Multiple scattering of flexural waves by random configurations of inclusions in thin plates

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ABSTRACT

Flexural waves are scattered by inclusions in a thin plate. For a single inclusion of arbitrary shape, reciprocity relations are obtained connecting coefficients in circular multipole expansions. Then, a formula for the effective wavenumber in a random arrangement of identical circular inclusions is derived, using the Lax quasi-crystalline approximation.

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1. Introduction

The simplest model for wave propagation in a thin plate is based on Kirchhoff theory, which gives a partial differential equation, \((\nabla^4 - k^4)w = 0\), for the out-of-plane component of displacement, w. (Notation, formulation, and boundary conditions will be described fully in Section 2.) More refined models are available, but we limit ourselves here to the Kirchhoff theory; extensions to other models seem feasible.

Flexural waves in the plate will be scattered by inclusions, including the cases of holes (cavities) and defects, for example. We are interested in describing the scattering by many small inclusions, arranged randomly.

Early work on scattering by one inclusion is discussed in the book by Pao and Mow [23, Section III.6]. Since then, several authors have analysed scattering by one circular inclusion. For example, Norris and Vemula [22] gave results for the limiting cases of a rigid inclusion and a cavity. Chou et al. [4] and Squire and Dixon [25] used similar methods but in the context of surfactant effects and ice dynamics, respectively. Lee and Chen have given numerical results for scattering by two circular inclusions [13,14] and by three circular cavities [15]. Evans and Porter [8] have discussed scattering by N point scatterers, located arbitrarily in an infinite plate. Movchan et al. [21] have studied waves in a plate perforated by an infinite square array of identical circular holes. Matus and Emets [19] have developed a T-matrix method for scattering by one non-circular inclusion in a plate.

The classical (deterministic) multiple scattering problem, with many identical circular inclusions in the plate, can be tackled by combining separated solutions with addition theorems. We give such an exact treatment in Section 3.1: it serves as a starting point for problems in which the inclusions are located randomly.

There is previous work on flexural waves in plates with various kinds of randomness. We mention Beran's paper [1] on random density fluctuations and Weaver's paper [27] on plates with randomly attached sprung masses. Dixon and Squire [7] have used several approaches, including a self-consistent approach in which the solution of a certain scattering problem, with one inclusion embedded in the (unknown) effective medium [25], is employed.

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The literature on multiple scattering of acoustic waves by many inclusions (in both deterministic and random configurations) is extensive. For a review, see [17]. The literature on acoustic scattering by random arrangements of identical circular scatterers is reviewed in the paper by Linton and Martin [16]. That paper starts with a classical, exact multiple scattering theory (built using separated solutions of the Helmholtz equation and appropriate addition theorems), as used earlier by, for example, Fikioris and Waterman [9] and by Bose and Mal [3], and ends with an estimate for the effective acoustic wavenumber, $K$, in the random medium.

\begin{equation}
\langle Ka \rangle^2 = \langle ka \rangle^2 - \frac{4i}{\pi} \Phi(0) + \frac{8\Phi^2}{\pi^2\langle ka \rangle^2} \int_0^{\pi} \cot(\theta/2) \frac{d}{d\theta} [f(\theta)]^2 d\theta + O(\Phi^3). \tag{1}
\end{equation}

In this formula, $f(\theta)$ is the far-field pattern for scattering of a plane wave (with wavenumber $k$) by a single inclusion of radius $a$, and $\Phi$ is the fractional area occupied by the scatterers. Formula (1) has been shown to compare well with experiments [6]. It was derived using the Lax quasi-crystalline approximation (QCA) [12], a heuristic closure assumption that has been shown to give good theoretical results in certain situations [18,28]. See Parnell and Abrahams [24] for a comparison of Eq. (1) with other multiple scattering theories in the low-frequency (homogenization) limit. The generalization of Eq. (1) to plane-strain elastodynamics was given recently by Conoir and Norris [5].

In this paper, we develop a thin plate theory analogous to the acoustic theory given by Linton and Martin [16]. We derive the following formula for the effective flexural wavenumber $K$ (in (58)) below:

\begin{equation}
\langle Ka \rangle^2 = \langle ka \rangle^2 - \frac{4i}{\pi} \Phi(0) + \frac{8\Phi^2}{\pi^2\langle ka \rangle^2} \int_0^{\pi} \cot(\theta/2) \frac{d}{d\theta} [f(\theta)]^2 d\theta + \frac{8\Phi^2}{\pi^2\langle ka \rangle^2} |g(0)|^2 + O(\Phi^3). \tag{2}
\end{equation}

Note that this is identical to Eq. (1) to $O(\Phi^2)$ except for the additional term involving a second far-field pattern $g(\theta)$: whereas $f(\theta)$ is generated by a plane wave, $e^{ik_0 y}, g(\theta)$ is generated by an incident field, $e^{-ik_0 y}$, a field that has no physical interest in itself. As an example, we calculate effective properties of a plate perforated by many small circular holes.

We also obtain reciprocity relations connecting coefficients in multipole expansions of the field scattered by a single inclusion. These relations (see Eq. (A.6) below) hold for inclusions of all shapes and composition, and so they are of independent interest. We use one of them in the derivation of our formula for $K$. Also, as pointed out by Conoir and Norris [5], when the inclusions are such that $f(\theta) \neq f(-\theta)$, $|f(\theta)|^2$ in Eq. (2) should be replaced by $f(\theta)f(-\theta)$.

### 2. Flexural wave scattering by a single inclusion in a thin plate

We use a Cartesian coordinate system $(x, y, z)$ where the $xy$ plane is the plane of the thin plate and $z$ is perpendicular to this. Under the assumptions of the Kirchhoff thin plate theory, the amplitude of flexural waves $W(x, y, t)$ normal to the plane of the plate is described by

\begin{equation}
D \nabla^2 \nabla^2 W + \rho h^3 \frac{\partial W}{\partial t^2} = q \tag{3}
\end{equation}

where $q$ is the external forcing, $\rho$ is the density, $h$ is the thickness of the plate, the bending rigidity is given by $D = Eh^3/[12(1 - \nu^2)]$, $E$ is Young’s modulus and $\nu$ is Poisson’s ratio.

Assuming time harmonic behaviour, $W(x, y, t) = \text{Re}[w(x, y)e^{-i\omega t}]$, and taking $q = 0$, from Eq. (3) we have

\begin{equation}
(\nabla^2 \nabla^2 - k^4)w = (\nabla^2 + k^2)(\nabla^2 - k^2)w = 0 \quad \text{with } k^4 = \frac{\rho h\omega^2}{D}. \tag{4}
\end{equation}

We consider the scattered field from an isolated circular inclusion with different material properties from the surrounding plate (of infinite extent). As noted in Section 1, this problem has been considered before, but we summarize it here so as to define notation for the main purpose of this article, which is the consideration of the multiple scattering problem.

Assume that the (circular) scatterer of radius $a$, has domain $S$ and assume that a polar coordinate system $x = r \cos \theta, y = r \sin \theta$ is aligned with the centre of the scatterer. The amplitude of flexural waves is then governed by the following equations,

\begin{equation}(\nabla^2 \nabla^2 - k^4)w = 0, \quad r > a, \tag{5}\end{equation}

\begin{equation}(\nabla^2 \nabla^2 - k_0^4)w_0 = 0, \quad r < a, \tag{6}\end{equation}

where $k_0^4 = \rho h_0^3/D_0$ and $k^4 = \rho h^3/D$, together with specified conditions on $r = a$ and associated conditions as $r \to \infty$. We note that in the exterior region, the total displacement field is given by

\begin{equation}w = w_{\text{inc}} + w_0 \tag{7}\end{equation}
where \( w_{\text{inc}} \) is the (specified) incident wave and \( w_0 \) is the scattered field. The field \( w_{\text{inc}} \) is regular in a neighbourhood of \( S \); in general, it has an expansion
\[
 w_{\text{inc}} = \sum_{n=-\infty}^{\infty} \left\{ A_n^1 J_n(kr) + A_n^2 I_n(kr) \right\} e^{in\theta},
\]
where \( J_n \) is a Bessel function, \( I_n \) is a modified Bessel function defined by \( I_n(x) = i^{-n} J_n(ix) \) with \( x \) real here, and the coefficients \( A_n^1 \) and \( A_n^2 \) are known. For example, if we have a plane wave propagating at an angle \( \theta_{\text{inc}} \) to the positive \( x \) axis, then
\[
 w_{\text{inc}} = \exp(ikr \cos(\theta - \theta_{\text{inc}})), \quad A_n^1 = i^n e^{-in\theta_{\text{inc}}} \quad \text{and} \quad A_n^2 = 0.
\]
For a second example, which is unphysical but useful later, we take
\[
 w_{\text{inc}} = \exp(-kr \sin \theta), \quad A_n^1 = 0 \quad \text{and} \quad A_n^2 = i^n.
\]
The scattered field outside \( S \) and the field inside \( S \) can be sought in the forms
\[
 w_0(r, \theta) = \sum_{n=-\infty}^{\infty} \left\{ A_n H_n^{(1)}(kr) + B_n K_n(kr) \right\} e^{in\theta}, \quad r \geq a.
\]
\[
 w_1(r, \theta) = \sum_{n=-\infty}^{\infty} \left\{ C_n J_n(kr) + E_n I_n(kr) \right\} e^{in\theta}, \quad r \leq a.
\]
where \( H_n^{(1)} \) is a Hankel function and \( K_n \) is a modified Bessel function defined by \( 2K_n(x) = m^n + 1 H_n^{(1)}(ix) \) with \( x \) real here. The coefficients \( A_n, B_n, C_n \) and \( E_n \) are determined from \( A_n^1 \) and \( A_n^2 \) using continuity conditions at \( r = a \). According to Kirchhoff theory, these conditions are those of continuity of displacement \( w \), its normal derivative \( \partial w / \partial r \), the bending moment \( M(w) \) and the (Kirchhoff) shear \( V(w) \), the latter two quantities being defined in polar coordinates by
\[
 M(w) = -D_1^i \frac{\partial^2 w}{\partial r^2} - D_i \nu_i \left( \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right),
\]
\[
 V(w) = -D_1^i \frac{\partial}{\partial r} \left( r^2 \nu_i \frac{\partial w}{\partial \theta} \right) - D_i \nu_i \left( \frac{\partial^2 w}{\partial r \partial \theta} - \frac{w}{r} \right),
\]
where \( D_i \) and \( \nu_i \) are the bending rigidity and Poisson ratio of the host \( (i = 0) \) and inclusion \( (i = 1) \).

Applying these continuity conditions leads to the system of equations
\[
 Ax = B a.
\]
where
\[
 A = \begin{pmatrix}
 H_n(\epsilon_0) & K_n(\epsilon_0) & -J_n(\epsilon_1) & -I_n(\epsilon_1) \\
 H_n'(\epsilon_0) & K_n'(\epsilon_0) & -J_n'(\epsilon_1) & -I_n'(\epsilon_1) \\
 S_k^0 & S_k^1 & -D_k^1 & -D_k^1 \\
 T_k^0 & T_k^1 & -D_k^1 & -D_k^1
\end{pmatrix},
\]
\[
 x = \begin{pmatrix}
 A_n^1 \\
 B_n^1 \\
 C_n \\
 E_n
\end{pmatrix}, \quad B = \begin{pmatrix}
 J_n(\epsilon_0) & I_n(\epsilon_0) \\
 J_n'(\epsilon_0) & I_n'(\epsilon_0) \\
 S_k^0 & S_k^1 \\
 T_k^0 & T_k^1
\end{pmatrix}, \quad a = \begin{pmatrix}
 A_n^1 \\
 A_n^1 \\
 A_n^1 \\
 A_n^1
\end{pmatrix}.
\]

Here, prime denotes differentiation with respect to the argument, \( \kappa = k_1/k_0 = k_a, \epsilon_1 = k_1 a = \kappa \epsilon_0 \), and \( D = D_1/D_0 \) is the ratio of bending stiffnesses and we have used the following notation [22]
\[
 S_k^0 = n^2 (1 - \nu_i) + \epsilon_i^2 X_k(\epsilon_i) - (1 - \nu_i) \epsilon_i X_k'(\epsilon_i),
\]
\[
 T_k^0 = n^2 (1 - \nu_i) X_k(\epsilon_i) - [n^2 (1 - \nu_i) + \epsilon_i^2] X_k'(\epsilon_i),
\]
in which the upper sign is taken when \( X_k = J_k \) or \( H_k \) and the lower sign is taken when \( X_k = I_k \) or \( K_k \). The differential equations satisfied by the various Bessel functions have been used in order to simplify the expressions for \( S_k^0 \) and \( T_k^0 \).
The 4 × 4 system, Eq. (15), can be solved for $A_n$, $B_n$, $C_n$ and $E_n$. Of importance later will be expressions for the coefficients $A_n$ and $B_n$. These expressions can be written in the form

$$
A_n = T_{11}^n A_n' + T_{12}^n A_n', \quad B_n = T_{21}^n A_n' + T_{22}^n A_n',
$$

(18)

where the (dimensionless) quantities $T_{ij}^n$ are computable. The expressions in Eq. (18) can also be used for cavities and for rigid inclusions (although the formulas for $T_{ij}^n$ will be different).

It is shown in Appendix A that

$$
(2i/\pi)T_{12}^n = T_{21}^n.
$$

(19)

This relation is a consequence of reciprocity. It is a special case of general results that hold for scatterers of any shape and composition.

In the far-field ($kr \rightarrow \infty$), we can use

$$
H_n(z) \sim \sqrt{2/(\pi z)}(-i)^n e^{-im/4} e^{i\theta}, \quad K_n(z) \sim \sqrt{\pi/(2z)}e^{-\theta}, \quad z \rightarrow \infty.
$$

These show that, physically, the first term in Eq. (11) is associated with the scattered field and the second term is the local, evanescent field. Thus,

$$
w_0(r, \theta) = -\sqrt{2/(\pi kr)}(-i)^n A_n e^{im} \text{ as } r \rightarrow \infty,
$$

where $A_n$ is given by Eq. (18). In particular, for an incident plane wave, given by Eq. (9), we have

$$
w_0(r, \theta) = \sqrt{2/(\pi kr)}f(\theta - \theta_{inc}) \exp(ikr - im) \text{ as } r \rightarrow \infty,
$$

where the far-field pattern $f$ is given by

$$
f(\theta) = \sum_{n=-\infty}^{\infty} T_{11}^n e^{im}\theta.
$$

(20)

Similarly, the far-field pattern associated with an incident wave of the form given in Eq. (10) is $g$, given by

$$
g(\theta) = \sum_{n=-\infty}^{\infty} T_{12}^n e^{im}\theta.
$$

(21)

3. Multiple scattering theory

3.1. Multiple scattering from $N$ inclusions

We now consider an exact theory for multiple scattering of flexural waves when there are $N$ identical circular inclusions embedded in the thin plate.

To begin with, the inclusions occupy a finite domain $D$, $(x, y) \in [0, L] \times [-H, 0]$, within an unbounded infinite plate. The area of $D$ is denoted by $|D|$. There are $n_0$ inclusions per unit area so that $N = n_0 |D|$. The fractional area occupied by the inclusions is denoted by

$$
\phi = n_0 \pi a^2 \quad (0 \leq \phi \leq 1).
$$

Later, we let $D$ expand to fill a half-space $\mathbb{R}_+$ by letting $L \rightarrow \infty$ and $H \rightarrow \infty$ whilst letting $N \rightarrow \infty$ with $n_0$ held fixed (Fig. 1).

The inclusions are located at $p_j = (j, q_j), j = 1, 2, \ldots, N$ with local polar coordinate system $(r_j, \theta_j)$ relating to the local Cartesian system $x_j = r_j \cos \theta_j, y_j = r_j \sin \theta_j$, see Fig. 2.

Suppose that the incident field which excites scattering from the $N$ inclusions is a plane wave propagating at an arbitrary angle $\theta_{inc}$ to the $x$ axis. Thus, $w_{inc}$ is given by Eq. (9) and it can be written in the form

$$
w_{inc}(r_j, \theta_j) = l_j \sum_{n=-\infty}^{\infty} i^n J_n(kr_j) e^{in(\theta_j - \theta_{inc})}
$$

(22)

local to the $j$th inclusion, where $l_j = e^{i(n\alpha_j + \beta_j)}$, $\alpha = k \cos \theta_{inc}$ and $\beta = k \sin \theta_{inc}$. (In [16], $l_n$ is denoted by $l_n$ (italics in the latter); here, we use $l_n$ to denote modified Bessel functions.)
The total field in $x > 0$, outside all the inclusions, is given by

$$w = w_{\text{inc}} + \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \left\{ A_{j} Z_{n} H_{n}(kr_{j}) + B_{j} W_{n} K_{n}(kr_{j}) \right\} e^{in\theta}, \quad r_{j} \geq a,$$

(23)

where the constants $Z_{n}$ and $W_{n}$ have been added for convenience and will be defined shortly. Inside the $j$th inclusion, the field is $w_{j}$, where

$$w_{j} = \sum_{n=-\infty}^{\infty} \left\{ C_{j}^{\prime} J_{n}(k_{1}r_{j}) + E_{j}^{\prime} I_{n}(k_{1}r_{j}) \right\} e^{in\theta}, \quad r_{j} \leq a.$$

With the benefit of Graf’s addition theorem, we can write down $w$ in the vicinity of the $s$th inclusion:

$$w(r_{s}, \theta_{s}) = \sum_{n=-\infty}^{\infty} \left\{ I_{s}^{\prime} J_{n}(k_{1}r_{s}) e^{-in\theta_{s}} + A_{s} Z_{n} H_{n}(kr_{s}) + B_{s} W_{n} K_{n}(kr_{s}) \right\} e^{in\theta_{s}}$$

$$+ \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \left( A_{j} Z_{n} H_{n-m}(kR_{js}) e^{i(n-m)\theta_{s}} J_{m}(kr_{s}) e^{im\theta_{s}} + B_{j} W_{n} K_{n-m}(kR_{js}) e^{i(n-m)\theta_{s}} I_{m}(kr_{s}) e^{im\theta_{s}} \right).$$

(24)
where with reference to Fig. 2, $R_m = |R_m| = |p_j - p_i|$ and $\theta_{ij}$ is the angle subtended between the $x_j$ axis and the $s$th inclusion (note that $\theta_{ij} = \pi + \theta_{ji}$). Rewrite Eq. (24) as

$$W(r_s, \theta_i) = \sum_{n = -\infty}^{\infty} \left( A_n^i J_m(k r_s) + B_n^i J_m(k r_s) + A_n^i Z_n H_m(k r_s) + B_n^i W_n K_m(k r_s) \right) e^{i(n-m)\theta_i},$$

for $r_s \geq a$, where (compare with Eq. (8), in which the roles of $A_n^i$ and $A_n^s$ are played here by $A_n^i$ and $B_n^i$, respectively)

$$A_n^i = I_1^m e^{-i m \pi},$$

$$B_n^i = \sum_{j=1}^{N} \sum_{n = -\infty}^{\infty} B_n^j W_n K_n(k r_j) e^{i(n-m)\theta_j}.$$  

Comparison with the solution for scattering by one inclusion, Eq. (18), gives

$$A_n^m = T_{11}^m A_n^s + T_{12}^m B_n^s, \quad B_n^m = T_{21}^m A_n^s + T_{22}^m B_n^s.$$  

To simplify these equations, we choose

$$Z_n = -T_{11}^n \quad \text{and} \quad W_n = -T_{21}^n.$$  

Furthermore, we define

$$Q_n = T_{12}^n / T_{11}^n \quad \text{and} \quad P_n = T_{22}^n / T_{21}^n,$$

so that Eq. (27) gives $A_n^m + Q_n B_n^m = 0$ and $B_n^m + A_n^m + P_m B_n^m = 0$. Writing these out explicitly, using Eqs. (25) and (26), we obtain

$$A_n^m + \sum_{j=1}^{N} \sum_{n = -\infty}^{\infty} \left( A_n^j Z_n H_n(k r_j) + Q_m B_n^j W_n K_{n-m}(k r_j) \right) e^{i(n-m)\theta_j} = -I_1^m e^{-i m \pi},$$

$$B_n^m + \sum_{j=1}^{N} \sum_{n = -\infty}^{\infty} \left( A_n^j Z_n H_n(k r_j) + P_m B_n^j W_n K_{n-m}(k r_j) \right) e^{i(n-m)\theta_j} = -I_1^m e^{-i m \pi}.$$  

These equations hold for $s = 1, 2, \ldots, N$ and for all integers $m$.

### 3.2. Averaging: random distribution of inclusions in a half-space

Let us introduce a probability density function, defining the statistics of the random distribution of inclusions [20,26]. This is written as

$$p(p_1, p_2, \ldots, p_N) = p(p_s) p(p_1, p_2, \ldots, p_s | p_i),$$

where $p_s$ indicates that $p_s$ is not present. Similarly,

$$p(p_1, p_2, \ldots, p_s, \ldots, p_N | p_i) = p(p_s | p_i) p(p_1, \ldots, p_s, \ldots, p_N | p_i),$$

where $p(p_s | p_i)$ denotes the probability density of finding an inclusion at $p_i$ given that there is already one located at $p_i$. We define the following ensemble average of a function $f(x, y, p_1, p_2, \ldots, p_N),$

$$\langle f \rangle_S = \int_D \int_{\partial D} \cdots \int_{\partial D} p(p_1, p_2, \ldots, p_s, \ldots, p_N | p_i) f(dp_1, dp_2, \ldots, dp_N),$$

which is the average with the $s$th inclusion fixed [16,26]. Similarly, we can define $\langle f \rangle_{js}$, the average taken with both the $s$th and $j$th inclusions fixed.
We are going to take the ensemble average of Eqs. (30) and (31) with the $s$th inclusion fixed; we start with Eq. (30). Noting the indistinguishability of inclusions in the $j$ sum after taking this average, we find that

$$\langle \mathcal{A}_m^s \rangle_s + (N-1) \sum_{n=0}^{\infty} \int_\mathcal{D} \rho(p_3 | p_1) \langle \mathcal{A}_n^s \rangle_g Z_n H_{n-m}(kR_0) e^{i(n-m)\theta} dp_3$$

$$+ Q_m(N-1) \sum_{n=0}^{\infty} \int_\mathcal{D} \rho(p_3 | p_1) \langle \mathcal{B}_n^s \rangle_g W_n K_{n-m}(kR_0) e^{i(n-m)\theta} dp_3 = -i^m e^{-im\phi},$$

where $\mathcal{D}$ is the region in which inclusions reside. Next we invoke the Lax quasi-crystalline closure approximation (QCA) [12], which amounts to taking

$$\langle \mathcal{A}_n^s \rangle_g = \langle \mathcal{A}_n^s \rangle_j \quad \text{and} \quad \langle \mathcal{B}_n^s \rangle_g = \langle \mathcal{B}_n^s \rangle_j,$$

and for notational convenience we set $s = 1$ and $j = 2$. We also make a simple choice for $\rho(p_2 | p_1)$: we choose the hole-correction pair-correlation function [9],

$$\rho(p_2 | p_1) = \begin{cases} |\mathcal{D}|^{-1} = n_0 / N, & \mathcal{R}_2 \geq b, \\ 0, & \mathcal{R}_2 < b, \end{cases}$$

where $b$ is the radius of the so-called exclusion zone, with $b = 2a$ classically. Finally, we let $N \rightarrow \infty$ and we let $\mathcal{D}$ expand to fill a half-space $\mathcal{H}$, but we keep $n_0$, the number of inclusions per unit area, fixed. The result is

$$\langle \mathcal{A}_m^s \rangle_s + n_0 \sum_{n=0}^{\infty} \frac{1}{\mathcal{N}} \left\{ \langle \mathcal{A}_n^s \rangle_g Z_n H_{n-m}(kR_21) + \langle \mathcal{B}_n^s \rangle_g W_n Q_m K_{n-m}(kR_21) \right\} e^{i(n-m)\theta} dp_2 dq_2 = -i^m e^{-im\phi},$$

where $\mathcal{N}$ is the half-space $p_2 \geq 0$ and the circle on the integral sign indicates that the hole centred at $p_2 = p_1$, of radius $b$, has to be cut out of the half-space. A similar calculation, starting from Eq. (31), leads to

$$\langle \mathcal{B}_m^s \rangle_s + n_0 \sum_{n=0}^{\infty} \frac{1}{\mathcal{N}} \left\{ \langle \mathcal{A}_n^s \rangle_g Z_n H_{n-m}(kR_21) + \langle \mathcal{B}_n^s \rangle_g W_n Q_m K_{n-m}(kR_21) \right\} e^{i(n-m)\theta} dp_2 dq_2 = -i^m e^{-im\phi}.$$
Here we have defined the integrals
\[ L_n(X) = \int_{-\infty}^{\infty} \psi_n(x,y)e^{iy} \, dy, \quad \tilde{L}_n(X) = \int_{-\infty}^{\infty} X_n(x,y)e^{iy} \, dy, \]
(40)
and
\[ M_n = \tilde{\Psi}(p_{21}, q_{21})\psi_n(p_{21}, q_{21}) dp_{21} dq_{21}, \]
\[ \tilde{M}_n = \tilde{\Psi}(p_{21}, q_{21})X_n(p_{21}, q_{21}) dp_{21} dq_{21}, \]
(41)
(42)
where \( \Psi(X,Y) = e^{i(X+Y)} \). The integrals in Eqs. (41) and (42) are over the half-plane \( p_2 > \gamma \) with the disc \( R_{21} < b \) removed.

The integrals \( L_0 \) and \( M_n \) were evaluated in [16]. The integrals \( \tilde{L}_0 \) and \( \tilde{M}_n \) can be evaluated similarly. From ([16], Eq. (66)), we have
\[ L_n(p_2 - p_1) = (2/\alpha)\pi e^{ip_1} e^{ip_1} \cdot \quad p_1 > p_2, \]
hence
\[ \sum_{n=-\infty}^{\infty} (-i)^{n-m} \int_{0}^{\infty} X_n(p_2) Z_n \, dp_2 = C_m e^{ip_1}, \]
where \( C_m \) is a constant.

Next, consider \( \tilde{L}_0(X) \), an even function of \( X \); we require \( \tilde{L}_0(X) \) for \( X < 0 \). We start with \( \tilde{L}_0(0) \). We have \( \tilde{L}_0(0) = 2\int_{0}^{\infty} K_0(ky) \cos \beta y \, dy = \pi / \gamma \) (using ([10], Eq. 6.671(14))) where
\[ \gamma = \sqrt{k^2 + \beta^2}. \]
As \( (\nabla^2 - k^2)K_0(ky) = 0 \) with \( r = (x^2 + y^2)^{1/2} \), we have
\[ \tilde{L}_0''(x) = k^2 \tilde{L}_0(x) - \int_{-\infty}^{\infty} e^{iy} \frac{\partial^2}{\partial y^2} K_0(ky) \, dy = \gamma^2 \tilde{L}_0(x). \]
After two integrations by parts. Then, as \( \tilde{L}_0(x) \to 0 \) when \( |x| \to \infty \), we obtain \( \tilde{L}_0(x) = Ce^{-\gamma|x|} \) for some constant \( C \). Hence, using \( \tilde{L}_0(0) = \pi / \gamma \),
\[ \tilde{L}_0(X) = (\pi / \gamma)e^{iN} \quad \text{and} \quad \tilde{L}_0(X) = \gamma \tilde{L}_0(X) \quad \text{for} \quad X < 0. \]

For \( \tilde{L}_n \) with \( n > 0 \), we use
\[ \tilde{L}_n(X) = \int_{-\infty}^{\infty} -\frac{1}{k} e^{iy} \left( \frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right) X_{n-1}(X,Y) \, dy. \]
Then, as in [16], we obtain \( k \tilde{L}_n = -\tilde{L}_{n-1}' - \beta \tilde{L}_{n-1} \), whence
\[ \tilde{L}_n(X) = (\pi / \gamma)(-1)^{n-1}(\beta + \gamma)^{n} k^n e^{iN}. \]
Similarly, for \( n = 0 \), we obtain \( \tilde{L}_0(X) = (\pi / \gamma)(\beta - \gamma)^{-n} k^n e^{iN} \). Hence,
\[ \sum_{n=-\infty}^{\infty} (-i)^{n-m} \int_{0}^{\infty} X_n(p_2) W_n Q_n \, dp_2 = \tilde{C}_m e^{-\gamma p_1}, \]
where \( \tilde{C}_m \) is a constant.

The double integrals \( M_n \) and \( \tilde{M}_n \) are evaluated using Green's theorem. We have
\[ M_n = M_n^a + M_n^b \quad \text{and} \quad \tilde{M}_n = \tilde{M}_n^a + \tilde{M}_n^b, \]
where \( M_n^a \) and \( M_n^b \) come from integrations along the line \( p_2 = \gamma \), and \( M_n^a \) and \( \tilde{M}_n^b \) come from integrations around the circle \( R_{21} = b \). From ([16], Eq. (67)), we have
\[ M_n^a = C_n e^{i(\alpha - \lambda)p_1} \quad \text{and} \quad M_n^b = 2\pi i^n (k^2 - K^2)^{-1} e^{i\beta N} X_n^a(Kb), \]
where \( C_n \) is a constant and \( X_n^a(Kb) = KbH_n^a(kb)J_n(kb) - KbH_n(kb)J_n^a(kb) \).
For $\tilde{M}_n$, we note that $\Psi \nabla^2 \chi_n - \chi_n \nabla^2 \Psi = (k^2 + K^2)\Psi \chi_n$ so that

$$(k^2 + K^2)\tilde{M}_n = \oint \left( \frac{\partial \chi_n}{\partial n} - \chi_n \frac{\partial \Psi}{\partial n} \right) dS$$

where $S$ consists of the line $p_2 = \rho$ (this gives $\tilde{M}^m_n$) and the circle $R_2 = b$ centred at $p_2 = p_1$ (this gives $\tilde{M}^b_n$), and the normal vector on $S$ points out of the half-space domain. On $p_2 = \rho$, $\partial \theta n = -\partial \phi p_2$ and so we have

$$-\int_{p_2 = \rho} \left( \Psi \frac{\partial \chi_n}{\partial p_2} - \chi_n \frac{\partial \Psi}{\partial p_2} \right) dp_2$$

$$= -e^{i\alpha(p_2 - p_1)} \int_{-\infty}^{\infty} e^{i\theta_1} \left[ \cos(\alpha_2) \frac{\partial \chi_n}{\partial \alpha_2} \sin(\alpha_2) \frac{\partial \chi_n}{\partial \alpha_2} - i \chi_n \right] dp_2$$

$$= -e^{i\alpha(p_2 - p_1)} \int_{-\infty}^{\infty} e^{i\theta_1} \left[ -\frac{k}{2} (\chi_{n+1} + \chi_{n+1}) \right] dp_2$$

$$= e^{i\alpha(p_2 - p_1)} \left( \tilde{\lambda}_{n-1} (\rho - p_1) + \tilde{\lambda}_{n+1} (\rho - p_1) \right) + i \chi_n$$

using $-2K_n(x) = \chi_n - (x + 1) + \chi_n + 1(x)$, $-2n/K_n(x) = \chi_n - (x + 1) - \chi_n + 1(x)$ and Eq. (40). Hence,

$$\tilde{M}^n_n = \tilde{\chi}_n e^{-(i \rho + \gamma \phi_1)}$$

for some constant $\tilde{\chi}_n$.

The contribution from the circle $R_2 = b$ is

$$-\int_{0}^{2\pi} \left[ e^{iK\cos(\theta - \phi)} \frac{\partial \chi_n}{\partial K} - \chi_n \frac{\partial e^{iK\cos(\theta - \phi)}}{\partial K} \right] \rho d\theta$$

$$= b \sum_{n=1}^{\infty} n^{2m} e^{i n \phi} \frac{\partial}{\partial \rho} (K_n(kb) \cos(\theta - \phi) - kK_n(kb)) d\theta$$

$$= -2\pi i^{m} e^{i n \phi} \tilde{\chi}_n (Kb)$$

where $\tilde{\chi}_n (Kb) = kbK_n(kb)J_n(Kb) - kK_n(kb)J_n(Kb)$. Hence

$$\tilde{M}^b_n = -2\pi i^{m} (k^2 + K^2)^{-1} e^{i n \phi} \tilde{\chi}_n (Kb).$$

Next, we substitute our results in Eq. (39), giving

$$-e^{i\phi_1} e^{-i\phi_1} = F_m e^{-i\phi_1} e^{i\phi_1} + C_m e^{i\phi_1} + \tilde{\chi}_n e^{-i\phi_1}$$

$$+ n_0 e^{i\phi_1} \sum_{n=-m}^{m} \left( -i \right)^{n-m} e^{-i n \phi} \left[ \tilde{E}_n Z\tilde{E}_m(p_1) + C_n W_n Q_m \tilde{E}_m(p_1) \right]$$

where

$$E_m(p_1) = C_{n-m} e^{i(n-m)\phi_1} + 2n_0 e^{i(n-m)\phi_1} \tilde{N}_n(m)(Kb)$$

$$\tilde{E}_m(p_1) = C_{n-m} e^{-(i \phi + \gamma \phi_1)} - 2n_0 e^{-(i \phi + \gamma \phi_1)} \tilde{N}_n(m)(Kb).$$

Eq. (43) contains terms that are proportional to $e^{i\phi_1}$, terms that are proportional to $e^{i\phi_1}$, and terms that are proportional to $e^{-i\phi_1}$: all should balance. Collecting the terms that are proportional to $e^{i\phi_1}$ gives

$$F_m + \frac{2n_0}{k^2 - K^2} \sum_{n=-m}^{m} F_n Z_n N_{n-m}(Kb) - \frac{2n_0 Q_m}{k^2 + K^2} \sum_{n=-m}^{m} C_n W_n \tilde{N}_n(m)(Kb) = 0.$$  (44)

(If the second sum is omitted, Eq. (44) reduces to ([16], Eq. (71)), as derived for an analogous acoustic problem.) A similar calculation, starting with Eq. (36), leads to

$$C_m + \frac{2n_0}{k^2 - K^2} \sum_{n=-m}^{m} F_n Z_n N_{n-m}(Kb) - \frac{2n_0 F_m}{k^2 + K^2} \sum_{n=-m}^{m} C_n W_n \tilde{N}_n(m)(Kb) = 0.$$  (45)

Eqs. (44) and (45) hold for all integers $m$. They provide an infinite homogeneous system of linear algebraic equations for $F_m$ and $C_m$. The existence of a non-trivial solution to this system will determine the effective wavenumber $K$.

It is worth noting that Eqs. (44) and (45) do not depend on $\rho$ or $\theta_n$. This may be regarded as a check: we would not expect the effective wavenumber to depend on the angle of incidence or on distance from the boundary. Indeed, Eqs. (44) and (45) could have been derived by assuming that $\rho = 0$ and $\theta_n = 0$ from the outset. Further information can be obtained by balancing the other exponential terms in Eq. (43), but this information will not be needed here.
3.3. Approximate determination of $K$ for small $\phi$

In [16], the expansion $K^2 = k^2 + \frac{\kappa_1}{n} + \frac{\kappa_2}{n^2} - \pi$ was used, and then expressions for $\kappa_1$ and $\kappa_2$ were sought. Here, we prefer to work with dimensionless quantities, so we write

$$(Ka)^2 = (ka)^2 + \kappa_1 \phi + \kappa_2 \phi^2 + \cdots$$

where $\phi = \frac{\pi a^2}{2n_0}$, $\kappa_1 = \frac{1}{n}$, and $\kappa_2 = \frac{\phi}{(\pi a)^2}$. Let $B = \frac{b}{a}$. From [16], we have

$$N_n(Kb) = \left(\frac{2i}{\pi} + (\phi/2)B^2\kappa_2d_n(kb) + O(\phi^2)\right).$$

where $d_n(x) = \sum_{n=-\infty}^{\infty} H_n(x) + \frac{1}{1+(n/x)^2} J_n(x) H_n(x)$. Similarly,

$$\tilde{N}_n(kb) = w_n(kb) + (\phi/2)B^2\kappa_3d_n(kb) + O(\phi^2),$$

where $\tilde{d}_n(x) = \sum_{n=-\infty}^{\infty} H_n(x) K_n(x) + [1-(n/x)^2] J_n(x) K_n(x)$ and

$$w_n(x) = \pi [K_n(x) J_n(x) - K_n(x) J_n(x)].$$

Hence

$$\frac{2m_0}{k^2-K^2}N_n(kb) = -\frac{4i}{\pi \kappa_1} + \phi \left[\frac{4i \kappa_2}{\pi \kappa_1} B^2 d_n(kb)\right] + O(\phi^2)$$

whereas

$$\frac{2m_0}{k^2+K^2} \tilde{N}_n(kb) = \phi \frac{1}{(ka)^2} w_n(kb) + O(\phi^2).$$

Substituting Eqs. (48) and (49) in Eq. (44), neglecting the second-order terms, we obtain

$$F_m + \sum_{n=-\infty}^{\infty} F_n Z_n \left[\frac{4i \kappa_2}{\pi \kappa_1} B^2 d_{n-m}(kb)\right] - \phi Q_m \sum_{n=-\infty}^{\infty} G_n W_n \frac{w_{n-m}(kb)}{(ka)^2} = 0.$$  \hspace{1cm} (50)

Eq. (45) leads to a very similar equation, with exactly the same sums, but with the leading $F_m$ replaced by $G_m$ and with $Q_m$ replaced by $P_m$.

At leading order in $\phi$, Eq. (50) gives

$$F_m = \frac{4i}{\pi \kappa_1} \sum_{n=-\infty}^{\infty} F_n Z_n,$$  \hspace{1cm} (51)

so that all the $F_m$ are equal. Then, if we write $F_m = F$, Eq. (51) gives

$$\kappa_1 = \frac{4i}{\pi} \sum_{n=-\infty}^{\infty} Z_n = -\frac{4i}{\pi} \bar{f}(0),$$  \hspace{1cm} (52)

where $f$ is the far-field pattern; see Eqs. (20) and (28). This result is exactly the same as the Foldy approximation for acoustic problems; see ([16], Eq. (77)). In addition, from Eq. (45), we obtain $G_m = F$, to leading order.

At next order, we can write

$$F_m = F + \phi \bar{F}_m \quad \text{and} \quad G_m = F + \phi \bar{G}_m.$$  \hspace{1cm} (53)

Then, the $O(\phi)$ terms in Eq. (50) are

$$\bar{F}_m + \sum_{n=-\infty}^{\infty} Z_n \left[\frac{4i \kappa_2}{\pi \kappa_1} B^2 d_{n-m}(kb)\right] - \phi Q_m \sum_{n=-\infty}^{\infty} W_n \frac{w_{n-m}(kb)}{(ka)^2} = 0.$$  \hspace{1cm} (54)

Put

$$S^2_m = \sum_{n=-\infty}^{\infty} Z_n d_{n-m}(kb) \quad \text{and} \quad S_m^W = \sum_{n=-\infty}^{\infty} W_n w_{n-m}(kb).$$
Then, rewrite Eq. (53), using Eq. (52):

\[
P_m - F B^2 \sum_{n=0}^{m} F(n) Q_n^{mW} = \frac{4i}{\pi \kappa_1} \sum_{n=-\infty}^{m} Z_n \hat{F}_n - \frac{F \kappa_2}{\kappa_1}. \tag{54}
\]

The right-hand side of Eq. (54) does not depend on \(m\); denote it by \(\hat{F}\). Thus

\[
\hat{F} = \frac{4i}{\pi \kappa_1} \sum_{n=-\infty}^{m} Z_n \hat{F}_n - \frac{F \kappa_2}{\kappa_1} = \frac{4i}{\pi \kappa_1} \sum_{n=-\infty}^{m} Z_n \left[ \hat{F} + F B^2 \sum_{n=0}^{m} F(n) Q_n^{mW} \right] - \frac{F \kappa_2}{\kappa_1}. \tag{55}
\]

The terms in \(\hat{F}\) cancel (using Eq. (52)), and then a factor of \(F/\kappa_1\) cancels leaving

\[
\kappa_2 = \frac{4i}{\pi} \sum_{n=-\infty}^{\infty} Z_n \left[ B^2 \hat{S}_n^f + \frac{1}{(ka)^2} Q_n^{mW} \right]
\]

\[
= \frac{4i B^2}{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} Z_n \delta_{m-n}(kb) + \frac{4i}{(ka)^2} \sum_{n=-\infty}^{\infty} Z_n Q_n W_w w_{-n}(kb). \tag{56}
\]

If we perform a similar analysis starting from Eq. (45), we arrive at

\[
\hat{C}_m - F B^2 \sum_{n=0}^{m} F(n) Q_n^{mW} = \hat{F}. \tag{57}
\]

instead of Eq. (54). As \(\hat{F}\) (defined by the right-hand side of Eq. (54)) does not involve \(\hat{C}_m\), we cannot back-substitute as we did with Eq. (55), so nothing is gained beyond a formula for \(\hat{C}_m\).

Returning to our formula (56) for the second-order correction, \(\kappa_2\), we notice that the first double-sum is exactly the same as in the acoustic case; see [16, Eq. (80)]. It involves \(Z_n\) and so it may be expressed in terms of the far-field pattern for plane-wave incidence on one inclusion, \(f(\theta)\).

The second double-sum in Eq. (56) involves both \(Z_n Q_n = -T_{12}^n\) and \(W_w = -T_{21}^n\) (see Eqs. (19), (28) and (29)). The quantity \(T_{12}^n\) also occurs in a far-field pattern but not for plane-wave incidence: it appears in the far-field pattern \(g\), defined by Eq. (21), corresponding to the incident field given by Eq. (10).

Let us approximate \(\kappa_2\) by approximating \(d_n(kb)\) and \(w_n(kb)\) for small \(kb\). We have \(x^2 d_n(x) \sim 2i|n|/\pi x \to 0\), and this leads to an expression for the first double-sum in (56) as a certain integral of \(f\); see [16, p. 3420] and Eq. (58). For \(w_n(kb)\), defined by Eq. (47), we find that \(w_n(x) \sim -1\) as \(x \to 0\) so that the second double-sum in Eq. (56) splits into the product of two single-sums:

\[
\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} Z_n Q_n W_w w_{-n}(kb) \approx \frac{2i}{\pi} \left[ \frac{g(0)}{g(0)} \right]^2.
\]

Combining these results, we obtain the approximation

\[
(Ka)^2 = (ka)^2 - \frac{4i}{\pi} \left[ \frac{df(0)}{d\theta} \right] + \frac{8 \phi^2}{\pi^2 (ka)^2} \int_0^{\theta} \left[ \frac{d(\theta)}{d\theta} \right]^2 d\theta + \frac{8 \phi^2}{\pi^2 (ka)^2} [g(0)]^2 + O(\phi^3). \tag{58}
\]

Apart from the term involving the far-field pattern \(g(0)\), this formula is reminiscent of the well known Lloyd–Brey formula for acoustics and its two-dimensional form given in Eq. (1).

4. Low-frequency results for cavities

For scattering by a cavity, we need the lower left 2 × 2 block in Eq. (15); dropping the redundant superscript 0, we have

\[
\left( \begin{array}{cc}
S_h & S_k \\
T_h & T_k \\
\end{array} \right) \left( \begin{array}{c}
A_n^{sv} \\
B_n^{sv} \\
\end{array} \right) = - \left( \begin{array}{cc}
S_h & S_k \\
T_h & T_k \\
\end{array} \right) \left( \begin{array}{c}
A_n \\
B_n \\
\end{array} \right). \tag{59}
\]

Hence,

\[
\Delta A_n^{sv} = [S_k T_h - S_h T_k] A_n + [S_k T_h - S_h T_k] A_n, \tag{60}
\]

\[
\Delta B_n^{sv} = [S_h T_h - S_h T_k] A_n + [S_h T_h - S_h T_k] A_n, \tag{61}
\]

where \(\Delta_n = S_h T_k - S_k T_h\). These equations are exact.
For low frequencies \((ka \ll 1)\), we obtain the following approximations:

\[
A^{aw}_{\nu} \approx \frac{\text{im}(ka)^2}{4(1-\nu)} (\nu' A\nu_0 - A\nu_0'),
\]

\[
B^{aw}_0 \approx \frac{(ka)^2}{2(1-\nu)} (\nu' A\nu_0 - A\nu_0'),
\]

\[
A^{aw}_{\ell+1} \approx \frac{\text{im}(ka)^2}{32(1-\nu)} (1 + \nu')_\ell A\nu_0 - 2 A\nu_0',
\]

\[
B^{aw}_\ell \approx \frac{(ka)^4}{16(1-\nu)} (2 A\nu_0 - (1 + \nu')_\ell A\nu_0'),
\]

\[
A^{aw}_{\ell+2} \approx \frac{\text{im}(1-\nu)(ka)^2}{2^{\ell+1}(3 + \nu)(n-1)!2^{n-2}} (A\nu_0 + A\nu_0'), \quad n \geq 2,
\]

\[
B^{aw}_{\ell+2} \approx \frac{(1-\nu)(ka)^2}{2^{\ell+1}(3 + \nu)(n-1)!2^{n-2}} (A\nu_0 + A\nu_0'), \quad n \geq 2.
\]

In particular,

\[
A^{aw}_{\nu} \approx \frac{\text{im}(ka)^2}{8(3 + \nu)} (\nu' A\nu_0 + A\nu_0'),
\]

\[
B^{aw}_{\nu} \approx -\frac{(1-\nu)(ka)^2}{4(3 + \nu)} (A\nu_0 + A\nu_0').
\]

We note that \(A^{aw}_{\nu}, B^{aw}_{\nu}, A^{aw}_{\ell+2}, B^{aw}_{\ell+2}\) occur at leading order, \(O((ka)^2)\), all other contributions being smaller. (Compare with acoustic/SH wave scattering from a cavity, where the monopole \((n=0)\) and dipole \((n=1)\) modes give the leading order contributions.) Norris and Venmula [22] also found that \(A^{aw}_{\nu}\) and \(A^{aw}_{\ell+2}\) gave the main contribution (for plane-wave incidence). However, although we agree with their expression for \(A^{aw}_{\nu}\), we disagree with their expressions for \(A^{aw}_{\ell+2}\); we computed these formulae both by hand and with a symbol manipulation package.

Comparison with Eq. (18), gives

\[
t^{0}_{11} = \frac{\text{im}(1-\nu)(ka)^2}{4(1-\nu)}, \quad t^{0}_{12} = -\frac{\text{im}(ka)^2}{4(1-\nu)}, \quad t^{0}_{21} = \frac{(ka)^2}{2(1-\nu)}, \quad t^{0}_{22} = \frac{-(1-\nu)(ka)^2}{2(1-\nu)},
\]

\[
t^{2}_{11} = t^{2}_{12} = \frac{\text{im}(1-\nu)(ka)^2}{8(3 + \nu)}, \quad t^{2}_{21} = t^{2}_{22} = -\frac{(1-\nu)(ka)^2}{4(3 + \nu)}.
\]

These satisfy the reciprocity relation, Eq. (19). Then, to leading order, we find that the far-field patterns are

\[
f(\theta) = \frac{\text{im}(ka)^2}{4} \left( \frac{\nu}{1-\nu} + \frac{1-\nu}{3 + \nu} \cos 2\theta \right),
\]

\[
g(\theta) = \frac{\text{im}(ka)^2}{4} \left( -\frac{1}{1-\nu} + \frac{1-\nu}{3 + \nu} \cos 2\theta \right).
\]

These approximations can be used in Eq. (58) to estimate the effective wavenumber, \(K\). We have

\[
f(0) = \frac{\text{im}(ka)^2}{4(1-\nu)(3 + \nu)} \left( 1 + \nu + 2\nu^2 \right), \quad g(0) = -\frac{\text{im}(ka)^2}{4(1-\nu)(3 + \nu)} \left( 2 + 3\nu - \nu^2 \right).
\]

The first of these gives the dilute (Foldy) approximation (linear in \(\phi\))

\[
K^2 = 1 + \frac{\phi(1 + \nu + 2\nu^2)}{(1-\nu)(3 + \nu)}.
\]

At second order in \(\phi\), we must evaluate the integral in Eq. (58); using

\[
\int_0^{\pi} \cot(\theta/2) \sin 2\theta d\theta = \pi, \quad \int_0^{\pi} \cot(\theta/2) \sin 2\theta \cos 2\theta d\theta = \frac{\pi}{2}
\]

and the approximation for \(f(\theta)\), Eq. (62), we find that

\[
\frac{8}{\pi^2 (ka)^2} \int_0^{\pi} \cot(\theta/2) \frac{d}{d\theta} f(\theta)^2 d\theta = (ka)^2 \left( \frac{1 + \nu}{3 + \nu} \right)^2.
\]
When this is substituted in Eq. (58), together with the expression for \( g(0) \), Eq. (64), we obtain an estimate for the coefficient \( \kappa_2 \) in Eq. (46):

\[
\frac{\kappa_2}{(ka)^2} = \frac{(1 + \nu)(1 + 3\nu)}{(3 + \nu)^2} - \frac{(2 + 3\nu - \nu^2)^2}{2(1-\nu)^2(3 + \nu)^2}
\]

The effective density, \( \rho_e \), and the effective bending stiffness, \( D_e \), are related to the effective wavenumber \( K \) by

\[
\frac{K^4}{K^2} = \frac{\rho_e}{\rho} \frac{D_e}{D} = 1 + \frac{2\kappa_1\phi}{(ka)^2} + \left( \frac{\kappa_1^2}{(ka)^4} + \frac{2\kappa_2}{(ka)^2} \right) \phi^2
\]

Finally, as it is well known that \( \rho_e/\rho = 1 - \phi \) in the quasistatic limit, we can estimate the effective bending stiffness, \( D_e/D \).

5. Conclusions

In this paper we have studied the multiple scattering of flexural waves by inclusions in thin plates. Reciprocity relations for a single inclusion have been established, connecting coefficients in circular multipole expansions. These relations are applicable regardless of the shape or composition of the inclusion. Using one of these relations and an approach analogous to that used by Linton and Martin [16] we derived an expression for the square of the effective wavenumber in the inhomogeneous plate region; this expression is of a similar form to the Lloyd–Berry expression associated with acoustics [16]. In the present case however, the expression involves two far-field scattering patterns, \( f(\theta) \) and \( g(\theta) \), associated with scattering from a single inclusion. The pattern \( f(\theta) \) is associated with an incident plane wave \( e^{i\theta} \); it is the pattern which usually arises in Lloyd–Berry expressions. However, the pattern \( g(\theta) \) is of no physical interest in itself, being associated with an incident wave of the form \( e^{-i\theta} \).

In order to show the applicability of the theory, we considered the low-frequency limit of the effective wavenumber for an inhomogeneous plate consisting of circular cylindrical cavities randomly distributed inside a homogeneous plate phase. It was shown how estimates for the effective density and effective bending stiffness can be derived.

Such theories can be useful in various contexts, for example the non-destructive evaluation of composite plates and the measurement of sea-ice thickness.

Appendix A. Proof of reciprocity relations

Consider scattering by a single scatterer \( S \) with (smooth) boundary \( \Gamma \). \( S \) could be an inclusion, a cavity, or a fixed rigid object. In this appendix, it is not assumed that \( S \) is circular. Surround \( S \) by a circle \( C \).

Outside \( S \), \( w \) satisfies \( (\nabla^2 - K^2)w = 0 \) (in the absence of body forces). Take a second flexural field, \( \hat{w} \). Reciprocity gives (see, for example, ([11], p. 258) or ([2], p. 235))

\[
\int_C R(w, \hat{w}) ds = \int_C R(w, \hat{w}) ds,
\]

where

\[
R(w, \hat{w}) = w\hat{w}(\theta) - \hat{w}w(\theta) - M(w) \frac{\partial \hat{w}}{\partial n} + M(\hat{w}) \frac{\partial w}{\partial n},
\]

\( M(w) \) is the bending moment and \( V(\hat{w}) \) is the Kirchhoff shear. We suppose that \( w \) and \( \hat{w} \) satisfy the same boundary conditions on \( \Gamma \), and that these ensure that \( \int_C R(w, \hat{w}) ds = 0 \). (If \( \Gamma \) has corners, additional corner conditions must be imposed.) Hence, if \( C \) has radius \( r \),

\[
\int_0^\pi \int_C R(w, \hat{w}) d\theta dr = 0 \quad \text{for any (sufficiently large) } r.
\]

We use this relation to prove the reciprocity relation, Eq. (19).

At \( C \), the field \( w(r, \theta) \) has the expansion

\[
w(r, \theta) = \sum_{m, n = \pm \infty} \{A_m H_m(0) + B_m K_m(0) + A_m^* J_m(0) + A_m^* J_m(0)\} e^{im\theta}.
\]

For the bending moment, defined by Eq. (13), we have

\[
M(w) = \frac{D}{\pi} \sum_{m, n = \pm \infty} \{A_m S_m + B_n T_n + A_m^* S_m + A_m^* T_m\} e^{im\theta}
\]

and for the Kirchhoff shear, defined by Eq. (14), we have

\[
V(w) = -\frac{D}{\pi} \sum_{m, n = \pm \infty} \{A_m T_n + B_n T_n + A_m^* T_n + A_m^* T_n\} e^{im\theta}.
\]
In these formulas, \( S_0 \) and \( T_0 \) are defined by Eqs. (16) and (17), respectively, with \( v_t = v' \) and \( v_{i} = kr \) therein. For \( \tilde{v} \), we choose similar expansions with care on the various coefficients and with \( e^{-i\omega t} \) instead of \( e^{i\omega t} \). Then

\[
\int_0^{\pi} (w\Psi(\tilde{u}) - \tilde{w}\Psi(w)) d\theta = -\frac{2\pi D}{r^2} \sum_{n m} P_n(kr)
\]  

(A.2)

where

\[
P_n = |A_n H_0 + B_n K_0 + A'_n J_0 + A'_n J_0|A_n T_0 + \hat{B}_n T_0 + A'_n T_0 + A'_n T_0|
\]

\[
-|A_n H_0 + B_n K_0 + A'_n J_0 + A'_n J_0|A_n T_0 + B_n T_0 + A'_n T_0 + A'_n T_0|
\]

\[
= |A_n B_n - A_n B_n|H_n T_0 - K_n T_0| + |A_n A'_n - A_n A'_n|H_n T_0 - K_n T_0|
\]

\[
+ |A_n A'_n - A_n A'_n|H_n T_0 - K_n T_0| + |A_n A'_n - A_n A'_n|H_n T_0 - K_n T_0|
\]

Similarly

\[
\int_0^{\pi} (\frac{\partial}{\partial r} \Psi(w) - \frac{\partial}{\partial r} \Psi(\tilde{u})) d\theta = -\frac{2\pi D}{r^2} \sum_{n m} (kr)Q_n(kr)
\]  

(A.3)

where

\[
Q_n = |A_n B_n - A_n B_n|H_n S_n - K_n S_n| + |A_n A'_n - A_n A'_n|H_n S_n - K_n S_n|
\]

\[
+ |A_n A'_n - A_n A'_n|H_n S_n - K_n S_n| + |A_n A'_n - A_n A'_n|H_n S_n - K_n S_n|
\]

To simplify notation, put \( x = kr, \alpha = n(1 - i), \beta = \alpha - \chi^2 \), and \( \gamma = \alpha - x^2 \). Let \( J_n \) denote \( J_n \) or \( H_n \), and let \( T_n \) denote \( T_n \) or \( K_n \). Then

\[
S_n = \chi J_n(1 - v')/\chi', \quad S_n = J_n(1 - v)x', \quad T_n = \chi J_n, \quad T_n = -\chi x/J_n
\]

Hence

\[J_n T_n - J_n T_n = x(J_n/T_n - T_n J_n) = J_n T_n - \chi J_n T_n'\]

Thus, in the combination \( P_n(kr) - krQ_n(kr) \) (required in Eq. (A.1)), all terms involving mixed combinations of \( T_n \) and \( J_n \) will cancel. For the remaining terms, we have

\[
H_n T_0 - H_n T_0 = 2i\beta/n, \quad K_n T_0 - K_n T_0 = -\gamma.
\]

\[
\alpha(H_n T_0 - K_n T_0) = 2i\gamma/n, \quad \alpha(H_n T_0 - K_n T_0) = -\beta.
\]

using the Wronskians \( \beta H_n - \gamma H_n = 2i/\alpha(n) \) and \( \alpha K_n - \beta K_n = 1/\alpha. \) Noting that \( \beta - \gamma = 2\chi^2 \), we find that

\[
P_n(kr) - krQ_n(kr) = 2(kr)^2 \left( |A_n A'_n - A_n A'_n|/2i/n + |A_n A'_n - A_n A'_n|/2i/n \right)
\]

Next, we use formulas relating \( A_n \) and \( B_n \) to \( A'_n \) and \( A'_n \), and the corresponding formulas for \( \tilde{v} \),

\[
A_n = T_{10}^m A_{10} + T_{11}^m A_{11} + T_{12}^m A_{12}, \quad B_n = T_{20}^m A_{20} + T_{21}^m A_{21}
\]

\[
A'_n = T_{10}^m A'_{10} + T_{11}^m A'_{11} + T_{12}^m A'_{12}, \quad B'_n = T_{20}^m A'_{20} + T_{21}^m A'_{21}
\]

(A.4)

(A.5)

where summation over all integer values of \( m \) is implied. (For circular scatterers, \( T_{10}^n = T_{20}^n s_{10}^n \) (no sum) and then Eq. (A.4) reduces to Eq. (18).) These relations give

\[
|A_n A'_n - A_n A'_n| = (T_{10}^m A_{10} + T_{11}^m A_{11} + T_{12}^m A_{12}) \left( A^m_{10} A'_n - A^m_{11} A'_n \right)
\]

\[
B_n A'_n - B_n A'_n = (T_{20}^m A_{20} + T_{21}^m A_{21} - T_{22}^m A_{22}) \left( A^m_{20} A'_n - A^m_{21} A'_n \right)
\]

whence \( P_n(kr) - krQ_n(kr) = 2(kr)^2 A_n \) where

\[
A_n = \left( 2i/n \right) \left( T_{10}^m A_{10} + T_{11}^m A_{11} + T_{12}^m A_{12} \right)
\]

\[
+ \left( 2i/n \right) \left( T_{20}^m A_{20} + T_{21}^m A_{21} - T_{22}^m A_{22} \right)
\]

Then, combining Eqs. (A.1), (A.2) and (A.3) gives \( \Lambda_n = 0 \). As the coefficients \( A'_n, A''_n, A'_n, \) and \( A'_{10} \) are all arbitrary (they correspond to different incident fields), we infer

\[
r_{10}^m = r_{11}^m \quad T_{12}^m = T_{22}^m = (2i/n)r_{12}^m = r_{12}^m
\]

(A.6)

The third of these simplifies to Eq. (19) for circular scatterers.

We note that Matas and Emett [19] have given similar reciprocity formulas but their proof is incomplete. They use the far-field pattern \( f \) only, so that their vector \( k(\vartheta) \) is not completely arbitrary: see their Eq. (23). Indeed, their proof shows that the block \( T_{11} \) is symmetric (the first of Eq. (A.6)) but it gives no information on the other three blocks; see their Eqs. (20) and (24).
References