/D. \tag{4}$$

We consider the scattered field from an isolated circular inclusion with different material properties from the surrounding plate (of infinite extent). As noted in Section 1, this problem has been considered before, but we summarize it here so as to define notation for the main purpose of this article, which is the consideration of the multiple scattering problem.

Assume that the (circular) scatterer of radius a, has domain S and assume that a polar coordinate system $x = r\cos\theta$, $y = r\sin\theta$ is aligned with the centre of the scatterer. The amplitude of flexural waves is then governed by the following equations,

$$(\nabla^2\nabla^2 - k^4)w = 0, \quad r > a, \tag{5}$$

$$(\nabla^2\nabla^2 - k_1^4)w_1 = 0, \quad r < a, \tag{6}$$

where $k^4 = \rho h\omega^2/D_0$ and $k_1^4 = \rho_1 h\omega^2/D_1$, together with specified conditions on $r = a$ and associated conditions as $r \to \infty$. We note that in the exterior region, the total displacement field is given by

$$w = w_{\text{inc}} + w_0 \tag{7}$$

where $w_{\rm inc}$ is the (specified) incident wave and w_0 is the scattered field. The field $w_{\rm inc}$ is regular in a neighbourhood of S; in general, it has an expansion

$$w_{\rm inc} = \sum_{n=-\infty}^{\infty} \left\{ \mathcal{A}_n^J J_n(kr) + \mathcal{A}_n^I I_n(kr) \right\} \mathrm{e}^{\mathrm{i}n\theta}, \tag{8}$$

where J_n is a Bessel function, I_n is a modified Bessel function defined by $I_n(x) = i^{-n} J_n(ix)$ with x real here, and the coefficients $\mathcal{A}_n^J$ and $\mathcal{A}_n^I$ are known. For example, if we have a plane wave propagating at an angle $\theta_{\rm inc}$ to the positive x axis, then

$$w_{\rm inc} = \exp\{ikr\cos(\theta - \theta_{inc})\}, \quad \mathcal{A}_n^J = i^n \mathrm{e}^{-in\theta_{\rm inc}} \quad \text{and} \quad \mathcal{A}_n^I = 0. \tag{9}$$

For a second example, which is unphysical but useful later, we take

$$w_{\rm inc} = \exp\{-kr\sin\theta\}, \quad \mathcal{A}_n^J = 0 \quad \text{and} \quad \mathcal{A}_n^I = \mathrm{i}^n. \tag{10}$$

The scattered field outside S and the field inside S can be sought in the forms

$$w_0(r,\theta) = \sum_{n=-\infty}^{\infty} \{A_n H_n(kr) + B_n K_n(kr)\} \mathrm{e}^{\mathrm{i}n\theta}, \quad r \geq a, \tag{11}$$

$$w_1(r,\theta) = \sum_{n=-\infty}^{\infty} \{C_n J_n(k_1 r) + E_n I_n(k_1 r)\} \mathrm{e}^{\mathrm{i}n\theta}, \quad r \leq a, \tag{12}$$

where $H_n \equiv H_n^{(1)}$ is a Hankel function and K_n is a modified Bessel function defined by $2K_n(x) = \pi i^{n+1} H_n^{(1)}(ix)$ with x real here. The coefficients A_n, B_n, C_n and E_n are determined from $\mathcal{A}_n^J$ and $\mathcal{A}_n^I$ using continuity conditions at $r = a$. According to Kirchhoff theory, these conditions are those of continuity of displacement w, its normal derivative $\partial w / \partial r$, the bending moment $M(w)$ and the (Kirchhoff) shear force $V(w)$, the latter two quantities being defined in polar coordinates by

$$M(w) = -D_i \frac{\partial^2 w}{\partial r^2} - D_i \nu_i \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right), \tag{13}$$

$$V(w) = -D_i \frac{\partial}{\partial r} \nabla^2 w - D_i (1 - \nu_i) \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial w}{\partial r} - \frac{w}{r} \right), \tag{14}$$

where D_i and ν_i are the bending rigidity and Poisson ratio of the host ($i = 0$) and inclusion ($i = 1$).

Applying these continuity conditions leads to the system of equations

$$\mathbf{Ax} = \mathbf{Ba}, \tag{15}$$

where

$$\mathbf{A} = \begin{pmatrix} H_n(\epsilon_0) & K_n(\epsilon_0) & -J_n(\epsilon_1) & -I_n(\epsilon_1) \\ H_n'(\epsilon_0) & K_n'(\epsilon_0) & -\kappa J_n'(\epsilon_1) & -\kappa I_n'(\epsilon_1) \\ S_{H_n}^0 & S_{K_n}^0 & -DS_{J_n}^1 & -DS_{I_n}^1 \\ T_{H_n}^0 & T_{K_n}^0 & -DT_{J_n}^1 & -DT_{I_n}^1 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} A_n \\ B_n \\ C_n \\ E_n \end{pmatrix}, \quad \mathbf{B} = -\begin{pmatrix} J_n(\epsilon_0) & I_n(\epsilon_0) \\ J_n'(\epsilon_0) & I_n'(\epsilon_0) \\ S_{J_n}^0 & S_{I_n}^0 \\ T_{J_n}^0 & T_{I_n}^0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \mathcal{A}_n^J \\ \mathcal{A}_n^I \end{pmatrix}.$$

Here, prime denotes differentiation with respect to the argument, $\kappa = k_1/k$, $\epsilon_0 = ka$, $\epsilon_1 = k_1 a = \kappa\epsilon_0$, and $D = D_1/D_0$ is the ratio of bending stiffnesses and we have used the following notation [22]

$$S_{X_n}^i = [n^2(1-\nu_i) \mp \epsilon_i^2] X_n(\epsilon_i) - (1-\nu_i)\epsilon_i X_n'(\epsilon_i), \tag{16}$$

$$T_{X_n}^i = n^2(1-\nu_i) X_n(\epsilon_i) - [n^2(1-\nu_i) \pm \epsilon_i^2]\epsilon_i X_n'(\epsilon_i), \tag{17}$$

in which the upper sign is taken when $X_n = J_n$ or H_n, and the lower sign is taken when $X_n = I_n$ or K_n. The differential equations satisfied by the various Bessel functions have been used in order to simplify the expressions for $S_{X_n}^i$ and $T_{X_n}^i$.

The 4×4 system, Eq. (15), can be solved for A_n, B_n, C_n and E_n. Of importance later will be expressions for the coefficients A_n and B_n. These expressions can be written in the form

$$A_n = T_{11}^n\mathcal{A}_n^J + T_{12}^n\mathcal{A}_n^I, \quad B_n = T_{21}^n\mathcal{A}_n^J + T_{22}^n\mathcal{A}_n^I, \tag{18}$$

where the (dimensionless) quantities T_{ij}^n are computable. The expressions in Eq. (18) can also be used for cavities and for rigid inclusions (although the formulas for T_{ij}^n will be different).

It is shown in Appendix A that

$$(2\mathrm{i}/\pi)T_{12}^n = T_{21}^n. \tag{19}$$

This relation is a consequence of reciprocity. It is a special case of general results that hold for scatterers of any shape and composition.

In the far-field ($kr\to\infty$), we can use

$$H_n(z)\sim\sqrt{2/(\pi z)}(-\mathrm{i})^n\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}z}, \quad K_n(z)\sim\sqrt{\pi/(2z)}\mathrm{e}^{-z}, \quad z\to\infty.$$

These show that, physically, the first term in Eq. (11) is associated with the scattered field and the second term is the local, evanescent field. Thus,

$$w_0(r,\theta)\sim\sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}kr-\mathrm{i}\pi/4}\sum_{n=-\infty}^{\infty}(-\mathrm{i})^n A_n\mathrm{e}^{\mathrm{i}n\theta} \quad \text{as } r\to\infty,$$

where A_n is given by Eq. $(18)_1$. In particular, for an incident plane wave, given by Eq. (9), we have

$$w_0(r,\theta)\sim\sqrt{2/(\pi kr)}f(\theta-\theta_{\mathrm{inc}})\exp(\mathrm{i}kr-\mathrm{i}\pi/4) \text{ as } r\to\infty,$$

where the *far-field pattern* f is given by

$$f(\theta) = \sum_{n=-\infty}^{\infty} T_{11}^n\mathrm{e}^{\mathrm{i}n\theta}. \tag{20}$$

Similarly, the far-field pattern associated with an incident wave of the form given in Eq. (10) is g, given by

$$g(\theta) = \sum_{n=-\infty}^{\infty} T_{12}^n\mathrm{e}^{\mathrm{i}n\theta}. \tag{21}$$

3. Multiple scattering theory

3.1. Multiple scattering from N inclusions

We now consider an exact theory for multiple scattering of flexural waves when there are N identical circular inclusions embedded in the thin plate.

To begin with, the inclusions occupy a finite domain $\mathcal{D}$, $(x,y)\in[0,L]\times[-H,H]$, within an unbounded infinite plate. The area of $\mathcal{D}$ is denoted by $|\mathcal{D}|$. There are n_0 inclusions per unit area so that $N = n_0|\mathcal{D}|$. The fractional area occupied by the inclusions is denoted by

$$\phi = n_0\pi a^2 \qquad (0\le\phi\le1).$$

Later, we let $\mathcal{D}$ expand to fill a half-space $\mathcal{H}$ by letting $L\to\infty$ and $H\to\infty$ whilst letting $N\to\infty$ with n_0 held fixed (Fig. 1).

The inclusions are located at $\mathbf{p}_j = (p_j, q_j)$, $j=1,2,\ldots,N$ with local polar coordinate system (r_j,θ_j) relating to the local Cartesian system $x_j=r_j\cos\theta_j$, $y_j=r_j\sin\theta_j$, see Fig. 2.

Suppose that the incident field which excites scattering from the N inclusions is a plane wave propagating at an arbitrary angle θ_{inc} to the x axis. Thus, w_{inc} is given by Eq. (9) and it can be written in the form

$$w_{\mathrm{inc}}(r_j,\theta_j) = \mathrm{I}_j\sum_{n=-\infty}^{\infty}\mathrm{i}^n J_n(kr_j)\mathrm{e}^{\mathrm{i}n(\theta_j-\theta_{\mathrm{inc}})} \tag{22}$$

local to the jth inclusion, where $\mathrm{I}_j = \mathrm{e}^{\mathrm{i}(\alpha p_j+\beta q_j)}$, $\alpha=k\cos\theta_{\mathrm{inc}}$ and $\beta=k\sin\theta_{\mathrm{inc}}$. (In [16], I_n is denoted by I_n (Italics in the latter); here, we use I_n to denote modified Bessel functions.)

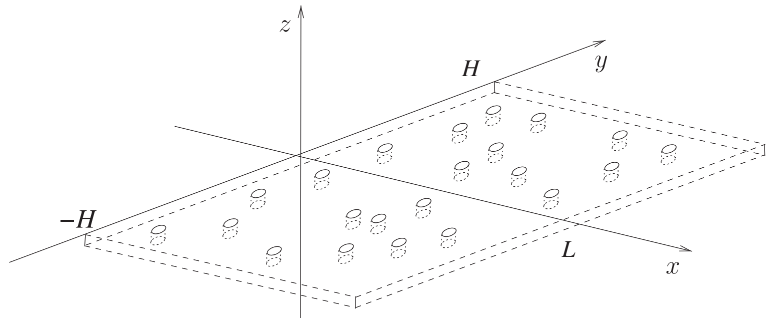

Fig. 1. Inclusions reside inside a domain $\mathcal{D}$ in an otherwise uniform and unbounded thin plate. We consider the limit where $L, H \to \infty$, whilst letting the number of inclusions $N \to \infty$ with the volume fraction ϕ held fixed.

The total field in $x>0$, outside all the inclusions, is given by

$$w = w_{\mathrm{inc}} + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \left\{ A_n^j Z_n H_n(kr_j) + B_n^j W_n K_n(kr_j) \right\} \mathrm{e}^{\mathrm{i}n\theta_j}, \quad r_j \geq a, \tag{23}$$

where the constants Z_n and W_n have been added for convenience and will be defined shortly. Inside the jth inclusion, the field is w_1^j, where

$$w_1^j = \sum_{n=-\infty}^{\infty} \left\{ C_n^j J_n(k_1 r_j) + E_n^j I_n(k_1 r_j) \right\} \mathrm{e}^{\mathrm{i}n\theta_j}, \quad r_j \leq a.$$

With the benefit of Graf's addition theorem, we can write down w in the vicinity of the sth inclusion:

$$\begin{aligned} w(r_s, \theta_s) = & \sum_{n=-\infty}^{\infty} \left\{ \mathrm{I}_s \mathrm{i}^n J_n(kr_s) \mathrm{e}^{-\mathrm{i}n\theta_{\mathrm{inc}}} + A_n^s Z_n H_n(kr_s) + B_n^s W_n K_n(kr_s) \right\} \mathrm{e}^{\mathrm{i}n\theta_s} \\ & + \sum_{\substack{j=1 \\ j \neq s}}^{N} \sum_{n=-\infty}^{\infty} \left(A_n^j Z_n \sum_{m=-\infty}^{\infty} H_{n-m}(kR_{js}) \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}} J_m(kr_s) \mathrm{e}^{\mathrm{i}m\theta_s} + B_n^j W_n \sum_{m=-\infty}^{\infty} K_{n-m}(kR_{js}) \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}} I_m(kr_s) \mathrm{e}^{\mathrm{i}m\theta_s} \right), \end{aligned} \tag{24}$$

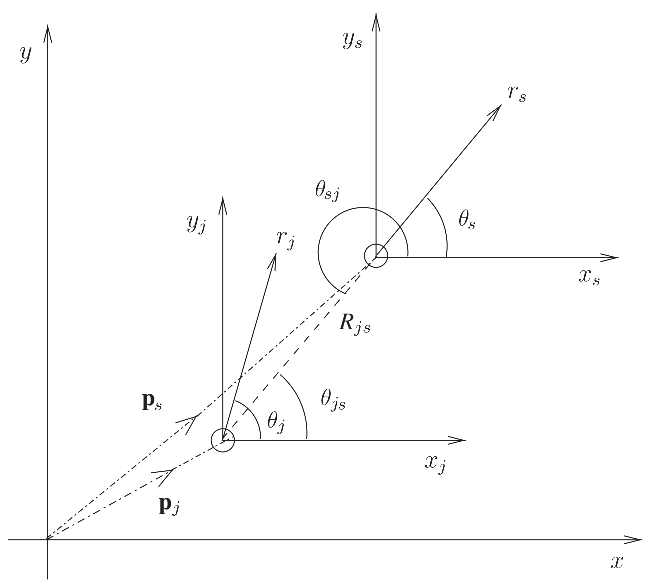

Fig. 2. Geometry of the location of inclusions and their local coordinate systems.

where with reference to Fig. 2, $R_{js} = |\mathbf{R}_{js}| = |\mathbf{p}_j - \mathbf{p}_s|$ and θ_{js} is the angle subtended between the x_j axis and the sth inclusion (note that $\theta_{sj} = \pi + \theta_{js}$). Rewrite Eq. (24) as

$$w(r_s,\theta_s) = \sum_{m=-\infty}^{\infty} \{\mathcal{A}_m^s J_m(kr_s) + \mathcal{B}_m^s I_m(kr_s) + A_m^s Z_m H_m(kr_s) + B_m^s W_m K_m(kr_s)\} \mathrm{e}^{\mathrm{i}m\theta_s}$$

for $r_s \geq a$, where (compare with Eq. (8), in which the roles of $\mathcal{A}_m^J$ and $\mathcal{A}_m^I$ are played here by $\mathcal{A}_m^s$ and $\mathcal{B}_m^s$, respectively)

$$\mathcal{A}_m^s = \mathrm{I}_s \mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}} + \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} A_n^j Z_n H_{n-m}(kR_{js}) \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}}, \tag{25}$$

$$\mathcal{B}_m^s = \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} B_n^j W_n K_{n-m}(kR_{js}) \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}}. \tag{26}$$

Comparison with the solution for scattering by one inclusion, Eq. (18), gives

$$A_m^s Z_m = T_{11}^m \mathcal{A}_m^s + T_{12}^m \mathcal{B}_m^s, \quad B_m^s W_m = T_{21}^m \mathcal{A}_m^s + T_{22}^m \mathcal{B}_m^s. \tag{27}$$

To simplify these equations, we choose

$$Z_n = -T_{11}^n \quad \text{and} \quad W_n = -T_{21}^n. \tag{28}$$

Furthermore, we define

$$Q_n = T_{12}^n / T_{11}^n \quad \text{and} \quad P_n = T_{22}^n / T_{21}^n, \tag{29}$$

so that Eq. (27) gives $A_m^s + \mathcal{A}_m^s + Q_m \mathcal{B}_m^s = 0$ and $B_m^s + \mathcal{A}_m^s + P_m \mathcal{B}_m^s = 0$. Writing these out explicitly, using Eqs. (25) and (26), we obtain

$$A_m^s + \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} \left\{ A_n^j Z_n H_{n-m}(kR_{js}) + Q_m B_n^j W_n K_{n-m}(kR_{js}) \right\} \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}} = -\mathrm{I}_s \mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}, \tag{30}$$

$$B_m^s + \sum_{\substack{j=1 \\ j\neq s}}^{N} \sum_{n=-\infty}^{\infty} \left\{ A_n^j Z_n H_{n-m}(kR_{js}) + P_m B_n^j W_n K_{n-m}(kR_{js}) \right\} \mathrm{e}^{\mathrm{i}(n-m)\theta_{js}} = -\mathrm{I}_s \mathrm{i}^m \mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}. \tag{31}$$

These equations hold for $s = 1, 2, \ldots, N$ and for all integers m.

3.2. Averaging: random distribution of inclusions in a half-space

Let us introduce a probability density function, defining the statistics of the random distribution of inclusions [20,26]. This is written as

$$p(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N) = p(\mathbf{p}_s) p(\mathbf{p}_1, \mathbf{p}_2, \ldots, \cancel{\mathbf{p}}_s \ldots, \mathbf{p}_N \,|\, \mathbf{p}_s), \tag{32}$$

where $\cancel{\mathbf{p}}_s$ indicates that $\mathbf{p}_s$ is not present. Similarly,

$$p(\mathbf{p}_1, \mathbf{p}_2, \ldots, \cancel{\mathbf{p}}_s, \ldots, \mathbf{p}_N | \mathbf{p}_s) = p(\mathbf{p}_j | \mathbf{p}_s) p(\mathbf{p}_1, \ldots, \cancel{\mathbf{p}}_j, \ldots, \cancel{\mathbf{p}}_s, \ldots, \mathbf{p}_N | \mathbf{p}_j, \mathbf{p}_s),$$

where $p(\mathbf{p}_j \,|\, \mathbf{p}_s)$ denotes the probability density of finding an inclusion at $\mathbf{p}_j \in \mathcal{D}$ given that there is already one located at $\mathbf{p}_s \in \mathcal{D}$. We define the following ensemble average of a function $f(x, y; \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)$,

$$\langle f \rangle_s = \int_{\mathrm{D}} \int_{\mathrm{D}} \cdots \int_{\mathrm{D}} p(\mathbf{p}_1, \mathbf{p}_2, \ldots, \cancel{\mathbf{p}}_s, \ldots, \mathbf{p}_N \,|\, \mathbf{p}_s) f \, \mathrm{d}\mathbf{p}_1 \ldots \mathrm{d}\cancel{\mathbf{p}}_s \ldots \mathrm{d}\mathbf{p}_N, \tag{33}$$

which is the average with the sth inclusion fixed [16,26]. Similarly, we can define $\langle f \rangle_{js}$, the average taken with both the sth and jth inclusions fixed.

We are going to take the ensemble average of Eqs. (30) and (31) with the sth inclusion fixed; we start with Eq. (30). Noting the indistinguishability of inclusions in the j sum after taking this average, we find that

$$\begin{aligned}\langle A_m^s\rangle_s + (N-1)\sum_{n=-\infty}^{\infty}\int_D p(\mathbf{p}_j\,|\,\mathbf{p}_s)\left\langle A_n^j\right\rangle_{sj} Z_n H_{n-m}(kR_{js})\mathrm{e}^{\mathrm{i}(n-m)\theta_{js}}d\mathbf{p}_j \\ + Q_m(N-1)\sum_{n=-\infty}^{\infty}\int_D p(\mathbf{p}_j\,|\,\mathbf{p}_s)\left\langle B_n^j\right\rangle_{sj} W_n K_{n-m}(kR_{js})\mathrm{e}^{\mathrm{i}(n-m)\theta_{js}}d\mathbf{p}_j = -\mathrm{I}_s\mathrm{i}^m\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}},\end{aligned} \tag{34}$$

where $\mathcal{D}$ is the region in which inclusions reside. Next we invoke the Lax quasi-crystalline closure approximation (QCA) [12], which amounts to taking

$$\left\langle A_n^j\right\rangle_{sj} = \left\langle A_n^j\right\rangle_j \quad \text{and} \quad \left\langle B_n^j\right\rangle_{sj} = \left\langle B_n^j\right\rangle_j,$$

and for notational convenience we set $s=1$ and $j=2$. We also make a simple choice for $p(\mathbf{p}_2\,|\,\mathbf{p}_1)$: we choose the *hole-correction* pair-correlation function [9],

$$p(\mathbf{p}_2\,|\,\mathbf{p}_1) = \begin{cases} |\mathcal{D}|^{-1} = n_0/N, & R_{21}\geq b, \\ 0, & R_{21}<b,\end{cases}$$

where b is the radius of the so-called *exclusion zone*, with $b=2a$ classically. Finally, we let $N\to\infty$ and we let $\mathcal{D}$ expand to fill a half-space $\mathcal{H}$, but we keep n_0, the number of inclusions per unit area, fixed. The result is

$$\left\langle A_m^1\right\rangle_1 + n_0\sum_{n=-\infty}^{\infty}\oint_{\mathcal{H}}\left\{\left\langle A_n^2\right\rangle_2 Z_n H_{n-m}(kR_{21}) + \left\langle B_n^2\right\rangle_2 W_n Q_m K_{n-m}(kR_{21})\right\}\mathrm{e}^{\mathrm{i}(n-m)\theta_{21}}dp_2 dq_2 = -\mathrm{I}_1\mathrm{i}^m\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}, \tag{35}$$

where $\mathcal{H}$ is the half-space $p_2\geq 0$ and the circle on the integral sign indicates that the hole centred at $\mathbf{p}_2 = \mathbf{p}_1$, of radius b, has to be cut out of the half-space. A similar calculation, starting from Eq. (31), leads to

$$\left\langle B_m^1\right\rangle_1 + n_0\sum_{n=-\infty}^{\infty}\oint_{\mathcal{H}}\left\{\left\langle A_n^2\right\rangle_2 Z_n H_{n-m}(kR_{21}) + \left\langle B_n^2\right\rangle_2 W_n P_m K_{n-m}(kR_{21})\right\}\mathrm{e}^{\mathrm{i}(n-m)\theta_{21}}dp_2 dq_2 = -\mathrm{I}_1\mathrm{i}^m\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}. \tag{36}$$

Eqs. (35) and (36) provide a system of integral equations which we must solve in order to derive information about the effective wave which propagates through the random medium. At the end of the calculations to follow, we shall let $kb\to 0$, leading to Eq. (2). As $\mathrm{I}_1=\mathrm{e}^{\mathrm{i}(\alpha p_1+\beta q_1)}$, we try

$$\langle A_m^s\rangle_s = \mathrm{i}^m\mathrm{e}^{\mathrm{i}\beta q_s}X_m(p_s), \qquad \langle B_m^s\rangle_s = \mathrm{i}^m\mathrm{e}^{\mathrm{i}\beta q_s}Y_m(p_s), \tag{37}$$

where $\beta=k\sin\theta_{\mathrm{inc}}$. Eq. (35) reduces to

$$X_m(p_1) + n_0\sum_{n=-\infty}^{\infty}(-\mathrm{i})^{n-m}\oint_{\mathcal{H}}\mathrm{e}^{\mathrm{i}\beta q_{21}}\{X_n(p_2)Z_n\psi_{n-m}(p_{21},q_{21}) + Y_n(p_2)W_nQ_m\chi_{n-m}(p_{21},q_{21})\}dp_2dq_2 = -\mathrm{e}^{\mathrm{i}\alpha p_1}\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}, \tag{38}$$

where $\alpha=k\cos\theta_{\mathrm{inc}}$, we have used $\theta_{21}=\theta_{12}-\pi$, we have written $p_{21}=p_2-p_1$ and $q_{21}=q_2-q_1$, and we have defined ψ_n and χ_n as follows: with $X=R\cos\Theta$ and $Y=R\sin\Theta$,

$$\psi_n(X,Y) = H_n(kR)\mathrm{e}^{\mathrm{i}n\Theta} \quad \text{and} \quad \chi_n(X,Y) = K_n(kR)\mathrm{e}^{\mathrm{i}n\Theta}.$$

Next, following [16], we suppose that for sufficiently large p_1 ($p_1>\ell$, say) we can write

$$X_m(p) = F_m\mathrm{e}^{-\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}\lambda p} \quad \text{and} \quad Y_m(p) = G_m\mathrm{e}^{-\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}\lambda p},$$

with coefficients F_m and G_m, where

$$\lambda = K\cos\varphi \quad \text{and} \quad \beta = K\sin\varphi = k\sin\theta_{\mathrm{inc}}.$$

(Note that $\lambda^2-\alpha^2=K^2-k^2$.) From the form of Eq. (37), we therefore identify K as an *effective wavenumber* and we now show how it may be derived.

If $p_1>\ell+b$, Eq. (38) becomes

$$F_m\mathrm{e}^{-\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}\lambda p_1} + n_0\sum_{n=-\infty}^{\infty}(-\mathrm{i})^{n-m}\int_0^{\ell}\mathcal{K}_{mn}(p_1,p_2)dp_2 + n_0\mathrm{e}^{\mathrm{i}\lambda p_1}\sum_{n=-\infty}^{\infty}(-\mathrm{i})^{n-m}\mathrm{e}^{-\mathrm{i}n\varphi}\widehat{M}_{mn} = -\mathrm{e}^{\mathrm{i}\alpha p_1}\mathrm{e}^{-\mathrm{i}m\theta_{\mathrm{inc}}}, \tag{39}$$

where $\widehat{M}_{mn} = F_nZ_nM_{n-m} + G_nW_nQ_m\tilde{M}_{n-m}$ and

$$\mathcal{K}_{mn}(p_1,p_2) = X_n(p_2)Z_nL_{n-m}(p_{21}) + Y_n(p_2)W_nQ_m\tilde{L}_{n-m}(p_{21}).$$

Here we have defined the integrals

$$L_n(X) = \int_{-\infty}^{\infty}\psi_n(X,Y)\mathrm{e}^{\mathrm{i}\beta Y}dY, \quad \tilde{L}_n(X) = \int_{-\infty}^{\infty}\chi_n(X,Y)\mathrm{e}^{\mathrm{i}\beta Y}dY, \tag{40}$$

and

$$M_n = \oint\Psi(p_{21},q_{21})\psi_n(p_{21},q_{21})dp_2dq_2, \tag{41}$$

$$\tilde{M}_n = \oint\Psi(p_{21},q_{21})\chi_n(p_{21},q_{21})dp_2dq_2, \tag{42}$$

where $\Psi(X,Y) = \mathrm{e}^{\mathrm{i}(\lambda X+\beta Y)}$. The integrals in Eqs. (41) and (42) are over the half-plane $p_2 > \ell$ with the disc $R_{21} < b$ removed.

The integrals L_n and M_n were evaluated in [16]. The integrals $\tilde{L}_n$ and $\tilde{M}_n$ can be evaluated similarly. From ([16], Eq. (66)), we have

$$L_n(p_2-p_1) = (2/\alpha)\mathrm{i}^n\mathrm{e}^{\mathrm{i}n\theta_{\mathrm{inc}}}\mathrm{e}^{\mathrm{i}\alpha(p_1-p_2)}, \quad p_1 > p_2,$$

hence

$$n_0\sum_{n=-\infty}^{\infty}(-\mathrm{i})^{n-m}\int_0^{\ell}X_n(p_2)Z_nL_{n-m}(p_{21})dp_2 = \mathcal{C}_m\mathrm{e}^{\mathrm{i}\alpha p_1},$$

where $\mathcal{C}_m$ is a constant.

Next, consider $\tilde{L}_n(X)$, an even function of X; we require $\tilde{L}_n(X)$ for $X<0$. We start with $\tilde{L}_0(x)$. We have $\tilde{L}_0(0) = 2\int_0^{\infty}K_0(ky)\cos\beta y\,dy = \pi/\gamma$ (using ([10], Eq. 6.671(14))) where

$$\gamma = \sqrt{k^2+\beta^2}.$$

As $(\nabla^2-k^2)K_0(kr) = 0$ with $r = (x^2+y^2)^{1/2}$, we have

$$\tilde{L}_0''(x) = k^2\tilde{L}_0(x)-\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\beta y}\frac{\partial^2}{\partial y^2}K_0(kr)dy = \gamma^2\tilde{L}_0(x),$$

after two integrations by parts. Then, as $\tilde{L}_0(x)\to 0$ when $|x|\to\infty$, we obtain $\tilde{L}_0(x) = C\mathrm{e}^{-\gamma|x|}$ for some constant C. Hence, using $\tilde{L}_0(0) = \pi/\gamma$,

$$\tilde{L}_0(X) = (\pi/\gamma)\mathrm{e}^{\gamma X} \quad \text{and} \quad \tilde{L}_0'(X) = \gamma\tilde{L}_0(X) \text{ for } X<0.$$

For $\tilde{L}_n$ with $n>0$, we use

$$\tilde{L}_n(X) = \int_{-\infty}^{\infty}-\frac{1}{k}\mathrm{e}^{\mathrm{i}\beta Y}\left(\frac{\partial}{\partial X}+\mathrm{i}\frac{\partial}{\partial Y}\right)\chi_{n-1}(X,Y)dY.$$

Then, as in [16], we obtain $k\tilde{L}_n = -\tilde{L}_{n-1}'-\beta\tilde{L}_{n-1}$, whence

$$\tilde{L}_n(X) = (\pi/\gamma)(-1)^n(\beta+\gamma)^nk^{-n}\mathrm{e}^{\gamma X}.$$

Similarly, for $n<0$, we obtain $\tilde{L}_n(X) = (\pi/\gamma)(\beta-\gamma)^{-n}k^n\mathrm{e}^{\gamma X}$. Hence,

$$n_0\sum_{n=-\infty}^{\infty}(-\mathrm{i})^{n-m}\int_0^{\ell}Y_n(p_2)W_nQ_m\tilde{L}_{n-m}(p_{21})dp_2 = \tilde{\mathcal{C}}_m\mathrm{e}^{-\gamma p_1}$$

where $\tilde{\mathcal{C}}_m$ is a constant.

The double integrals M_n and $\tilde{M}_n$ are evaluated using Green's theorem. We have

$$M_n = M_n^{\ell}+M_n^b \quad \text{and} \quad \tilde{M}_n = \tilde{M}_n^{\ell}+\tilde{M}_n^b,$$

where M_n^{ℓ} and $\tilde{M}_n^{\ell}$ come from integrations along the line $p_2 = \ell$, and M_n^b and $\tilde{M}_n^b$ come from integrations around the circle $R_{21} = b$. From ([16], Eq. (67)), we have

$$M_n^{\ell} = \mathcal{C}_n^{\ell}\mathrm{e}^{\mathrm{i}(\alpha-\lambda)p_1} \text{ and } M_n^b = 2\pi\mathrm{i}^n(k^2-K^2)^{-1}\mathrm{e}^{\mathrm{i}n\varphi}\mathcal{N}_n(Kb),$$

where $\mathcal{C}_n^{\ell}$ is a constant and $\mathcal{N}_n(Kb) = kbH_n'(kb)J_n(Kb)-KbH_n(kb)J_n'(Kb)$.

For $\tilde{M}_n$, we note that $\Psi\nabla^2\chi_n - \chi_n\nabla^2\Psi = (k^2+K^2)\Psi\chi_n$ so that

$$(k^2+K^2)\tilde{M}_m = \oint\left(\Psi\nabla^2\chi_m - \chi_m\nabla^2\Psi\right)dp_2 dq_2 = \int_S\left(\Psi\frac{\partial\chi_m}{\partial n} - \chi_m\frac{\partial\Psi}{\partial n}\right)dS$$

where S consists of the line $p_2 = \ell$ (this gives $\tilde{M}_m^{\ell}$) and the circle $R_{21} = b$ centred at $\mathbf{p}_2 = \mathbf{p}_1$ (this gives $\tilde{M}_m^{b}$), and the normal vector on S points out of the half-space domain. On $p_2 = \ell$, $\partial/\partial n = -\partial/\partial p_2$ and so we have

$$\begin{aligned}
&-\int_{p_2=\ell}\left(\Psi\frac{\partial\chi_n}{\partial p_2} - \chi_n\frac{\partial\Psi}{\partial p_2}\right)dq_2\\
&= -e^{i\lambda(\ell-p_1)}\int_{-\infty}^{\infty} e^{i\beta q_{21}}\left[\cos\alpha_{12}\frac{\partial\chi_n}{\partial R_{12}} - \frac{\sin\alpha_{12}}{R_{12}}\frac{\partial\chi_n}{\partial\alpha_{12}} - i\lambda\chi_n\right]_{p_2=\ell} dq_2\\
&= -e^{i\lambda(\ell-p_1)}\int_{-\infty}^{\infty} e^{i\beta q_{21}}\left[-\frac{k}{2}(\chi_{n-1}+\chi_{n+1}) - i\lambda\chi_n\right]_{p_2=\ell} dq_2\\
&= e^{i\lambda(\ell-p_1)}\left[\frac{k}{2}\left(\tilde{L}_{n-1}(\ell-p_1) + \tilde{L}_{n+1}(\ell-p_1)\right) + i\lambda\tilde{L}_n(\ell-p_1)\right],
\end{aligned}$$

using $-2K_n'(x) = K_{n-1}(x) + K_{n+1}(x)$, $-(2n/x)K_n(x) = K_{n-1}(x) - K_{n+1}(x)$ and Eq. (40). Hence,

$$\tilde{M}_n^{\ell} = \tilde{\mathcal{C}}_n^{\ell} e^{-(i\lambda+\gamma)p_1} \quad \text{for some constant } \tilde{\mathcal{C}}_n^{\ell}.$$

The contribution from the circle $R_{21} = b$ is

$$\begin{aligned}
&-\int_0^{2\pi}\left[e^{iKR\cos(\Theta-\varphi)}\frac{\partial\chi_n}{\partial R} - \chi_n\frac{\partial}{\partial R}e^{iKR\cos(\Theta-\varphi)}\right]_{R=b} b\,d\Theta\\
&= b\int_0^{2\pi} e^{iKb\cos(\Theta-\varphi)}e^{in\Theta}[iKK_n(kb)\cos(\Theta-\varphi) - kK_n'(kb)]d\Theta\\
&= -2\pi i^n e^{in\varphi}\tilde{\mathcal{N}}_n(Kb)
\end{aligned}$$

where $\tilde{\mathcal{N}}_n(Kb) = kbK_n'(kb)J_n(Kb) - KbK_n(kb)J_n'(Kb)$. Hence

$$\tilde{M}_n^{b} = -2\pi i^n(k^2+K^2)^{-1}e^{in\varphi}\tilde{\mathcal{N}}_n(Kb).$$

Next, we substitute our results in Eq. (39), giving

$$\begin{aligned}
&-e^{i\alpha p_1}e^{-im\theta_{\mathrm{inc}}} = F_m e^{-im\varphi}e^{i\lambda p_1} + \mathcal{C}_m e^{i\alpha p_1} + \tilde{\mathcal{C}}_m e^{-\gamma p_1}\\
&+ n_0 e^{i\lambda p_1}\sum_{n=-\infty}^{\infty}(-i)^{n-m}e^{-in\varphi}\left[F_n Z_n \mathcal{B}_{mn}(p_1) + G_n W_n Q_m \tilde{\mathcal{B}}_{mn}(p_1)\right],
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
\mathcal{B}_{mn}(p_1) &= \mathcal{C}_{n-m}^{\ell}e^{i(\alpha-\lambda)p_1} + 2\pi i^{n-m}(k^2-K^2)^{-1}e^{i(n-m)\varphi}\mathcal{N}_{n-m}(Kb)\\
\tilde{\mathcal{B}}_{mn}(p_1) &= \tilde{\mathcal{C}}_{n-m}^{\ell}e^{-(i\lambda+\gamma)p_1} - 2\pi i^{n-m}(k^2+K^2)^{-1}e^{i(n-m)\varphi}\tilde{\mathcal{N}}_{n-m}(Kb).
\end{aligned}$$

Eq. (43) contains terms that are proportional to $e^{i\alpha p_1}$, terms that are proportional to $e^{i\lambda p_1}$ and terms that are proportional to $e^{-\gamma p_1}$: all should balance. Collecting the terms that are proportional to $e^{i\lambda p_1}$ gives

$$F_m + \frac{2\pi n_0}{k^2-K^2}\sum_{n=-\infty}^{\infty}F_n Z_n\mathcal{N}_{n-m}(Kb) - \frac{2\pi n_0 Q_m}{k^2+K^2}\sum_{n=-\infty}^{\infty}G_n W_n\tilde{\mathcal{N}}_{n-m}(Kb) = 0. \tag{44}$$

(If the second sum is omitted, Eq. (44) reduces to ([16], Eq. (71)), as derived for an analogous acoustic problem.) A similar calculation, starting with Eq. (36), leads to

$$G_m + \frac{2\pi n_0}{k^2-K^2}\sum_{n=-\infty}^{\infty}F_n Z_n\mathcal{N}_{n-m}(Kb) - \frac{2\pi n_0 P_m}{k^2+K^2}\sum_{n=-\infty}^{\infty}G_n W_n\tilde{\mathcal{N}}_{n-m}(Kb) = 0. \tag{45}$$

Eqs. (44) and (45) hold for all integers m. They provide an infinite homogeneous system of linear algebraic equations for F_m and G_m. The existence of a non-trivial solution to this system will determine the effective wavenumber K.

It is worth noting that Eqs. (44) and (45) do not depend on ℓ or θ_{inc}. This may be regarded as a check: we would not expect the effective wavenumber to depend on the angle of incidence or on distance from the boundary. Indeed, Eqs. (44) and (45) could have been derived by assuming that $\ell = 0$ and $\theta_{\mathrm{inc}} = 0$ from the outset. Further information can be obtained by balancing the other exponential terms in Eq. (43), but this information will not be needed here.

3.3. Approximate determination of K for small ϕ

In [16], the expansion $K^2 = k^2 + \delta_1 n_0 + \delta_2 n_0^2 + \cdots$ was used, and then expressions for δ_1 and δ_2 were sought. Here, we prefer to work with dimensionless quantities, so we write

$$(Ka)^2 = (ka)^2 + \kappa_1\phi + \kappa_2\phi^2 + \cdots \tag{46}$$

where $\phi = \pi a^2 n_0$, $\kappa_1 = \delta_1/\pi$ and $\kappa_2 = \delta_2/(\pi a)^2$. Let $B = b/a$. From [16], we have

$$\mathcal{N}_n(Kb) = (2\mathrm{i}/\pi) + (\phi/2)B^2\kappa_1 d_n(kb) + O(\phi^2),$$

where $d_n(x) = J_n'(x)H_n'(x) + [1-(n/x)^2]J_n(x)H_n(x)$. Similarly,

$$\tilde{\mathcal{N}}_n(Kb) = w_n(kb) + (\phi/2)B^2\kappa_1\tilde{d}_n(kb) + O(\phi^2),$$

where $\tilde{d}_n(x) = J_n'(x)K_n'(x) + [1-(n/x)^2]J_n(x)K_n(x)$ and

$$w_n(x) = x[K_n'(x)J_n(x) - K_n(x)J_n'(x)]. \tag{47}$$

Hence

$$\frac{2\pi n_0}{k^2-K^2}\mathcal{N}_n(Kb) = -\frac{4\mathrm{i}}{\pi\kappa_1} + \phi\left[\frac{4\mathrm{i}\kappa_2}{\pi\kappa_1^2} - B^2 d_n(kb)\right] + O(\phi^2) \tag{48}$$

whereas

$$\frac{2\pi n_0}{k^2+K^2}\tilde{\mathcal{N}}_n(Kb) = \phi\frac{1}{(ka)^2}w_n(kb) + O(\phi^2). \tag{49}$$

Substituting Eqs. (48) and (49) in Eq. (44), neglecting the second-order terms, we obtain

$$F_m + \sum_{n=-\infty}^{\infty} F_n Z_n\left(\frac{4}{\mathrm{i}\pi\kappa_1} + \phi\left[\frac{4\mathrm{i}\kappa_2}{\pi\kappa_1^2} - B^2 d_{n-m}(kb)\right]\right) - \phi Q_m\sum_{n=-\infty}^{\infty} G_n W_n\frac{w_{n-m}(kb)}{(ka)^2} = 0. \tag{50}$$

Eq. (45) leads to a very similar equation, with exactly the same sums, but with the leading F_m replaced by G_m and with Q_m replaced by P_m.

At leading order in ϕ, Eq. (50) gives

$$F_m = \frac{4\mathrm{i}}{\pi\kappa_1}\sum_{n=-\infty}^{\infty} F_n Z_n, \tag{51}$$

so that all the F_m are equal. Then, if we write $F_m = F$, Eq. (51) gives

$$\kappa_1 = \frac{4\mathrm{i}}{\pi}\sum_{n=-\infty}^{\infty} Z_n = -\frac{4\mathrm{i}}{\pi}f(0), \tag{52}$$

where f is the far-field pattern; see Eqs. (20) and (28). This result is exactly the same as the Foldy approximation for acoustic problems; see ([16], Eq. (77)). In addition, from Eq. (45), we obtain $G_m = F$, to leading order.

At next order, we can write

$$F_m = F + \phi\tilde{F}_m \quad \text{and} \quad G_m = F + \phi\tilde{G}_m.$$

Then, the $O(\phi)$ terms in Eq. (50) are

$$\tilde{F}_m + \sum_{n=-\infty}^{\infty} Z_n\left(\frac{4\tilde{F}_n}{\mathrm{i}\pi\kappa_1} + F\left[\frac{4\mathrm{i}\kappa_2}{\pi\kappa_1^2} - B^2 d_{n-m}(kb)\right]\right) - FQ_m\sum_{n=-\infty}^{\infty} W_n\frac{w_{n-m}(kb)}{(ka)^2} = 0. \tag{53}$$

Put

$$S_m^Z = \sum_{n=-\infty}^{\infty} Z_n d_{n-m}(kb) \quad \text{and} \quad S_m^W = \sum_{n=-\infty}^{\infty} W_n w_{n-m}(kb).$$

Then, rewrite Eq. (53), using Eq. (52):

$$\tilde{F}_m - FB^2 S_m^Z - \frac{F}{(ka)^2} Q_m S_m^W = \frac{4\mathrm{i}}{\pi\kappa_1} \sum_{n=-\infty}^{\infty} Z_n \tilde{F}_n - \frac{F\kappa_2}{\kappa_1}. \tag{54}$$

The right-hand side of Eq. (54) does not depend on m; denote it by $\hat{F}$. Thus

$$\hat{F} = \frac{4\mathrm{i}}{\pi\kappa_1} \sum_{n=-\infty}^{\infty} Z_n \tilde{F}_n - \frac{F\kappa_2}{\kappa_1} = \frac{4\mathrm{i}}{\pi\kappa_1} \sum_{n=-\infty}^{\infty} Z_n \left[\hat{F} + FB^2 S_n^Z + \frac{F}{(ka)^2} Q_n S_n^W\right] - \frac{F\kappa_2}{\kappa_1}. \tag{55}$$

The terms in $\hat{F}$ cancel (using Eq. (52)), and then a factor of F/κ_1 cancels leaving

$$\begin{aligned}\kappa_2 &= \frac{4\mathrm{i}}{\pi} \sum_{n=-\infty}^{\infty} Z_n \left[B^2 S_n^Z + \frac{1}{(ka)^2} Q_n S_n^W\right] \\ &= \frac{4\mathrm{i}}{\pi} B^2 \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Z_s d_{s-n}(kb) + \frac{4\mathrm{i}}{\pi(ka)^2} \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Q_n W_s w_{s-n}(kb).\end{aligned} \tag{56}$$

If we perform a similar analysis starting from Eq. (45), we arrive at

$$\tilde{G}_m - FB^2 S_m^Z - F(ka)^{-2} P_m S_m^W = \hat{F}, \tag{57}$$

instead of Eq. (54). As $\hat{F}$ (defined by the right-hand side of Eq. (54)) does not involve $\tilde{G}_m$, we cannot back-substitute as we did with Eq. (55), so nothing is gained beyond a formula for $\tilde{G}_m$.

Returning to our formula (56) for the second-order correction, κ_2, we notice that the first double-sum is exactly the same as in the acoustic case; see ([16], Eq. (80)). It involves Z_n and so it may be expressed in terms of the far-field pattern for plane-wave incidence on one inclusion, $f(\theta)$.

The second double-sum in Eq. (56) involves both $Z_n Q_n = -T_{12}^n$ and $W_n = -T_{21}^n = -(2\mathrm{i}/\pi)T_{12}^n$ (see Eqs. (19), (28) and (29)). The quantity T_{12}^n also occurs in a far-field pattern *but not for plane-wave incidence*: it appears in the far-field pattern g, defined by Eq. (21), corresponding to the incident field given by Eq. (10).

Let us approximate κ_2 by approximating $d_n(kb)$ and $w_n(kb)$ for small kb. We have $x^2 d_n(x) \sim 2\mathrm{i}|n|/\pi$ as $x \to 0$, and this leads to an expression for the first double-sum in (56) as a certain integral of f; see ([16], p. 3420) and Eq. (58). For $w_n(kb)$, defined by Eq. (47), we find that $w_n(x) \sim -1$ as $x \to 0$ so that the second double-sum in Eq. (56) splits into the product of two single-sums:

$$\sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} Z_n Q_n W_s w_{s-n}(kb) \simeq -\frac{2\mathrm{i}}{\pi} [g(0)]^2.$$

Combining these results, we obtain the approximation

$$(Ka)^2 = (ka)^2 - \frac{4\mathrm{i}}{\pi} \phi f(0) + \frac{8\phi^2}{\pi^3 (ka)^2} \int_0^\pi \cot(\theta/2) \frac{d}{d\theta} [f(\theta)]^2 d\theta + \frac{8\phi^2}{\pi^2 (ka)^2} [g(0)]^2 + O(\phi^3). \tag{58}$$

Apart from the term involving the far-field pattern g(0), this formula is reminiscent of the well known Lloyd–Berry formula for acoustics and its two-dimensional form given in Eq. (1).

4. Low-frequency results for cavities

For scattering by a cavity, we need the lower left 2×2 block in Eq. (15); dropping the redundant superscript 0, we have

$$\begin{pmatrix} S_{H_n} & S_{K_n} \\ T_{H_n} & T_{K_n} \end{pmatrix} \begin{pmatrix} A_n^{\mathrm{cav}} \\ B_n^{\mathrm{cav}} \end{pmatrix} = -\begin{pmatrix} S_{J_n} & S_{I_n} \\ T_{J_n} & T_{I_n} \end{pmatrix} \begin{pmatrix} \mathcal{A}_n^J \\ \mathcal{A}_n^I \end{pmatrix}. \tag{59}$$

Hence,

$$\Delta_n A_n^{\mathrm{cav}} = \left[S_{K_n} T_{J_n} - S_{J_n} T_{K_n}\right] \mathcal{A}_n^J + \left[S_{K_n} T_{I_n} - S_{I_n} T_{K_n}\right] \mathcal{A}_n^I, \tag{60}$$

$$\Delta_n B_n^{\mathrm{cav}} = \left[S_{J_n} T_{H_n} - S_{H_n} T_{J_n}\right] \mathcal{A}_n^J + \left[S_{I_n} T_{H_n} - S_{H_n} T_{I_n}\right] \mathcal{A}_n^I, \tag{61}$$

where $\Delta_n = S_{H_n} T_{K_n} - S_{K_n} T_{H_n}$. These equations are exact.

For low frequencies ($ka \ll 1$), we obtain the following approximations:

$$A_0^{\text{cav}} \sim \frac{\text{i}\pi(ka)^2}{4(1-\nu)}\left(\nu\mathcal{A}_o^J - \mathcal{A}_o^I\right), \quad B_0^{\text{cav}} \sim \frac{(ka)^2}{2(1-\nu)}(\mathcal{A}_o^J - \nu\mathcal{A}_o^I),$$
$$A_{\pm1}^{\text{cav}} \sim \frac{\text{i}\pi(ka)^4}{32(1-\nu)}\left((1+\nu)\mathcal{A}_{\pm1}^J - 2\mathcal{A}_{\pm1}^I\right),$$
$$B_{\pm1}^{\text{cav}} \sim \frac{(ka)^4}{16(1-\nu)}(2\mathcal{A}_{\pm1}^J - (1+\nu)\mathcal{A}_{\pm1}^I),$$
$$A_{\pm n}^{\text{cav}} \sim \frac{\text{i}\pi(1-\nu)(ka)^{2n-2}}{2^{2n-1}(3+\nu)(n-1)!(n-2)!}(\mathcal{A}_{\pm n}^J + \mathcal{A}_{\pm n}^I), \quad n \geq 2,$$
$$B_{\pm n}^{\text{cav}} \sim -\frac{(1-\nu)(ka)^{2n-2}}{2^{2n-2}(3+\nu)(n-1)!(n-2)!}(\mathcal{A}_{\pm n}^J + \mathcal{A}_{\pm n}^I), \quad n \geq 2.$$

In particular,

$$A_{\pm2}^{\text{cav}} \sim \frac{\text{i}\pi(1-\nu)}{8(3+\nu)}(ka)^2(A_{\pm2}^J + A_{\pm2}^I),$$
$$B_{\pm2}^{\text{cav}} \sim -\frac{(1-\nu)}{4(3+\nu)}(ka)^2(A_{\pm2}^J + A_{\pm2}^I).$$

We note that A_0^{cav}, B_0^{cav}, $A_{\pm2}^{\text{cav}}$ and $B_{\pm2}^{\text{cav}}$ occur at leading order, $O((ka)^2)$, all other contributions being smaller. (Compare with acoustic/SH wave scattering from a cavity, where the monopole ($n=0$) and dipole ($n=1$) modes give the leading order contributions.) Norris and Vemula [22] also found that A_0^{cav} and $A_{\pm2}^{\text{cav}}$ gave the main contribution (for plane-wave incidence). However, although we agree with their expression for A_0^{cav}, we disagree with their expressions for $A_{\pm1}^{\text{cav}}$ and $A_{\pm2}^{\text{cav}}$; we computed these formulae both by hand and with a symbol manipulation package.

Comparison with Eq. (18), gives

$$T_{11}^0 = \frac{\text{i}\pi\nu(ka)^2}{4(1-\nu)}, \quad T_{12}^0 = \frac{-\text{i}\pi(ka)^2}{4(1-\nu)}, \quad T_{21}^0 = \frac{(ka)^2}{2(1-\nu)}, \quad T_{22}^0 = \frac{-\nu(ka)^2}{2(1-\nu)},$$
$$T_{11}^{\pm2} = T_{12}^{\pm2} = \frac{\text{i}\pi(1-\nu)}{8(3+\nu)}(ka)^2, \quad T_{21}^{\pm2} = T_{22}^{\pm2} = -\frac{(1-\nu)}{4(3+\nu)}(ka)^2.$$

These satisfy the reciprocity relation, Eq. (19). Then, to leading order, we find that the far-field patterns are

$$f(\theta) = \frac{\text{i}\pi}{4}(ka)^2\left(\frac{\nu}{1-\nu} + \frac{1-\nu}{3+\nu}\cos 2\theta\right), \tag{62}$$

$$g(\theta) = \frac{\text{i}\pi}{4}(ka)^2\left(-\frac{1}{1-\nu} + \frac{1-\nu}{3+\nu}\cos 2\theta\right). \tag{63}$$

These approximations can be used in Eq. (58) to estimate the effective wavenumber, K. We have

$$f(0) = \text{i}\pi(ka)^2\frac{1+\nu+2\nu^2}{4(1-\nu)(3+\nu)}, \quad g(0) = -\text{i}\pi(ka)^2\frac{2+3\nu-\nu^2}{4(1-\nu)(3+\nu)}. \tag{64}$$

The first of these gives the dilute (Foldy) approximation (linear in ϕ)

$$\frac{K^2}{k^2} \simeq 1 + \frac{\phi(1+\nu+2\nu^2)}{(1-\nu)(3+\nu)}.$$

At second order in ϕ, we must evaluate the integral in Eq. (58); using

$$\int_0^\pi \cot(\theta/2)\sin 2\theta\, d\theta = \pi, \quad \int_0^\pi \cot(\theta/2)\sin 2\theta \cos 2\theta\, d\theta = \frac{\pi}{2}$$

and the approximation for $f(\theta)$, Eq. (62), we find that

$$\frac{8}{\pi^3(ka)^2}\int_0^\pi \cot(\theta/2)\frac{d}{d\theta}[f(\theta)]^2 d\theta \simeq (ka)^2\frac{(1+\nu)(1+3\nu)}{(3+\nu)^2}.$$

When this is substituted in Eq. (58), together with the expression for g(0), Eq. (64), we obtain an estimate for the coefficient κ_2 in Eq. (46):

$$\frac{\kappa_2}{(ka)^2} = \frac{(1+\nu)(1+3\nu)}{(3+\nu)^2} - \frac{(2+3\nu-\nu^2)^2}{2(1-\nu)^2(3+\nu)^2}$$

The effective density, ρ_*, and the effective bending stiffness, D_*, are related to the effective wavenumber K by

$$\frac{K^4}{k^4} = \frac{\rho_*}{\rho}\frac{D}{D_*} \simeq 1 + \frac{2\kappa_1\phi}{(ka)^2} + \left[\frac{\kappa_1^2}{(ka)^4} + \frac{2\kappa_2}{(ka)^2}\right]\phi^2 \tag{65}$$

$$= 1 + \frac{2(1+\nu+2\nu^2)}{(1-\nu)(3+\nu)}\phi - \frac{(1+\nu)(1+3\nu)^2}{(1-\nu)(3+\nu)^2}\phi^2.$$

Finally, as it is well known that $\rho_*/\rho = 1-\phi$ in the quasistatic limit, we can estimate the effective bending stiffness, D_*/D.

5. Conclusions

In this paper we have studied the multiple scattering of flexural waves by inclusions in thin plates. Reciprocity relations for a single inclusion have been established, connecting coefficients in circular multipole expansions. These relations are applicable regardless of the shape or composition of the inclusion. Using one of these relations and an approach analogous to that used by Linton and Martin [16] we derived an expression for the square of the effective wavenumber in the inhomogeneous plate region; this expression is of a similar form to the Lloyd–Berry expression associated with acoustics [16]. In the present case however, the expression involves *two* far-field scattering patterns, $f(\theta)$ and $g(\theta)$, associated with scattering from a single inclusion. The pattern $f(\theta)$ is associated with an incident plane wave e^{ikx}: it is the pattern which usually arises in Lloyd–Berry expressions. However, the pattern $g(\theta)$ is of no physical interest in itself, being associated with an incident wave of the form e^{-ky}.

In order to show the applicability of the theory, we considered the low-frequency limit of the effective wavenumber for an inhomogeneous plate consisting of circular cylindrical cavities randomly distributed inside a homogeneous plate phase. It was shown how estimates for the effective density and effective bending stiffness can be derived.

Such theories can be useful in various contexts, for example the non-destructive evaluation of composite plates and the measurement of sea-ice thickness.

Appendix A. Proof of reciprocity relations

Consider scattering by a single scatterer S with (smooth) boundary Γ; S could be an inclusion, a cavity, or a fixed rigid object. In this appendix, it is *not* assumed that S is circular. Surround S by a circle C.

Outside S, w satisfies $(\nabla^4 - k^4)w = 0$ (in the absence of body forces). Take a second flexural field, $\hat{w}$. Reciprocity gives (see, for example, ([11], p. 258) or ([2], p. 235))

$$\int_C R(w,\hat{w})ds = \int_\Gamma R(w,\hat{w})ds,$$

where

$$R(w,\hat{w}) = wV(\hat{w}) - \hat{w}V(w) - M(\hat{w})\frac{\partial w}{\partial n} + M(w)\frac{\partial \hat{w}}{\partial n},$$

$M(w)$ is the bending moment and $V(w)$ is the Kirchhoff shear. We suppose that w and $\hat{w}$ satisfy the same boundary conditions on Γ, and that these ensure that $\int_\Gamma R(w,\hat{w})ds = 0$. (If Γ has corners, additional corner conditions must be imposed.) Hence, if C has radius r,

$$\int_0^{2\pi} R(w,\hat{w})d\theta = 0 \quad \text{for any (sufficiently large) } r. \tag{A.1}$$

We use this relation to prove the reciprocity relation, Eq. (19).

At C, the field $w(r,\theta)$ has the expansion

$$w(r,\theta) = \sum_{n=-\infty}^{\infty}\{A_nH_n(kr) + B_nK_n(kr) + \mathcal{A}_n^J J_n(kr) + \mathcal{A}_n^I I_n(kr)\}e^{in\theta}.$$

For the bending moment, defined by Eq. (13), we have

$$M(w) = -\frac{D}{r^2}\sum_{n=-\infty}^{\infty}\{A_nS_{H_n} + B_nS_{K_n} + \mathcal{A}_n^J S_{J_n} + \mathcal{A}_n^I S_{I_n}\}e^{in\theta.}$$

and for the Kirchhoff shear, defined by Eq. (14), we have

$$V(w) = -\frac{D}{r^3}\sum_{n=-\infty}^{\infty}\{A_nT_{H_n} + B_nT_{K_n} + \mathcal{A}_n^J T_{J_n} + \mathcal{A}_n^I T_{I_n}\}e^{in\theta}.$$

In these formulas, S_X and T_X are defined by Eqs. (16) and (17), respectively, with $\nu_i=\nu$ and $\epsilon_i=kr$ therein. For $\hat{w}$, we choose similar expansions with carets on the various coefficients and with $e^{-in\theta}$ instead of $e^{in\theta}$. Then

$$\int_0^{2\pi}\{wV(\hat{w})-\hat{w}V(w)\}d\theta=-\frac{2\pi D}{r^3}\sum_{n=-\infty}^{\infty}P_n(kr) \tag{A.2}$$

where

$$\begin{aligned}P_n &= [A_nH_n+B_nK_n+\mathcal{A}_n^JJ_n+\mathcal{A}_n^II_n][\hat{A}_nT_{H_n}+\hat{B}_nT_{K_n}+\hat{\mathcal{A}}_n^JT_{J_n}+\hat{\mathcal{A}}_n^IT_{I_n}]\\ &\quad-[\hat{A}_nH_n+\hat{B}_nK_n+\hat{\mathcal{A}}_n^JJ_n+\hat{\mathcal{A}}_n^II_n][A_nT_{H_n}+B_nT_{K_n}+\mathcal{A}_n^JT_{J_n}+\mathcal{A}_n^IT_{I_n}]\\ &=[A_n\hat{B}_n-\hat{A}_nB_n][H_nT_{K_n}-K_nT_{H_n}]+[A_n\hat{\mathcal{A}}_n^J-\hat{A}_n\mathcal{A}_n^J][H_nT_{J_n}-J_nT_{H_n}]\\ &\quad+[A_n\hat{\mathcal{A}}_n^I-\hat{A}_n\mathcal{A}_n^I][H_nT_{I_n}-I_nT_{H_n}]+[B_n\hat{\mathcal{A}}_n^J-\hat{B}_n\mathcal{A}_n^J][K_nT_{J_n}-J_nT_{K_n}]\\ &\quad+[B_n\hat{\mathcal{A}}_n^I-\hat{B}_n\mathcal{A}_n^I][K_nT_{I_n}-I_nT_{K_n}]+[\mathcal{A}_n^J\hat{\mathcal{A}}_n^I-\hat{\mathcal{A}}_n^J\mathcal{A}_n^I][J_nT_{I_n}-I_nT_{J_n}].\end{aligned}$$

Similarly

$$\int_0^{2\pi}\left\{\frac{\partial w}{\partial r}M(\hat{w})-\frac{\partial\hat{w}}{\partial r}M(w)\right\}d\theta=-\frac{2\pi D}{r^3}\sum_{n=-\infty}^{\infty}(kr)Q_n(kr) \tag{A.3}$$

where

$$\begin{aligned}Q_n &= [A_n\hat{B}_n-\hat{A}_nB_n][H_n'S_{K_n}-K_n'S_{H_n}]+[A_n\hat{\mathcal{A}}_n^J-\hat{A}_n\mathcal{A}_n^J][H_n'S_{J_n}-J_n'S_{H_n}]\\ &\quad+[A_n\hat{\mathcal{A}}_n^I-\hat{A}_n\mathcal{A}_n^I][H_n'S_{I_n}-I_n'S_{H_n}]+[B_n\hat{\mathcal{A}}_n^J-\hat{B}_n\mathcal{A}_n^J][K_n'S_{J_n}-J_n'S_{K_n}]\\ &\quad+[B_n\hat{\mathcal{A}}_n^I-\hat{B}_n\mathcal{A}_n^I][K_n'S_{I_n}-I_n'S_{K_n}]+[\mathcal{A}_n^J\hat{\mathcal{A}}_n^I-\hat{\mathcal{A}}_n^J\mathcal{A}_n^I][J_n'S_{I_n}-I_n'S_{J_n}].\end{aligned}$$

To simplify notation, put $x=kr$, $\alpha=n^2(1-\nu)$, $\beta=\alpha+x^2$ and $\gamma=\alpha-x^2$. Let $\mathcal{J}_n$ denote J_n or H_n, and let $\mathcal{I}_n$ denote I_n or K_n. Then

$$\begin{aligned}S_{\mathcal{J}_n}&=\gamma\mathcal{J}_n-(1-\nu)x\mathcal{J}_n', & S_{\mathcal{I}_n}&=\beta\mathcal{I}_n-(1-\nu)x\mathcal{I}_n',\\ T_{\mathcal{J}_n}&=\alpha\mathcal{J}_n-\beta x\mathcal{J}_n', & T_{\mathcal{I}_n}&=\alpha\mathcal{I}_n-\gamma x\mathcal{I}_n'.\end{aligned}$$

Hence

$$\mathcal{J}_nT_{\mathcal{I}_n}-\mathcal{I}_nT_{\mathcal{J}_n}=x[\mathcal{J}_n'S_{\mathcal{I}_n}-\mathcal{I}_n'S_{\mathcal{J}_n}]=\beta x\mathcal{J}_n'\mathcal{I}_n-\gamma x\mathcal{J}_n\mathcal{I}_n'.$$

Thus, in the combination $P_n(kr)-krQ_n(kr)$ (required in Eq. (A.1)), all terms involving mixed combinations of $\mathcal{I}_n$ and $\mathcal{J}_n$ will cancel. For the remaining terms, we have

$$\begin{aligned}H_nT_{J_n}-J_nT_{H_n}&=2i\beta/\pi, & K_nT_{I_n}-I_nT_{K_n}&=-\gamma,\\ x[H_n'S_{J_n}-J_n'S_{H_n}]&=2i\gamma/\pi, & x[K_n'S_{I_n}-I_n'S_{K_n}]&=-\beta,\end{aligned}$$

using the Wronskians $J_nH_n'-J_n'H_n=2i/(\pi x)$ and $K_nI_n'-K_n'I_n=1/x$. Noting that $\beta-\gamma=2x^2$, we find that

$$P_n(kr)-krQ_n(kr)=2(kr)^2\{[A_n\hat{\mathcal{A}}_n^J-\hat{A}_n\mathcal{A}_n^J](2i/\pi)+[B_n\hat{\mathcal{A}}_n^I-\hat{B}_n\mathcal{A}_n^I]\}.$$

Next, we use formulas relating A_n and B_n to $\mathcal{A}_n^J$ and $\mathcal{A}_n^I$, and the corresponding formulas for $\hat{w}$,

$$A_n=T_{11}^{nm}\mathcal{A}_m^J+T_{12}^{nm}\mathcal{A}_m^I,\quad B_n=T_{21}^{nm}\mathcal{A}_m^J+T_{22}^{nm}\mathcal{A}_m^I, \tag{A.4}$$

$$\hat{A}_n=T_{11}^{nm}\hat{\mathcal{A}}_m^J+T_{12}^{nm}\hat{\mathcal{A}}_n^I,\quad \hat{B}_n=T_{21}^{nm}\hat{\mathcal{A}}_m^J+T_{22}^{nm}\hat{\mathcal{A}}_m^I, \tag{A.5}$$

where summation over all integer values of m is implied. (For circular scatterers, $T_{ij}^{nm}=T_{ij}^n\delta_{nm}$ (no sum) and then Eq. (A.4) reduces to Eq. (18).) These relations give

$$\begin{aligned}A_n\hat{\mathcal{A}}_n^J-\hat{A}_n\mathcal{A}_n^J&=(T_{11}^{nm}-T_{11}^{mn})\hat{\mathcal{A}}_n^J\mathcal{A}_m^J+T_{12}^{nm}[\hat{\mathcal{A}}_n^J\mathcal{A}_m^I-\mathcal{A}_n^J\hat{\mathcal{A}}_m^I],\\ B_n\hat{\mathcal{A}}_n^I-\hat{B}_n\mathcal{A}_n^I&=(T_{22}^{nm}-T_{22}^{mn})\hat{\mathcal{A}}_n^I\mathcal{A}_m^I-T_{21}^{nm}[\hat{\mathcal{A}}_n^I\mathcal{A}_m^J-\mathcal{A}_n^I\hat{\mathcal{A}}_m^J],\end{aligned}$$

whence $P_n(kr)-krQ_n(kr)=2(kr)^2\Lambda_n$, where

$$\begin{aligned}\Lambda_n&=(2i/\pi)(T_{11}^{nm}-T_{11}^{mn})\hat{\mathcal{A}}_n^JA_m^J+(T_{22}^{nm}-T_{22}^{mn})\hat{\mathcal{A}}_n^I\mathcal{A}_m^I\\ &\quad+[(2i/\pi)T_{12}^{nm}-T_{21}^{mn}][\mathcal{A}_m^I\hat{\mathcal{A}}_n^J-\hat{\mathcal{A}}_m^I\mathcal{A}_n^J].\end{aligned}$$

Then, combining Eqs. (A.1), (A.2) and (A.3) gives $\Lambda_n=0$.

As the coefficients $\mathcal{A}_n^I$, $\hat{\mathcal{A}}_n^J$, $\hat{\mathcal{A}}_n^I$ and $\mathcal{A}_n^J$ are all arbitrary (they correspond to different incident fields), we infer

$$T_{11}^{nm}=T_{11}^{mn},\quad T_{22}^{nm}=T_{22}^{mn},\quad (2i/\pi)T_{12}^{nm}=T_{21}^{mn}. \tag{A.6}$$

The third of these simplifies to Eq. (19) for circular scatterers.

We note that Matus and Emets [19] have given similar reciprocity formulas but their proof is incomplete. They use the far-field pattern f only, so that their vector $\mathbf{b}(\theta)$ is not completely arbitrary: see their Eq. (23). Indeed, their proof shows that the block $\mathbf{T}_{1,1}$ is symmetric (the first of Eq. (A.6)) but it gives no information on the other three blocks; see their Eqs. (20) and (24).

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