Multiple scattering by random configurations of circular cylinders: Reflection, transmission, and effective interface conditions

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In a previous paper, Linton and Martin [J. Acoust. Soc. Am. **117**, 3413–3423 (2005)] obtained two formulas for the effective wavenumber in a dilute random array of circular scatterers. They emerged from a study of the problem of the reflection of a plane wave at oblique incidence to a half-space containing the scatterers. Here, their study is extended to obtain formulas for the reflection and transmission coefficients and to investigate the average fields near the boundary of the half-space. Comparisons with previous work are made. © *2011 Acoustical Society of America*. [DOI: 10.1121/1.3546098]

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I. INTRODUCTION

In recent years, there has been a revival of interest in methods for estimating the effective wavenumber for propagation through random composites. In this paper, two-dimensional acoustic problems are considered, with the Helmholtz equation, $(\nabla^2+k^2)u=0$, outside circular cylinders. Inside each cylinder, there is another Helmholtz equation, $(\nabla^2+k_0^2)u_0=0$, with transmission conditions connecting u and u_0 across the circular boundaries. It is assumed that every circle has radius a.

To analyze multiple scattering[1] by random configurations of scatterers, the theory given by Linton and Martin[2] is developed further; their paper and formulas taken from it will be identified by LM below. In LM, formulas for the effective wavenumber, K, were derived. These formulas take the form

$$K^2=k^2+n_0\delta_1+n_0^2\delta_2, \tag{1}$$

where n_0 is the number of cylinders per unit area; the area fraction occupied by the scatterers, $\phi\equiv n_0\pi a^2$, is assumed to be small. Explicit formulas for δ_1 and δ_2 were given in LM. They were obtained using the Lax quasicrystalline approximation (QCA), they compare favorably with experiments,[3,4] they have been confirmed by an independent method that is valid for weak scattering[5] (see Sec. VIII), and they have been used to estimate the dynamic effective mass density of random composites.[6] The LM formulas are accurate to second order in ϕ. They require the solution of a scattering problem for one scatterer; this scalar (transmission) problem is discussed briefly in Sec. II.

The basic problem considered in LM consists of circular scatterers distributed randomly in the (right) half-plane $x>0$ with a plane wave incident obliquely from the (left) half-plane $x<0$. The goal of LM was to estimate the effective wavenumber, K, within the right half-plane. No information on the reflected and transmitted wavefields was given. Here, we show that these wavefields can be determined by extending the analysis in LM.

There is a specularly reflected plane wave. The (average) reflection coefficient is found to be

$$R=n_0(\mathrm{i}/k^2)f(\pi-2\theta_{\mathrm{in}})+O(n_0^2), \tag{2}$$

where θ_{in} is the angle of incidence and $f(\theta)$ is the far-field pattern for scattering by one cylinder. An explicit expression for the $O(n_0^2)$ contribution is also obtained, see Eq. (42).

The average field transmitted to the right half-plane has a more complicated form. For example, at normal incidence, it is found to be

$$\mathrm{e}^{\mathrm{i}Kx}+n_0\pi a^2\{\mathcal{A}_1\mathrm{e}^{\mathrm{i}Kx}-\mathrm{e}^{\mathrm{i}kx}\}+O(n_0^2), \tag{3}$$

where an explicit formula for $\mathcal{A}_1$ is obtained, see Eq. (52). To verify these results, comparisons with the independent weak-scattering results in Ref. 5 are made, and agreement is found.

In Sec. IX, the behavior of the fields across $x=0$ is examined. It is found that the fields themselves are continuous but the slopes are discontinuous. An estimate for the slope discontinuity is derived. This is then used to estimate the effective density of the right half-plane.

II. LINTON–MARTIN (LM) FORMULATION

The LM approach (described in Sec. IV of LM[2]) starts with an exact multiple-scattering method for an arbitrary deterministic arrangement of N identical circular scatterers. The total field outside the cylinders, u, is written as LM(47),

$$u=u_{\mathrm{in}}+\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty}A_n^jZ_nH_n(kr_j)\,\mathrm{e}^{\mathrm{i}n\theta_j}, \tag{4}$$

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for $r_j > a$, $j = 1, 2, \ldots, N$, where (r_j, θ_j) are the polar coordinates centered at $\mathbf{r}_j = (x_j, y_j)$, the center of the jth cylinder. The incident field is taken as a plane wave at oblique incidence, LM(24), so that

$$u_{\mathrm{in}} = \mathrm{e}^{\mathrm{i}(\alpha x + \beta y)} \quad \text{with} \quad \alpha = k\cos\theta_{\mathrm{in}},\; \beta = k\sin\theta_{\mathrm{in}}. \tag{5}$$

The coefficients A_n^j are to be found, $H_n \equiv H_n^{(1)}$ is a Hankel function and Z_n, which characterizes scattering by one isolated cylinder, is given by LM(49),

$$Z_n = (\mathrm{Re}\,\Delta_n)/\Delta_n = Z_{-n} \tag{6}$$

with

$$\Delta_n = (\rho_0/\rho)H_n'(ka)J_n(k_0a) - (k_0/k)H_n(ka)J_n'(k_0a). \tag{7}$$

Here, k_0 is the interior wavenumber, ρ_0 is the interior density, and ρ is the exterior density. The interface conditions are LM(50),

$$u = u_0 \quad \text{and} \quad \frac{1}{\rho}\frac{\partial u}{\partial r} = \frac{1}{\rho_0}\frac{\partial u_0}{\partial r} \quad \text{on} \quad r = a, \tag{8}$$

where u_0 is the field inside the cylinder; these are appropriate for a fluid cylinder surrounded by a different fluid, so that u is the pressure. The far-field pattern, f, is given by LM(53) as

$$f(\theta) = -\sum_{n=-\infty}^{\infty} Z_n\,\mathrm{e}^{\mathrm{i}n\theta} = -Z_0 - 2\sum_{n=1}^{\infty} Z_n \cos n\theta. \tag{9}$$

In LM, an exact linear system is derived for A_n^j. Then, ensemble averages are taken, followed by letting $N \to \infty$ and imposition of the Lax QCA. (All the cylinders are in the right half-plane, $x > 0$.) The key quantity is $\langle A_n^j \rangle_j$, the average of A_n^j conditional on there being a cylinder at $\mathbf{r}_j$. It is expressed as LM(56),

$$\langle A_n^j \rangle_j = \mathrm{i}^n\,\mathrm{e}^{\mathrm{i}\beta y_j}\,\Phi_n(x_j), \quad x_j > 0. \tag{10}$$

Then, in order to avoid possible difficulties near the "interface" at $x = 0$, Linton and Martin assumed that $\boldsymbol{\Phi}_\mathrm{n}$ could be written as LM(58),

$$\Phi_n(x) = F_n\,\mathrm{e}^{-\mathrm{i}n\varphi}\,\mathrm{e}^{\mathrm{i}\lambda x} \quad \text{for } x > \ell, \tag{11}$$

where the length ℓ is not (and need not be) specified; later, we shall take $\ell = 0$ (see Sec. III). The quantities K and φ are defined by LM(31),

$$\lambda = K\cos\varphi \quad \text{and} \quad \beta = K\sin\varphi = k\sin\theta_{\mathrm{in}}. \tag{12}$$

The coefficients F_n were then shown to solve an infinite homogeneous system of linear algebraic equations, LM(71). Analysis of this system led to formulas for δ_1 and δ_2 in Eq. (1), LM(77) and LM(80), respectively,

$$\delta_1 = 4\mathrm{i}\sum_{n=-\infty}^{\infty} Z_n = -4\mathrm{i}f(0), \tag{13}$$

$$\delta_2 = 4\pi\mathrm{i}b^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} Z_m Z_n d_{m-n}(kb), \tag{14}$$

where b is the "hole radius" introduced in the conditional averaging to prevent cylinders overlapping, the function $d_n(x)$ is defined by LM(73),

$$d_n(x) = J_n'(x)H_n'(x) + [1 - (n/x)^2]J_n(x)H_n(x), \tag{15}$$

and J_n is a Bessel function. Equation (13) is the well-known Foldy–Lax estimate. The second-order contribution, Eq. (14), is approximated further in LM by taking the limit $kb \to 0$; the result is LM(7), a formula that is reminiscent of the Lloyd–Berry formula for analogous three-dimensional problems.[7,8]

Concerning the dependence of F_n on n_0, the analysis in Sec. IV C of LM shows that

$$F_n = F + n_0 q_n + O(n_0^2), \tag{16}$$

$$q_n = \tilde{Q} + \pi b^2 F \sum_{m=-\infty}^{\infty} Z_m d_{m-n}(kb), \tag{17}$$

where F and $\tilde{Q}$ are n_0-independent constants. The values of these two constants were not determined; this will be done in Sec. IV.

III. DEPENDENCE ON ℓ

On p. 3419 of LM, it is shown that $\mathcal{B} = -1$,

$$-1 = \mathcal{B} = \frac{2n_0}{\alpha}\sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n\theta_{\mathrm{in}}} C_n(\ell), \tag{18}$$

$$C_n(\ell) = \int_0^\ell \Phi_n(t)\mathrm{e}^{-\mathrm{i}\alpha t}\,\mathrm{d}t + \frac{\mathrm{i}F_n\mathrm{e}^{-\mathrm{i}n\varphi}}{\lambda - \alpha}\,\mathrm{e}^{\mathrm{i}(\lambda-\alpha)\ell}. \tag{19}$$

This relation was not used in LM. As the main results in LM do not depend on ℓ, we shall assume that a continuity argument can be used to assert that Eq. (11) should hold for all $\ell > 0$, leading to a useful simplification. So, setting $\ell = 0$ in Eq. (18) gives

$$-1 = \frac{2\mathrm{i}n_0}{\alpha(\lambda - \alpha)}\sum_{n=-\infty}^{\infty} F_n Z_n \mathrm{e}^{\mathrm{i}n(\theta_{\mathrm{in}} - \varphi)}. \tag{20}$$

This equation will be used in Sec. IV to obtain information on F_n.

IV. INFERENCES FROM K

Given the LM formula for K, Eq. (1) with Eqs. (13) and (14), estimates for φ and λ can be obtained. To begin, rewrite Eq. (1) as

$$(K/k)^2 = 1 + \phi\kappa_1 + \phi^2\kappa_2, \tag{21}$$

where $\phi = \pi a^2 n_0$, and

$$\kappa_1 = \frac{\delta_1}{\pi(ka)^2} = -\frac{4\mathrm{i}f(0)}{\pi(ka)^2} \quad \text{and} \quad \kappa_2 = \frac{\delta_2}{(\pi k a^2)^2} \tag{22}$$

are dimensionless. Ignoring terms that are $O(\phi^3)$ as $\phi \to 0$, Eq. (21) gives $K/k = 1 + \frac{1}{2}\phi\kappa_1 + \frac{1}{8}\phi^2(4\kappa_2 - \kappa_1^2)$.

Put $C = \cos\theta_{\text{in}}$, $S = \sin\theta_{\text{in}}$, and $T = \tan\theta_{\text{in}}$. To solve Eq. $(12)_2$, $(K/k)\sin\varphi = S$, for φ, write

$$\varphi = \theta_{\text{in}} + \phi p_1 + \phi^2 p_2. \tag{23}$$

Then

$$\begin{aligned} S &= (K/k)\left\{S\cos(\phi p_1 + \phi^2 p_2) + C\sin(\phi p_1 + \phi^2 p_2)\right\} \\ &= (K/k)\left\{S\left(1 - \frac{1}{2}\phi^2 p_1^2\right) + C(\phi p_1 + \phi^2 p_2)\right\} \\ &= S + \frac{1}{2}\phi(Sk_1 + 2Cp_1) \\ &\quad + \frac{1}{8}\phi^2[S(4\kappa_2 - \kappa_1^2) + 4Cp_1\kappa_1 + 8Cp_2 - 4Sp_1^2]. \end{aligned}$$

Hence,

$$p_1 = -\frac{1}{2}T\kappa_1 \quad \text{and} \quad p_2 = \frac{1}{8}T[(3 + T^2)\kappa_1^2 - 4\kappa_2]. \tag{24}$$

Next, calculate $\lambda = K\cos\varphi$, using Eqs. (23) and (24),

$$\lambda/k = C + \frac{1}{2}\phi\kappa_1 C^{-1} + \frac{1}{8}\phi^2(4\kappa_2 C^{-1} - \kappa_1^2 C^{-3}). \tag{25}$$

This approximation is used in Eq. (20). Thus,

$$\frac{2\mathrm{i}n_0}{\alpha(\lambda - \alpha)} = \frac{4\mathrm{i}}{\pi(ka)^2\kappa_1}\left[1 + \phi\left(\frac{\kappa_1}{4C^2} - \frac{\kappa_2}{\kappa_1}\right)\right]. \tag{26}$$

From Eqs. (16) and (17),

$$\frac{F_n}{F} = 1 + \phi Q + \phi(b/a)^2 \sum_{m=-\infty}^{\infty} Z_m d_{m-n}(kb), \tag{27}$$

where $Q = \tilde{Q}/(\pi a^2 F)$, a dimensionless constant. Hence

$$\begin{aligned} &\frac{1}{F}\sum_{n=-\infty}^{\infty} F_n Z_n \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}} - \varphi)} \\ &= \sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}} - \varphi)} + \phi Q \sum_{n=-\infty}^{\infty} Z_n \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}} - \varphi)} \\ &\quad + \phi\frac{\pi(ka)^2}{4\mathrm{i}}\mathcal{K}(\theta_{\text{in}} - \varphi) \\ &= -f(\theta_{\text{in}} - \varphi) - \phi Q f(0) - \frac{1}{4}\mathrm{i}\phi\pi(ka)^2\mathcal{K}(0) \\ &= -\{f(0) - \phi p_1 f'(0)\} - \phi Q f(0) - \frac{1}{4}\mathrm{i}\phi\pi(ka)^2\kappa_2 \\ &= -\frac{1}{4}\mathrm{i}\pi(ka)^2\kappa_1 - \frac{1}{4}\mathrm{i}\phi Q\pi(ka)^2\kappa_1 - \frac{1}{4}\mathrm{i}\phi\pi(ka)^2\kappa_2 \\ &= \frac{\pi\kappa_1}{4\mathrm{i}}(ka)^2\left[1 + \phi\left(Q + \frac{\kappa_2}{\kappa_1}\right)\right], \end{aligned} \tag{28}$$

where $f'(0) = 0$ has been used [see Eq. (9)] and the function $\mathcal{K}$ is defined by

$$\mathcal{K}(\delta) = \frac{4\mathrm{i}(b/a)^2}{\pi(ka)^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} Z_m Z_n d_{m-n}(kb)\,\mathrm{e}^{\mathrm{i}n\delta}, \tag{29}$$

so that $\mathcal{K}(0) = \kappa_2$ [see Eqs. (14) and (22)]. (Note that $\lim_{kb\to 0}\mathcal{K}(\delta)$ can be expressed as an integral, similar to that for $\mathcal{K}(0)$ as in Sec. IV C of LM.) Substituting Eqs. (26) and (28) in Eq. (20) gives

$$\begin{aligned} -1 &= F\left[1 + \phi\left(\frac{\kappa_1}{4C^2} - \frac{\kappa_2}{\kappa_1}\right)\right]\left[1 + \phi\left(Q + \frac{\kappa_2}{\kappa_1}\right)\right] \\ &= F\left[1 + \phi\left(Q + \frac{\kappa_1}{4C^2}\right)\right] + O(\phi^2). \end{aligned}$$

Thus,

$$F = -1 \quad \text{and} \quad Q = -\frac{\kappa_1}{4C^2} = \frac{\mathrm{i}f(0)}{\pi(ka)^2 C^2}.$$

Substituting back in Eq. (28) gives

$$\begin{aligned} \sum_{n=-\infty}^{\infty} F_n Z_n \mathrm{e}^{\mathrm{i}n(\theta_{\text{in}} - \varphi)} &= \frac{\mathrm{i}}{4}\pi(ka)^2\kappa_1\left[1 - \phi\left(\frac{\kappa_1}{4C^2} - \frac{\kappa_2}{\kappa_1}\right)\right] \\ &\quad + O(\phi^2). \end{aligned} \tag{30}$$

This approximation will be used in Sec. VII.

V. RESULTS FROM FOLDY THEORY

Classical Foldy theory assumes isotropic scattering. The effective wavenumber is given by LM(1),

$$K^2 = k^2 - 4\mathrm{i}gn_0, \tag{31}$$

where g is the (dimensionless) scattering coefficient. For the scattering problem, we have

$$\langle u(x,y)\rangle = \begin{cases} \mathrm{e}^{\mathrm{i}(\alpha x + \beta y)} + R\mathrm{e}^{\mathrm{i}(-\alpha x + \beta y)}, & x < 0, \\ \mathcal{A}\mathrm{e}^{\mathrm{i}(\lambda x + \beta y)}, & x > 0, \end{cases} \tag{32}$$

where R is the (average) reflection coefficient and $\mathcal{A}$ is the (average) transmission coefficient.

Foldy theory (see Sec. III A in LM) predicts that $R = R_{\text{F}}$ and $\mathcal{A} = \mathcal{A}_{\text{F}}$, where

$$R_{\text{F}} = \frac{\alpha - \lambda}{\lambda + \alpha} \quad \text{and} \quad \mathcal{A}_{\text{F}} = \frac{2\alpha}{\lambda + \alpha} = 1 + R_{\text{F}}. \tag{33}$$

From these formulas and Eq. (32), it follows that both $\langle u\rangle$ and $(\partial/\partial x)\langle u\rangle$ are continuous across $x = 0$.

Working to first order in ϕ, $\varphi = \theta_{\text{in}} + \phi p_1$, $p_1 = 2\mathrm{i}gT/[\pi(ka)^2]$, $\lambda = \alpha - \phi k p_1/S$, and Eq. (33) gives

$$R_{\text{F}} = \frac{\mathrm{i}\phi g}{\pi(ka)^2 C^2} = \frac{\mathrm{i}n_0 g}{\alpha^2} \quad \text{and} \quad \mathcal{A}_{\text{F}} = 1 + R_{\text{F}}. \tag{34}$$

If we put $g = f(0)$, which is the correct non-isotropic extension of Eq. (31), it turns out that Eq. (34) does not give the correct estimates for R or $\mathcal{A}$, as we show below. [See the text above Eq. (43) for R and the text below Eq. (52) for $\mathcal{A}$.]

VI. THE AVERAGE REFLECTED FIELD

Calculating the ensemble average of Eq. (4) for $x < 0$ gives

$$\langle u(x,y)\rangle = u_{\text{in}} + \frac{n_0}{N}\sum_{j=1}^{N}\sum_{n=-\infty}^{\infty} Z_n \int\!\!\int \langle A_n^j\rangle_j H_n(kr_j) \times \mathrm{e}^{\mathrm{i}n\theta_j}\,\mathrm{d}x_j\,\mathrm{d}y_j. \tag{35}$$

Here, $x - x_j = r_j \cos\theta_j$ and $y - y_j = r_j \sin\theta_j$. Then, use the indistinguishability of the scatterers, let $N \to \infty$, and use Eq. (10):

$$\langle u(x,y)\rangle = u_{\text{in}} + n_0 \sum_{n=-\infty}^{\infty} \mathrm{i}^n Z_n \int_0^{\infty} \Phi_n(x_1) \times \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\beta y_1} H_n(kr_1)\,\mathrm{e}^{\mathrm{i}n\theta_1}\,\mathrm{d}y_1\,\mathrm{d}x_1. \tag{36}$$

The inner integral can be evaluated. In LM, it is shown that

$$L_n(X) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\beta Y} H_n(k\mathcal{R})\mathrm{e}^{\mathrm{i}n\Theta}\,\mathrm{d}Y = \begin{cases} (2/\alpha)(-\mathrm{i})^n \mathrm{e}^{-\mathrm{i}n\theta_{\text{in}}}\mathrm{e}^{\mathrm{i}\alpha X}, & X > 0, \\ (2/\alpha)\mathrm{i}^n \mathrm{e}^{\mathrm{i}n\theta_{\text{in}}}\mathrm{e}^{-\mathrm{i}\alpha X}, & X < 0, \end{cases} \tag{37}$$

where α and β are defined by Eq. (5), $X = \mathcal{R}\cos\Theta$ and $Y = \mathcal{R}\sin\Theta$. Comparison with the inner integral in Eq. (36) shows that we should take $Y = y_1 - y$, $\mathcal{R} = r_1$, $\Theta = \theta_1 + \pi$, and $X = x_1 - x$. As $x < 0$ and $x_1 > 0$, $X > 0$ and so

$$\langle u(x,y)\rangle = u_{\text{in}} + \frac{2n_0}{\alpha}\mathrm{e}^{\mathrm{i}\beta y} \sum_{n=-\infty}^{\infty} (-1)^n Z_n \times \int_0^{\infty} \Phi_n(x_1)\mathrm{e}^{-\mathrm{i}n\theta_{\text{in}}}\mathrm{e}^{\mathrm{i}\alpha(x_1 - x)}\,\mathrm{d}x_1. \tag{38}$$

Hence, comparison with Eq. (32) gives

$$R = \frac{2n_0}{\alpha}\sum_{n=-\infty}^{\infty} (-1)^n \mathrm{e}^{-\mathrm{i}n\theta_{\text{in}}} Z_n \int_0^{\infty} \Phi_n(x_1)\mathrm{e}^{\mathrm{i}\alpha x_1}\,\mathrm{d}x_1 = \frac{2\mathrm{i}n_0}{\alpha(\lambda+\alpha)}\sum_{n=-\infty}^{\infty} F_n Z_n \mathrm{e}^{\mathrm{i}n(\pi - \theta_{\text{in}} - \varphi)}. \tag{39}$$

Le Bas *et al.*[9] have obtained a formula for R for a slab, $0 < x < d$; letting $d \to \infty$ in their Eq. (41) gives agreement with Eq. (39). The same paper[9] also contains Eq. (20); see their Eq. (31).

Equation (39) is an exact formula for the reflection coefficient. It can be used to approximate R for small ϕ. From Eq. (25),

$$\frac{2\mathrm{i}n_0}{\alpha(\lambda+\alpha)} = \frac{\mathrm{i}\phi}{\pi(ka)^2 C^2}\left(1 - \frac{\phi\kappa_1}{4C^2}\right). \tag{40}$$

Also,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{F_n}{F} Z_n \mathrm{e}^{\mathrm{i}n(\pi-\theta_{\text{in}}-\varphi)} &= \sum_{n=-\infty}^{\infty} (1 + \phi Q) Z_n \mathrm{e}^{\mathrm{i}n(\pi-\theta_{\text{in}}-\varphi)} - \frac{1}{4}\mathrm{i}\phi\pi(ka)^2\mathcal{K}(\pi - \theta_{\text{in}} - \varphi) \\ &= -f(\pi - \theta_{\text{in}} - \varphi) - \phi Q f(\theta_{\text{re}}) - \frac{1}{4}\mathrm{i}\phi\pi(ka)^2\mathcal{K}(\theta_{\text{re}}),\end{aligned}$$

with $f(\pi - \theta_{\text{in}} - \varphi) = f(\theta_{\text{re}}) + \frac{1}{2}\phi T\kappa_1 f'(\theta_{\text{re}}) + O(\phi^2)$ and $\theta = \pi - 2\theta_{\text{in}}$.

As $F = -1$, Eq. (39) gives

$$R = \phi R_1 + \phi^2 R_2,$$

where

$$R_1 = \frac{\mathrm{i}f(\theta_{\text{re}})}{\pi(ka)^2 C^2}, \tag{41}$$

$C = \cos\theta_{\text{in}}$, and

$$R_2 = \frac{\mathrm{i}}{\pi(ka)^2 C^2}\left[\frac{\kappa_1}{2} T f'(\theta_{\text{re}}) + \frac{\mathrm{i}}{4}\pi(ka)^2\mathcal{K}(\theta_{\text{re}}) - \frac{\kappa_1}{2C^2} f(\theta_{\text{re}})\right]. \tag{42}$$

Bose[10] has obtained similar formulas [see Eq. (24) in Ref. 10] for a slab, $0 < x < H$. They involve the quantity $e^{2\mathrm{i}kHC}$, so that the limit $H \to \infty$ cannot be taken. It appears that this is due to the use of a "Born-type approximation" at an early stage in the analysis.

The estimate $R = \phi R_1$ agrees with the Foldy estimate, R_F [see Eq. (34)], but only for isotropic scattering [where $f(\theta)$ does not depend on θ]: particularly, using $g = f(0)$ gives an incorrect result.

At normal incidence, we have $\theta_{\text{in}} = 0$, $\theta_{\text{re}} = \pi$, $C = 1$, and $T = 0$, whence

$$R_1 = \frac{\mathrm{i}f(\pi)}{\pi(ka)^2} \quad \text{and} \quad R_2 = -\frac{\mathcal{K}(\pi)}{4} - \frac{2f(0)f(\pi)}{[\pi(ka)^2]^2}, \tag{43}$$

where we have used Eq. (22) for κ_1. The formula for R_1 agrees with an estimate from Angel *et al.*[11] (see Appendix A for comparisons with the work of Aristégui, Angel, and their colleagues).

VII. THE AVERAGE TRANSMITTED FIELD

The field in the region to the right of $x = 0$ is given by Eq. (4) outside the cylinders. Inside the jth cylinder, the field, u_j, is given by LM(48),

$$u_j = \sum_{n=-\infty}^{\infty} B_n^j J_n(k_0 r_j)\,\mathrm{e}^{\mathrm{i}n\theta_j}, \quad r_j < a. \tag{44}$$

The fact that there are different expansions in different regions makes the calculation of $\langle u(x,\ y)\rangle$ for $x>0$ less straightforward than when $x<0$.

In Appendix B, it is shown that

$$\langle u(x,y)\rangle = (1-\phi)u_{\rm in} + \langle u(x,y)\rangle_{\rm ext} + \langle u(x,y)\rangle_{\rm int}, \quad (45)$$

for $x>a$, where

$$\langle u(x,y)\rangle_{\rm ext} = n_0 \sum_{n=-\infty}^{\infty} Z_n \iint_{x_1>0,\, r_1>a} \langle A_n^1\rangle_1 H_n(kr_1) \times {\rm e}^{{\rm i}n\theta_1}\,{\rm d}x_1\,{\rm d}y_1, \quad (46)$$

$$\langle u(x,y)\rangle_{\rm int} = n_0 \sum_{n=-\infty}^{\infty} \iint_{r_1<a} \langle B_n^1\rangle_1 J_n(k_0 r_1){\rm e}^{{\rm i}n\theta_1}\,{\rm d}x_1\,{\rm d}y_1. \quad (47)$$

(As $x>a$, the disc $r_1<a$ lies in $x_1>0$.)

A. Calculation of $\langle u\rangle_{\rm ext}$

From Eqs. (10) and (11),

$$\langle A_n^1\rangle_1 = {\rm i}^n F_n\, {\rm e}^{-{\rm i}n\varphi}\, {\rm e}^{{\rm i}(\lambda x_1+\beta y_1)}. \quad (48)$$

Substitution in Eq. (46) gives

$$\langle u(x,y)\rangle_{\rm ext} = n_0 \sum_{n=-\infty}^{\infty} {\rm i}^n F_n Z_n {\rm e}^{-{\rm i}n\varphi} \times \iint_{x_1>0,\, r_1>a} {\rm e}^{{\rm i}(\lambda x_1+\beta y_1)} H_n(kr_1)\, {\rm e}^{{\rm i}n\theta_1}\,{\rm d}x_1\,{\rm d}y_1.$$

The double integral is similar to M_n in Sec. IV B of LM. Its value is found to be

$$\frac{2{\rm i}(-{\rm i})^n}{\alpha(\lambda-\alpha)}{\rm e}^{{\rm i}(\alpha x+\beta y)}\,{\rm e}^{{\rm i}n\theta_{\rm in}} + \frac{2\pi(-{\rm i})^n}{k^2-K^2}{\rm e}^{{\rm i}(\lambda x+\beta y)}{\rm e}^{{\rm i}n\varphi}\mathcal{N}_n(Ka,ka),$$

where

$$\mathcal{N}_n(Ka,ka) = kaH_n'(ka)J_n(Ka) - KaH_n(ka)J_n'(Ka).$$

Hence

$$\langle u(x,y)\rangle_{\rm ext} = \mathcal{P}{\rm e}^{{\rm i}(\alpha x+\beta y)} + \mathcal{Q}_{\rm ext}{\rm e}^{{\rm i}(\lambda x+\beta y)},$$

where

$$\mathcal{P} = \frac{2{\rm i}n_0}{\alpha(\lambda-\alpha)} \sum_{n=-\infty}^{\infty} F_n Z_n {\rm e}^{{\rm i}n(\theta_{\rm in}-\varphi)},$$

$$\mathcal{Q}_{\rm ext} = \frac{2\pi n_0}{k^2-K^2} \sum_{n=-\infty}^{\infty} F_n Z_n \mathcal{N}_n(Ka,ka).$$

Combining Eqs. (26) and (30) shows that

$$\mathcal{P} = -1 + O(\phi^2);$$

there is no linear term in ϕ. Thus,

$$(1-\phi)u_{\rm in} + \mathcal{P}{\rm e}^{{\rm i}(\alpha x+\beta y)} = -\phi{\rm e}^{{\rm i}(\alpha x+\beta y)} + O(\phi^2) \quad \text{as } \phi\to 0.$$

For $\mathcal{Q}_{\rm ext}$, use $K^2-k^2=\lambda^2-\alpha^2$, Eqs. (26) and (40), and $\alpha=kC$ to obtain

$$\frac{2\pi n_0}{k^2-K^2} = -\frac{2}{(ka)^2\kappa_1}\left(1-\phi\frac{\kappa_2}{\kappa_1}\right) + O(\phi^2).$$

From LM(72),

$$\mathcal{N}_n(Ka,ka) = (2{\rm i}/\pi)\left[1-\frac{1}{4}{\rm i}\phi\pi(ka)^2\kappa_1 d_n(ka)\right] + O(\phi^2),$$

where d_n is defined by Eq. (15). Then, using Eqs. (27) and (29),

$$\begin{aligned}\sum_{n=-\infty}^{\infty} F_n Z_n \mathcal{N}_n &= \frac{2}{{\rm i}\pi}\sum_{n=-\infty}^{\infty} Z_n(1+\phi Q) \times\left[1-\frac{{\rm i}}{4}\phi\pi(ka)^2\kappa_1 d_n(ka)\right] - \frac{\phi}{2}(ka)^2\kappa_2\\ &= -\frac{1}{2}(ka)^2\kappa_1\left\{1+\phi Q+\phi\frac{\kappa_2}{\kappa_1} + \phi\sum_{n=-\infty}^{\infty} Z_n d_n(ka)\right\}.\end{aligned}$$

Hence,

$$\mathcal{Q}_{\rm ext} = 1+\phi Q+\phi\sum_{n=-\infty}^{\infty} Z_n d_n(ka) + O(\phi^2).$$

B. Calculation of $\langle u\rangle_{\rm int}$

Next, consider $\langle u\rangle_{\rm int}$, defined by Eq. (47) in terms of the coefficients B_n^1 in Eq. (44). In Sec. IV A of LM, a linear system for A_n^j was obtained by applying the pair of transmission conditions, Eq. (8), on each cylinder followed by elimination of B_n^j. Those calculations also yield a simple relation between A_n^j and B_n^j, namely

$$B_n^j = c_n A_n^j, \quad \text{with} \quad c_n = \frac{2(\rho_0/\rho)}{\pi{\rm i}ka\Delta_n};$$

for $\Delta_{\rm n}$, see Eq. (7). Using this relation and Eq. (48) in Eq. (47) gives

$$\begin{aligned}\langle u(x,y)\rangle_{\rm int} &= n_0\sum_{n=-\infty}^{\infty} {\rm i}^n F_n c_n {\rm e}^{-{\rm i}n\varphi} \times\iint_{r_1<a} {\rm e}^{{\rm i}(\lambda x_1+\beta y_1)} J_n(k_0 r_1)\,{\rm e}^{{\rm i}n\theta_1}\,{\rm d}x_1\,{\rm d}y_1\\ &= n_0{\rm e}^{{\rm i}(\lambda x+\beta y)}\sum_{n=-\infty}^{\infty} F_n c_n I_n = \mathcal{Q}_{\rm int}{\rm e}^{{\rm i}(\lambda x+\beta y)},\end{aligned}$$

say, where

$$\begin{aligned} I_n &= (-\mathrm{i})^n \mathrm{e}^{-\mathrm{i}n\varphi} \int_0^a \int_0^{2\pi} \mathrm{e}^{\mathrm{i}K\mathcal{R}\cos(\Theta-\varphi)} J_n(k_0\mathcal{R})\,\mathrm{e}^{\mathrm{i}n\Theta}\,\mathcal{R}\,\mathrm{d}\Theta\,\mathrm{d}\mathcal{R} \\ &= 2\pi\int_0^a J_n(K\mathcal{R})J_n(k_0\mathcal{R})\,\mathcal{R}\,\mathrm{d}\mathcal{R} \\ &= 2\pi(K^2-k_0^2)^{-1}\mathcal{M}_n(Ka,k_0a) \end{aligned}$$

and

$$\mathcal{M}_n(Ka,k_0a) = k_0aJ_n'(k_0a)J_n(Ka) - KaJ_n(k_0a)J_n'(Ka). \tag{49}$$

Hence,

$$\mathcal{Q}_{\mathrm{int}} = 2\phi\sum_{n=-\infty}^{\infty} c_n \frac{\mathcal{M}_n(ka,k_0a)}{(k_0a)^2-(ka)^2} + O(\phi^2).$$

C. Synthesis

Substituting the results for $\langle u\rangle_{\mathrm{ext}}$ and $\langle u\rangle_{\mathrm{int}}$ back in Eq. (45) then gives the transmitted field as

$$\langle u(x,y)\rangle = \mathcal{A}\,\mathrm{e}^{\mathrm{i}(\lambda x+\beta y)} - \phi\,\mathrm{e}^{\mathrm{i}(\alpha x+\beta y)}, \tag{50}$$

with

$$\mathcal{A} = 1 + \phi\mathcal{A}_1 + O(\phi^2), \tag{51}$$

$$\begin{aligned} \mathcal{A}_1 = \frac{\mathrm{i}f(0)}{\pi(ka)^2C^2} + \sum_{n=-\infty}^{\infty} Z_n d_n(ka) \\ + 2\sum_{n=-\infty}^{\infty} c_n\frac{\mathcal{M}_n(ka,k_0a)}{(k_0a)^2-(ka)^2}. \end{aligned} \tag{52}$$

The first term in Eq. (52) constitutes the Foldy estimate [see $\mathcal{A}_{\mathrm{F}}$, given by Eq. (34)] if we take $g = f(0)$, but it is seen here that the correct estimate of $\mathcal{A}$ at $O(\phi)$ contains two additional terms.

Notice that the dependence of $\mathcal{A}_1$ on the angle of incidence appears only in the first term, via $C = \cos\theta_{\mathrm{in}}$.

The formula for the transmitted field, Eq. (50) with Eqs. (51) and (52), is surprisingly complicated, especially as it is only first order in ϕ. (Indeed, the analysis above does not give any information on the $O(\phi^2)$ contribution, unlike in Sec. VI where we obtained the second-order contribution to R.) Fortunately, we can check our calculations with an independent analysis that is valid for weak scattering: we do this next.

VIII. WEAK SCATTERING

The term "weak scattering" means here that

$$\rho = \rho_0 \quad \text{and} \quad |m_0| \ll 1, \quad \text{where} \quad m_0 = 1-(k_0/k)^2.$$

Martin and Maurel[5] (MM) have given results for weak scattering, correct to second order in both ϕ and m_0; their paper and formulas taken from it will be identified by MM below. Particularly, MM confirms the LM formula for K^2 and it contains an estimate for the transmitted field when $\theta_{\mathrm{in}} = 0$, MM(5.25),

$$\langle u\rangle = \mathcal{A}_{\mathrm{MM}}\mathrm{e}^{\mathrm{i}Kx} \quad \text{with} \tag{53}$$

$$\mathcal{A}_{\mathrm{MM}} = 1 + \frac{1}{4}m_0\phi + \frac{1}{4}m_0^2\{P_0 - \pi(ka)^2\mathcal{H}\}\phi + O(\phi^2), \tag{54}$$

where

$$\mathcal{H} = \frac{\mathrm{i}}{4}\sum_{n=-\infty}^{\infty}\mathcal{J}_n d_n, \tag{55}$$

$$\mathcal{J}_n = J_n^2 - J_{n-1}J_{n+1} = J_n'^2 - \{[n/(ka)]^2 - 1\}J_n^2, \tag{56}$$

$$4P_0 = (ka)^2 + 2\pi\mathrm{i}(ka)^2\sum_{n=-\infty}^{\infty}\mathcal{J}_n(J_nH_n - d_n), \tag{57}$$

and all functions have argument ka. (MM also contains a formula for the $O(\phi^2)$ correction to $\mathcal{A}_{\mathrm{MM}}$.) Here, Eq. (54) will be compared with the estimate found in Sec. VII, Eq. (50) with Eqs. (51) and (52).

From MM(2.24), we have an estimate for $f(0)$ that can be used in the first term in Eq. (52) (with $C = 1$),

$$T_1 \equiv \frac{\mathrm{i}f(0)}{\pi(ka)^2} = \frac{1}{4}m_0 - \frac{1}{4}m_0^2\pi(ka)^2\mathcal{H};$$

these contributions can be seen in Eq. (54).

From MM(2.18), we have an estimate for Z_n that can be used in the second term in Eq. (52),

$$\begin{aligned} T_2 &\equiv \sum_{n=-\infty}^{\infty} Z_n d_n \\ &= m_0\pi(ka)^2\mathcal{H} - \frac{m_0^2}{16}\sum_{n=-\infty}^{\infty}\pi ka\{\mathrm{i}S_n - ka\mu_n\mathcal{J}_n\}d_n, \end{aligned} \tag{58}$$

where $\mu_n = \pi(ka)^2 d_n$ and $S_n = 2kaJ_{n-1}J_{n+1}$.

The third term in Eq. (52), denoted by T_3, is more complicated. To begin, MM(2.12) and MM(2.14) give

$$\begin{aligned} c_n &= 2/(\pi\mathrm{i}ka\Delta_n) \\ &= -1 + \frac{1}{4}\mathrm{i}m_0\mu_n + \frac{1}{16}m_0^2(\mu_n^2 - \pi\mathrm{i}kaU_n), \end{aligned} \tag{59}$$

where $U_n = 2ka(J_nH_n - d_n) + 2[i/(\pi ka)][n^2 - (ka)^2]$. Then, as $(k_0a)^2 - (ka)^2 = -m_0(ka)^2$ and $\mathcal{M}_n(ka,ka) = 0$, it is necessary to expand $\mathcal{M}_n(ka,k_0a)$ to third order in m_0. Thus, from Eqs. (6), (7), and (49),

$$\begin{aligned} \mathcal{M}_n(ka,k_0a) &= -ka\,\mathrm{Re}\,\Delta_n \\ &= \frac{1}{2}m_0(ka)^2\left\{\mathcal{J}_n - \frac{1}{4}m_0S_n/(ka) + \frac{1}{16}m_0^2\Omega_n\right\}, \end{aligned} \tag{60}$$

where the first two terms can be found below MM(2.15),

$$\Omega_n = \frac{2}{3}(8+n^2-z^2)J_n'^2 + \frac{4}{3}zJ_nJ_n' \\ -\frac{2}{3}\{(n^2-z^2)(n^2-z^2+8)+8z^2\}z^{-2}J_n^2, \quad (61)$$

and we have written $z \equiv ka$. Then, using Eqs. (59) and (60), the third term in Eq. (52) becomes

$$T_3 = \sum_{n=-\infty}^{\infty}\left[1-\frac{\mathrm{i}}{4}m_0\mu_n - \frac{m_0^2}{16}(\mu_n^2 - \pi\mathrm{i}kaU_n)\right] \\ \times\left[\mathcal{J}_n - \frac{m_0}{4ka}S_n + \frac{m_0^2}{16}\Omega_n\right] \\ = \sum_{n=-\infty}^{\infty}\mathcal{J}_n - \frac{m_0}{4}\sum_{n=-\infty}^{\infty}\left\{\mathrm{i}\mu_n\mathcal{J}_n + \frac{1}{ka}S_n\right\} \\ + \frac{m_0^2}{16}\sum_{n=-\infty}^{\infty}\left\{\mathrm{i}\mu_n\frac{S_n}{ka} - \mathcal{J}_n\mu_n^2 + \pi\mathrm{i}ka\mathcal{J}_nU_n + \Omega_n\right\}.$$

As $\sum_n \mathcal{J}_n = 1$ and $\sum_n S_n = 0$ [see Sec. 2.2 of MM or Eq. (C1)],

$$T_3 = 1 - m_0\pi(ka)^2\mathcal{H} \\ + \frac{m_0^2}{16}\sum_{n=-\infty}^{\infty}\left\{\mathrm{i}\mu_n\frac{S_n}{ka} - \mathcal{J}_n\mu_n^2\right\} + \frac{m_0^2}{16}\mathcal{S}_1, \quad (62)$$

where $\mathcal{S}_1 = \sum_n(\pi\mathrm{i}ka\mathcal{J}_nU_n + \Omega_n)$ and we have used Eq. (55). Substituting for U_n and comparison with Eq. (57) gives $\mathcal{S}_1 = 4P_0 + \mathcal{S}_2$, where

$$\mathcal{S}_2 = -(ka)^2 + \sum_{n=-\infty}^{\infty}\{2[(ka)^2 - n^2]\mathcal{J}_n + \Omega_n\} \\ = \frac{1}{2}(ka)^2 + \sum_{n=-\infty}^{\infty}\Omega_n, \quad (63)$$

after use of Eq. (C2). Hence, adding Eqs. (58) and (62),

$$T_2 + T_3 = 1 + \frac{1}{4}m_0^2P_0 + \frac{1}{16}m_0^2\mathcal{S}_2.$$

It is shown in Appendix C that $\mathcal{S}_2 = 0$. Thus, for weak scattering and normal incidence,

$$\phi\mathcal{A}_1 = \phi(T_1 + T_2 + T_3) = (\mathcal{A}_{\mathrm{MM}} - 1) + \phi,$$

which gives

$$\langle u\rangle = \mathcal{A}_{\mathrm{MM}}\mathrm{e}^{\mathrm{i}Kx} + \phi(\mathrm{e}^{\mathrm{i}Kx} - \mathrm{e}^{\mathrm{i}kx}). \quad (64)$$

This shows agreement with the MM estimate, correct to first order in ϕ and second order in m_0. Note that the method used in MM is based on an iterative solution of the governing Lippmann–Schwinger equation. It leads to an expression of the form

$$\langle u\rangle = \mathrm{e}^{\mathrm{i}kx}(\text{polynomial in } x),$$

which is then set equal to $\mathcal{A}_{\mathrm{MM}}\mathrm{e}^{\mathrm{i}Kx}$; expanding about $x=0$ leads to expressions for both K and $\mathcal{A}_{\mathrm{MM}}$. If this process is applied to Eq. (64), it is easily seen that the last term is $O(\phi^2)$ and so should be ignored if the goal is to determine the amplitude correct to first order in ϕ.

IX. EFFECTIVE INTERFACE CONDITIONS

In this section, the fields near the "interface" at $x=0$ are investigated, working to first order in ϕ.

The average total field in $x<0$, evaluated at $x=0$, u_-, is

$$u_- = (1+\phi R_1)\mathrm{e}^{\mathrm{i}\beta y}$$

and the corresponding x-derivative, u'_-, is

$$u'_- = \mathrm{i}kC(1-\phi R_1)\mathrm{e}^{\mathrm{i}\beta y}.$$

From Eq. (50), the transmitted field at $x=\delta$, say, u_+, is

$$u_+ = \{(1+\phi\mathcal{A}_1)\mathrm{e}^{\mathrm{i}(\lambda-\alpha)\delta} - \phi\}\mathrm{e}^{\mathrm{i}\alpha\delta}\mathrm{e}^{\mathrm{i}\beta y} \\ = \{1+\mathrm{i}(\lambda-\alpha)\delta + \phi\mathcal{A}_1 - \phi\}\mathrm{e}^{\mathrm{i}\alpha\delta}\mathrm{e}^{\mathrm{i}\beta y} \\ = \left\{1+\phi\left[\frac{1}{2}\mathrm{i}k\delta\kappa_1/C + \mathcal{A}_1 - 1\right]\right\}\mathrm{e}^{\mathrm{i}\alpha\delta}\mathrm{e}^{\mathrm{i}\beta y} \quad (65)$$

and the corresponding x-derivative, u'_+, is

$$u'_+ = \mathrm{i}kC\left\{(\lambda/\alpha)(1+\phi\mathcal{A}_1)\mathrm{e}^{\mathrm{i}(\lambda-\alpha)\delta} - \phi\right\}\mathrm{e}^{\mathrm{i}\alpha\delta}\mathrm{e}^{\mathrm{i}\beta y} \\ = \mathrm{i}kC\left\{1+\phi\left[\frac{1}{2}\kappa_1/C^2 + \frac{1}{2}\mathrm{i}k\delta\kappa_1/C + \mathcal{A}_1 - 1\right]\right\}\mathrm{e}^{\mathrm{i}\alpha\delta}\mathrm{e}^{\mathrm{i}\beta y}. \quad (66)$$

To estimate these quantities, suppose further that $ka \ll 1$ and $k_0a \ll 1$. Then, the terms containing $k\delta\kappa_1$ in Eqs. (65) and (66) are smaller than the other terms (see below): ignoring them and letting $\delta \to 0$ gives

$$u_+ - u_- = \phi(\mathcal{A}_1 - R_1 - 1)\mathrm{e}^{\mathrm{i}\beta y}, \quad (67)$$

$$u'_+ - u'_- = \phi\mathrm{i}kC\left(\mathcal{A}_1 + R_1 + \frac{1}{2}\kappa_1/C^2 - 1\right)\mathrm{e}^{\mathrm{i}\beta y}. \quad (68)$$

These give the discontinuities in $\langle u\rangle$ and its normal derivative across $x=0$.

For small ka and k_0a, Z_0 and $Z_{\mp 1}$ are dominant, in general, and they are $O((ka)^2)$ (see Sec. III A in Ref. 6). Thus, $f(\theta) \simeq -Z_0 - 2Z_1\cos\theta$.

From Eq. (22), $\kappa_1 = -4\mathrm{i}f(0)/[\pi(ka)^2] = O(1)$ as $ka \to 0$, which justifies discarding the $k\delta\kappa_1$ terms above.

From Eq. (41), $f(\theta_{\mathrm{re}}) = f(\pi - 2\theta_{\mathrm{in}})$ is needed to calculate R_1. As $\cos\theta_{\mathrm{re}} = 1 - 2C^2$,

$$R_1 = \frac{\mathrm{i}}{\pi(ka)^2C^2}[-Z_0 - 2Z_1(1-2C^2)] \\ = \frac{\mathrm{i}f(0)}{\pi(ka)^2C^2} + \frac{4\mathrm{i}Z_1}{\pi(ka)^2}.$$

For $\mathcal{A}_1$, use Eq. (52), containing three terms. For the second term, use $d_n(ka) \sim 2\mathrm{i}|n|/[\pi(ka)^2]$ [see above LM(82)] to obtain $\sum_n Z_n d_n(ka) \sim 4\mathrm{i}Z_1/[\pi(ka)^2]$. For the third term, use

$$\mathcal{M}_n(ka, k_0a) \sim (ka)^n (k_0a)^n \frac{(ka)^2 - (k_0a)^2}{2^{2n+1} n!\,(n+1)!}, \quad n \geq 0,$$

with $\mathcal{M}_{-n} = -\mathcal{M}_n$. Also, $c_0 \sim -1$ and $c_n \sim -2(k/k_0)^n$ for $n > 0$, with $c_{-n} = -c_n$. Hence, the dominant contribution to the third term in Eq. (52) comes from $n = 0$; as $\mathcal{M}_0(ka, k_0a) \sim \frac{1}{2}[(ka)^2 - (k_0a)^2]$,

$$\mathcal{A}_1 = \frac{\mathrm{i}f(0)}{\pi(ka)^2 C^2} + \frac{4\mathrm{i}Z_1}{\pi(ka)^2} + 1.$$

Use of these approximations for R_1 and $\mathcal{A}_1$ gives

$$\mathcal{A}_1 = 1 + R_1, \tag{69}$$

so that Eq. (67) gives $u_+ - u_- = O(\phi^2)$. In other words, there is no discontinuity in $\langle u \rangle$ across $x = 0$, for any angle of incidence.

Similarly, from Eq. (68),

$$u'_+ - u'_- = -\phi kC \frac{8Z_1}{\pi(ka)^2} \mathrm{e}^{\mathrm{i}\beta y} \sim 2\phi \mathrm{i}kC \frac{\rho_0 - \rho}{\rho_0 + \rho} \mathrm{e}^{\mathrm{i}\beta y}, \tag{70}$$

using Eq. (23) from Ref. 6. Thus, at this level of approximation, there is a jump in the normal (x) derivative of $\langle u \rangle$ across $x = 0$ (unless $\rho_0 = \rho$). Moreover, for normal incidence, it is seen that Eq. (70) agrees with the estimates of Aristégui and Angel[12] [see Eq. (A3)], in the low-frequency, small-ϕ limit.

As a reviewer noted, the discontinuity in slope at $x = 0$ could be used to predict the effective density, ρ_{eff}, of the effective medium occupying $x > 0$:

$$\rho^{-1} u'_- = \rho_{\mathrm{eff}}^{-1} u'_+. \tag{71}$$

Using the estimates for $u'_\pm$ given above,

$$\frac{\rho_{\mathrm{eff}}}{\rho} = \frac{1 + \phi(R_1 + \frac{1}{2}\kappa_1/C^2)}{1 - \phi R_1} \sim 1 + \phi\left(2R_1 + \frac{\kappa_1}{2C^2}\right).$$

Substituting for R_1 and κ_1 gives

$$\rho_{\mathrm{eff}}/\rho \sim 1 + 8\mathrm{i}\phi Z_1/[\pi(ka)^2] \sim 1 - 2\phi(\rho - \rho_0)/(\rho + \rho_0),$$

in agreement with Ament's formula for the effective density; see Eq. (11) in Ref. 6. This agreement provides a further check on the calculations.

X. CONCLUSIONS

A plane wave is incident on a half-space containing a dilute random arrangement of identical scatterers. An expression for the average reflection coefficient has been derived: it involves the far-field pattern for a single scatterer. The average field within the half-space has also been calculated: it is found that a small amount of the incident wave penetrates [the second term on the right-hand side of Eq. (50)]. This result was checked by comparing with an independent calculation, valid for weak scattering (and normal incidence). Effective interface conditions at the boundary of the half-space were also obtained. It is anticipated that extensions to three-dimensional problems can be made.

APPENDIX A: ARISTÉGUI AND ANGEL

Aristégui, Angel, and their colleagues have written several papers[11–14] in which waves are normally incident on a finite slab, $-h < y_2 < h$, containing circular scatterers. Comparisons with their work, in the limit of a semi-infinite slab, will be given here.

To begin, write the averaged field as

$$U(y_2) = \begin{cases} u_0 \mathrm{e}^{\mathrm{i}ky_2} + u_0 R' \mathrm{e}^{-\mathrm{i}ky_2}, & y_2 < -h, \\ C_+ \mathrm{e}^{\mathrm{i}Ky_2} + C_- \mathrm{e}^{-\mathrm{i}Ky_2}, & -h < y_2 < h, \\ u_0 T' \mathrm{e}^{\mathrm{i}ky_2}, & y_2 > h. \end{cases}$$

Put $x = y_2 + h$, $U(y_2) = u(x)$ and $u_0 = \mathrm{e}^{ikh}$:

$$u(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R' \mathrm{e}^{2\mathrm{i}kh} \mathrm{e}^{-\mathrm{i}kx}, & x < 0, \\ C_+ \mathrm{e}^{-\mathrm{i}Kh} \mathrm{e}^{\mathrm{i}Kx} + C_- \mathrm{e}^{\mathrm{i}Kh} \mathrm{e}^{-\mathrm{i}Kx}, & 0 < x < 2h, \\ T' \mathrm{e}^{\mathrm{i}kx}, & x > 2h. \end{cases}$$

Aristégui and Angel[13] have given expressions for R', $C_\pm$ and T'. Using these gives

$$R \equiv R' \mathrm{e}^{2\mathrm{i}kh} = \mathrm{e}^{2\mathrm{i}(k-K)h}\left(1 - \mathrm{e}^{4\mathrm{i}Kh}\right)(k^2 - K^2)/D,$$

$$\mathcal{A} \equiv C_+ \mathrm{e}^{-\mathrm{i}Kh} = 2k\mathrm{e}^{2\mathrm{i}(k-K)h}(K + k)/D,$$

$$\mathcal{A}_- \equiv C_- \mathrm{e}^{\mathrm{i}Kh} = 2k\mathrm{e}^{2\mathrm{i}(k-K)h} \mathrm{e}^{4\mathrm{i}Kh}(K - k)/D,$$

and $T' = 4kK/D$, where

$$D = \mathrm{e}^{2\mathrm{i}(k-K)h}\{(k + K)^2 - (k - K)^2 \mathrm{e}^{4\mathrm{i}Kh}\}.$$

Letting $h \to \infty$ (using $\mathrm{Im}\, K > 0$) gives

$$u(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx}, & x < 0, \\ \mathcal{A}\mathrm{e}^{\mathrm{i}Kx}, & x > 0, \end{cases} \tag{A1}$$

where

$$R = \frac{1 - \Theta}{1 + \Theta}, \quad \mathcal{A} = \frac{2}{1 + \Theta} = 1 + R \quad \text{and} \quad \Theta = \frac{K}{k}.$$

These agree with the Foldy estimates, Eq. (34), when $\theta_{\mathrm{in}} = 0$ and $K^2 = k^2 - 4\mathrm{i}gn_0$.

In later papers,[11,14] formulas for non-isotropic scattering were obtained, using $K^2 = k^2 - 4\mathrm{i}n_0 f(0) + O(n_0^2)$. In particular,[11]

$$\begin{aligned} \Theta &= (K/k)\left\{1 - (2\mathrm{i}n_0/k^2)[f(0) - f(\pi)]\right\}^{-1} \\ &= 1 - (2\mathrm{i}n_0/k^2) f(\pi) + O(n_0^2), \end{aligned}$$

giving

$$R = \frac{\mathrm{i}n_0}{k^2} f(\pi) \quad \text{and} \quad \mathcal{A} = 1 + R = 1 + \frac{\mathrm{i}n_0}{k^2} f(\pi). \tag{A2}$$

This expression for R agrees with Eq. (43) but the estimate for $\mathcal{A}$ is incorrect.

Evidently, Eqs. (A1) and (A2) show that $u(x)$ is continuous across $x = 0$, whereas

$$\begin{aligned} u'(0+) - u'(0-) &= \mathrm{i}K\mathcal{A} - \mathrm{i}k(1-R) \\ &= \mathrm{i}(K-k) + \mathrm{i}(K+k)R \\ &\simeq \mathrm{i}k\{(K/k) - 1 + 2R\} \\ &\simeq (2n_0/k)\{f(0) - f(\pi)\}. \end{aligned} \tag{A3}$$

These results are consistent with those found in Sec. IX.

APPENDIX B: SOME ENSEMBLE AVERAGING

In this appendix, notation from Sec. II of LM[2] is used. Start with N scatterers located at $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_N$; denote this configuration by Λ_N. The ensemble average of any quantity $F(\mathbf{r} \mid \Lambda_N)$ is defined by LM(8),

$$\langle F(\mathbf{r}) \rangle = \int_{(N)} p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, F(\mathbf{r}|\Lambda_N)\, \mathrm{d}V_{1\cdots N}, \tag{B1}$$

where the subscript (N) indicates that the integration is over N copies of the region B_N containing N scatterers, and $\mathrm{d}V_{1\cdots N} = \mathrm{d}V_1 \cdots \mathrm{d}V_N$. ($B_N$ has area N/n_0.) Similarly, the average of $F(\mathbf{r} \mid \boldsymbol{\Lambda}_N)$ over all configurations for which the first scatterer is fixed at $\mathbf{r}_1$ is given by LM(9),

$$\langle F(\mathbf{r}) \rangle_1 = \int_{(N-1)} p(\mathbf{r}_2, \ldots, \mathbf{r}_N|\mathbf{r}_1)\, F(\mathbf{r}|\Lambda_N)\, \mathrm{d}V_{2\cdots N},$$

where $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = p(\mathbf{r}_1)\, p(\mathbf{r}_2, \ldots, \mathbf{r}_N \mid \mathbf{r}_1)$ defines $p(\mathbf{r}_2, \ldots, \mathbf{r}_N \mid \mathbf{r}_1)$ and $p(\mathbf{r}) = n_0/N$.

For clarity, suppose first that $N = 2$. Equation (B1) reduces to

$$\begin{aligned} \langle F(\mathbf{r}) \rangle &= \int\!\!\int p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12} \\ &= \int\!\!\int_{r_1<a} p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12} \\ &\quad + \int\!\!\int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12}. \end{aligned} \tag{B2}$$

The first term in Eq. (B2) is

$$\int_{r_1<a} p(\mathbf{r}_1) \int p(\mathbf{r}_2|\mathbf{r}_1) F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{21} = \frac{n_0}{2} \int_{r_1<a} \langle F(\mathbf{r}) \rangle_1\, \mathrm{d}V_1.$$

The second term in Eq. (B2) is split as

$$\begin{aligned} &\int_{r_2>a} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12} \\ &\quad + \int_{r_2<a} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12}. \end{aligned} \tag{B3}$$

The first term in this expression involves integration for which both $r_1 > a$ and $r_2 > a$. In that case, an expansion of the following form is available [cf. Eq. (4)],

$$F(\mathbf{r}|\Lambda_N) = F_0(\mathbf{r}) + \sum_{j=1}^{N} F_j(\mathbf{r}|\Lambda_N), \tag{B4}$$

with $N = 2$, where F_0 does not depend on Λ_N and F_1, … ,F_N are small, $O(n_0)$. Hence

$$\begin{aligned} &\int_{r_2>a} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_{12} \\ &= F_0(\mathbf{r}) \int_{r_2>a} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, \mathrm{d}V_{12} \\ &\quad + \int_{r_1>a} \int_{r_2>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F_1\, \mathrm{d}V_{21} \\ &\quad + \int_{r_2>a} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2)\, F_2\, \mathrm{d}V_{12} \\ &\simeq F_0(\mathbf{r})(n_0/2)^2[(2/n_0) - \pi a^2]^2 \\ &\quad + \int_{r_1>a} \int p(\mathbf{r}_1, \mathbf{r}_2)\, F_1\, \mathrm{d}V_{21} \\ &\quad + \int_{r_2>a} \int p(\mathbf{r}_1, \mathbf{r}_2)\, F_2\, \mathrm{d}V_{12} \\ &\simeq (1 - n_0\pi a^2) F_0(\mathbf{r}) \\ &\quad + \frac{n_0}{2} \int_{r_1>a} \langle F_1 \rangle_1\, \mathrm{d}V_1 + \frac{n_0}{2} \int_{r_2>a} \langle F_2 \rangle_2\, \mathrm{d}V_2 \\ &= (1 - n_0\pi a^2) F_0(\mathbf{r}) + n_0 \int_{r_1>a} \langle F_1 \rangle_1\, \mathrm{d}V_1, \end{aligned}$$

using the indistinguishability of the scatterers in the last step. Here, two approximations were made, in which $O(n_0^2)$ contributions were discarded. First, the inner integrals $\int_{r_j>a} \mathrm{d}V_j$ $(j = 1, 2)$ of small quantities (F_1 or F_2) over the (large) region B_2 with a (small) hole were replaced by integrals over B_2 (no hole). Second, the term $(n_0/2)^2(\pi a^2)^2 F_0$ was ignored.

Similarly, the second term in Eq. (77) is approximately

$$\int_{r_2<a} \int p(\mathbf{r}_1, \mathbf{r}_2)\, F(\mathbf{r}|\Lambda_2)\, \mathrm{d}V_1\, \mathrm{d}V_2 = \frac{n_0}{2} \int_{r_2<a} \langle F \rangle_2\, \mathrm{d}V_2.$$

Substituting back yields

$$\begin{aligned} \langle F(\mathbf{r}) \rangle =& (1 - n_0\pi a^2) F_0(\mathbf{r}) + n_0 \int_{r_1>a} \langle F_1 \rangle_1\, \mathrm{d}V_1 \\ &+ n_0 \int_{r_1<a} \langle F(\mathbf{r}) \rangle_1\, \mathrm{d}V_1. \end{aligned} \tag{B5}$$

This is the result for $N = 2$. It holds for any $N \geq 2$, as will be shown next.

From the definition, Eq. (B1),

$$\begin{aligned} \langle F(\mathbf{r}) \rangle =& \int_{(N-1)} \int_{r_1<a} p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, F(\mathbf{r}|\Lambda_N)\, \mathrm{d}V_{1\ldots N} \\ &+ \int_{(N-1)} \int_{r_1>a} p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, F(\mathbf{r}|\Lambda_N)\, \mathrm{d}V_{1\ldots N}. \end{aligned} \tag{B6}$$

The first term is $(n_0/N)\int_{r_1<a}\langle F\rangle_1\,\mathrm{d}V_1$. The second term is split as

$$\int_{(N-2)}\int_{r_2>a}\int_{r_1>a} p(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\,F(\mathbf{r}|\Lambda_N)\,\mathrm{d}V_{1\cdots N} + \int_{(N-2)}\int_{r_2<a}\int_{r_1>a} p(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\,F(\mathbf{r}|\Lambda_N)\,\mathrm{d}V_{1\cdots N}. \tag{B7}$$

The second term in Eq. (B7) is approximately

$$\int_{r_2<a}\int_{(N-1)} p(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\,F(\mathbf{r}|\Lambda_N)\,\mathrm{d}V_{13\ldots N2} = \frac{n_0}{N}\int_{r_2<a}\langle F\rangle_2\,\mathrm{d}V_2.$$

The pattern is now clear. The splitting process is repeated on the first term in Eq. (B7). This shows that the second term in Eq. (B6) is approximately

$$\int_{r_N>a}\cdots\int_{r_1>a} p(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\,F(\mathbf{r}|\Lambda_N)\,\mathrm{d}V_{1\cdots N} + \sum_{j=2}^{N}\frac{n_0}{N}\int_{r_j<a}\langle F\rangle_j\,\mathrm{d}V_j. \tag{B8}$$

Using the expansion (B4), the first term in Eq. (B8) becomes, approximately,

$$\left(\frac{n_0}{N}\right)^N\left(\frac{N}{n_0}-\pi a^2\right)^N + \frac{n_0}{N}\sum_{j=1}^{N}\int_{r_j>a}\langle F_j\rangle_j\,\mathrm{d}V_j \simeq (1-n_0\pi a^2)F_0(\mathbf{r}) + n_0\int_{r_1>a}\langle F_1\rangle_j\,\mathrm{d}V_1.$$

Collecting up the results, Eq. (B5) is obtained again, but now for any N.

APPENDIX C: SOME SUMS OF PRODUCTS OF BESSEL FUNCTIONS

In this appendix, all functions have argument z and all sums are from $n=-\infty$ to $n=+\infty$. The basic sums are[15]

$$\sum J_n^2 = 1 \quad\text{and}\quad \sum J_nJ_{n+m} = 0,\quad m\neq 0. \tag{C1}$$

Differentiating the first of these gives $\sum J_nJ_n' = 0$.

The differential equation for $J_n(z)$ gives

$$\begin{aligned}4(n^2-z^2)J_n &= 4z^2J_n'' + 4zJ_n'\\ &= 2z^2(J_{n-1}'-J_{n+1}') + 4zJ_n'\\ &= z^2(J_{n-2}-2J_n+J_{n+2}) + 2z(J_{n-1}-J_{n+1}),\end{aligned}$$

using $2J_\nu' = J_{\nu-1}-J_{\nu+1}$. Hence

$$4\sum(n^2-z^2)J_n^2 = -2z^2.$$

Also, squaring,

$$16(n^2-z^2)^2J_n^2 = z^4(J_{n-2}^2+4J_n^2+J_{n+2}^2) + 4z^2(J_{n-1}^2+J_{n+1}^2) + \text{cross terms},$$

where "cross terms" denotes terms of the form J_mJ_n with $m\neq n$. Hence, using Eq. (C1),

$$16\sum(n^2-z^2)^2J_n^2 = 6z^4+8z^2.$$

As $4[J_n']^2 = J_{n-1}^2 - 2J_{n-1}J_{n+1} + J_{n+1}^2$,

$$2\sum J_n'^2 = 1 \quad\text{and}\quad \sum_n J_n'J_n'' = 0.$$

Differentiating the differential equation for $J_n(z)$ gives

$$(n^2-z^2)J_n' = z^2J_n''' + 3zJ_n'' + J_n' + 2zJ_n$$

whence

$$\begin{aligned}\sum(n^2-z^2)J_n'^2 &= \sum\{z^2J_n'''+3zJ_n''+J_n'+2zJ_n\}J_n'\\ &= \frac{1}{2}+z^2\sum_n J_n'''J_n'\end{aligned}$$

As $16J_n'''J_n' = -8[J_n']^2 - J_{n-1}^2 - J_{n+1}^2 + \text{cross terms}$,

$$16\sum(n^2-z^2)J_n'^2 = 8-6z^2.$$

These sums are sufficient to evaluate the sums needed in Sec. VIII. First,

$$\begin{aligned}\sum 2(z^2-n^2)\mathcal{J}_n(z) &= \frac{2}{z^2}\sum(n^2-z^2)^2J_n^2 - 2\sum(n^2-z^2)J_n'^2\\ &= \frac{1}{8z^2}(6z^4+8z^2) - \frac{1}{8}(8-6z^2) = \frac{3}{2}z^2.\end{aligned} \tag{C2}$$

Then, using Eq. (61),

$$\begin{aligned}\frac{3}{2}z^2\sum\Omega_n &= \sum[(8+n^2-z^2)z^2J_n'^2 + 2z^3J_nJ_n'\\ &\quad - \{(n^2-z^2)(n^2-z^2+8)+8z^2\}J_n^2]\\ &= 8z^2\sum J_n'^2 + z^2\sum(n^2-z^2)J_n'^2\\ &\quad + 2z^3\sum J_nJ_n' - \sum(n^2-z^2)^2J_n^2\\ &\quad - 8\sum(n^2-z^2)J_n^2 - 8z^2\sum J_n^2\\ &= 4z^2 + \frac{z^2}{16}(8-6z^2) - \frac{1}{16}(6z^4+8z^2)\\ &\quad + 4z^2 - 8z^2\\ &= -\frac{3}{4}z^4.\end{aligned}$$

Thus, $\sum\Omega_n = -z^2/2$ and so Eq. (63) gives $\mathcal{S}_2 = 0$.

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