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Scattering by defects in an exponentially graded layer and misuse of the method of images

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ABSTRACT

An inhomogeneous layer is sandwiched between homogeneous half-spaces. The layer contains a defect. There are incident waves and the problem is to calculate the scattered waves. Five recent publications are criticised, mainly because of their misuse of the method of images.

1. Introduction

The method of images is a classical technique whereby the effects of a boundary are found by introducing certain image solutions (Jackson, 1975, Section 2.1). The simplest applications concern Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in the region $x > 0$ with a rigid wall at $x = 0$, so that $\partial u/\partial x = 0$ on the wall. For example, for a source at $(x,y) = (x_0,0)$, with $x_0 > 0$, a solution is

$$u(x,y) = U(x,y) + U(-x,y) \quad \text{with} \quad U(x,y) = \log\left\{(x-x_0)^2 + y^2\right\}; \tag{1}$$

the term $U(x,y)$ is the solution in the absence of the wall and the term $U(-x,y)$ is the (mirror) image. In generalizing the method to other partial differential equations (PDEs) (but the same boundary condition), it should be noted that the image term, $U(-x,y)$, must also satisfy the same PDE as $U(x,y)$, for $x > 0$. Overlooking this condition implies that several published treatments of scattering by defects in graded materials are incorrect. Undoubtedly, the associated physical problems are of interest, so it seems worthwhile to discuss how they can be solved correctly: this is the main purpose of this short note.

The first paper to be criticised is by Fang (2008) in this journal. It was followed by three more with co-authors (Fang et al., 2009, 2010b; Yang et al., 2010). All four papers concern the scattering of waves by circular defects in an inhomogeneous layer. The layer is bonded to one or two homogeneous half-spaces, and a plane wave is normally incident on the layer from one of the half-spaces. The materials themselves are piezoelectric (Fang, 2008; Fang et al., 2009, 2010b) or purely elastic (Yang et al., 2010). (Most of our discussion will focus on the slightly simpler elastic problem, with antiplane motions (Yang et al., 2010).) Within the layer, the material properties are assumed to vary exponentially. It is claimed that the scattering problems have been solved exactly: we show below that this is false. The problems themselves can be solved by exact methods, but the solutions would be extremely complicated: it is unclear whether or not they would be valuable. In the process of discussing the four cited papers, we also extract a sensible physical problem and we outline how it could be treated.

2. Formulation and critique

2.1. Formulating a problem

To start the discussion, we fix labels using Cartesian coordinates (x,y): the "left half-space" occupies $x < -h_1$ whereas the "right half-space" occupies $x > h_2$, with $h_1 > 0$ and $h_2 > 0$; see Fig. 1 for a sketch of the geometry.

According to Fig. 1 in each of the four cited papers, the incident wave comes from the left half-space. Two of the papers (Fang, 2008; Fang et al., 2010b) state that the layer is bonded to one homogeneous "material", whereas the other two (Fang et al., 2009; Yang et al., 2010) state that the layer is bonded to homogeneous "materials". All include the homogeneous right half-space. The scattering problem is easier if the left half-space is absent because then the continuity conditions across the left (welded) interface at $x = -h_1$ are replaced by a boundary condition at $x = -h_1$.

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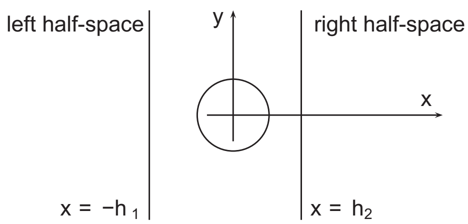

Fig. 1. The scattering problem. There is a circular cavity in an inhomogeneous layer, $-h_1 < x < h_2$. The right half-space is homogeneous, and the material properties are continuous across the interface at $x = h_2$. Initially, the left half-space is also homogeneous (with continuity conditions across $x = -h_1$). Later, it is discarded and then boundary conditions are imposed at $x = -h_1$, with a plane wave incident from the right.

However, if the left half-space is absent, we cannot then have a wave incident from the left!

2.2. Simple images

All four papers describe the line $x = -h_1$ as a free surface, and two of them (Fang et al., 2010b; Yang et al., 2010) state the boundary condition there. This condition suggests the introduction of images. Explicitly, an appeal is made to the following fact: the combination

$$u(x,y) = U(x+h_1,y) + U(-x-h_1,y)$$
$$\text{satisfies} \quad \frac{\partial u}{\partial x} = 0 \text{ on } x = -h_1, \tag{2}$$

for any smooth function U. (Compare with Eq. (1).) However, to use this fact, we must also ensure that both $U(x+h_1,y)$ and $U(-x-h_1,y)$ satisfy the governing PDE in the region of interest, which in our case is $x > -h_1$. We shall see that the nature of the inhomogeneity in the layer (see Eq. (3) below) means that, in general, if $u(x,y)$ is a solution, then $u(-x,y)$ is not a solution (see Eq. (5) below): simple images cannot be used, not even at the free surface of a graded material.

2.3. Exponential grading

At this stage, it appears that we have discarded the left-half space. However, all four papers proceed as follows. Suppose that the shear modulus, $\mu(x)$, and the density, $\rho(x)$, vary exponentially in the layer,

$$\mu(x) = \mu_0 e^{2\beta x} \quad \text{and} \quad \rho(x) = \rho_0 e^{2\beta x}, \tag{3}$$

where μ_0, ρ_0 and β are constants. (This is the formulation in Yang et al. (2010); the other three papers consider piezoelectic materials, so there are then four material properties to be modelled as exponentials.) Suppose that the right half-space has shear modulus μ_2 and density ρ_2. Enforcing continuity of the material properties across the right interface gives

$$\mu_2 = \mu_0 e^{2\beta h_2} \quad \text{and} \quad \rho_2 = \rho_0 e^{2\beta h_2}. \tag{4}$$

Now, let us *define* $\mu_1 = \mu(-h_1)$ and $\rho_1 = \rho(-h_1)$. Then, a short calculation gives $2\beta(h_1 + h_2) = \log(\mu_2/\mu_1) = \log(\rho_2/\rho_1)$. All four papers regard the second equality as implying an *assumption* on the material properties. This would be correct if μ_1 and ρ_1 represented specified properties of the left half-space and continuity was being enforced across $x = -h_1$ (or if one wanted to specify the values of both μ and ρ at the boundary, $x = -h_1$).

So, now we have discarded the left half-space, and we use Eqs. (3) and (4) within the layer, with continuity across the welded interface at $x = h_2$ and a boundary condition at $x = -h_1$.

2.4. An incident field

To obtain a meaningful scattering problem, we define a sensible incident field. For a plane wave coming from the right, we can write (the antiplane component of the displacement as)

$$u(x,y) = e^{-ik(x-h_2)} + \mathscr{R} e^{ik(x-h_2)}, \quad x > h_2,$$

where $k^2 = \rho_2\omega^2/\mu_2 = \rho_0\omega^2/\mu_0$, the time-dependence is $e^{-i\omega t}$ and $\mathscr{R}$ is the reflection coefficient: energy conservation implies that $|\mathscr{R}| = 1$. Within the layer,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\beta \frac{\partial u}{\partial x} + k^2 u = 0; \tag{5}$$

solutions have the form $u(x,y) = e^{-\beta x} w(x,y)$ where w solves a two-dimensional Helmholtz equation, $(\nabla^2 + \kappa^2)w = 0$, $\kappa = \sqrt{k^2 - \beta^2}$ and it is assumed that κ is real and positive. Thus, we write

$$u(x,y) = \mathscr{A}\left\{(\beta + i\kappa)e^{-(\beta - i\kappa)(x+h_1)} - (\beta - i\kappa)e^{-(\beta + i\kappa)(x+h_1)}\right\},$$
$$-h_1 < x < h_2;$$

this satisfies $\partial u/\partial x = 0$ at $x = -h_1$ for any choice of the constant $\mathscr{A}$. Then, we enforce continuity of u and $\partial u/\partial x$ across $x = h_2$ so as to obtain

$$\mathscr{R} = \frac{(i\beta - k)\sin\kappa h + i\kappa\cos\kappa h}{(i\beta + k)\sin\kappa h + i\kappa\cos\kappa h} \quad \text{and}$$
$$\mathscr{A} = \frac{e^{\beta h}}{(i\beta + k)\sin\kappa h + i\kappa\cos\kappa h},$$

where $h = h_1 + h_2$ is the thickness of the layer. This defines a sensible incident field, one that can be scattered by any defects that are introduced into the bimaterial structure.

It is worth noting that the solution we have constructed is not of the "image form", given in Eq. (2). By contrast, in the four cited papers, the authors take their incident field as

$$e^{-(\beta - i\kappa)(x+h_1)} + e^{(\beta - i\kappa)(x+h_1)}; \tag{6}$$

this is of the form Eq. (2) but the second term does not satisfy the governing PDE, Eq. (5). The authors also use simple mirror images to represent the field scattered by the circular defect in the presence of the free surface; again, this is erroneous.

2.5. Two special cases

We are going to consider scattering by a circular defect within the inhomogeneous layer. However, let us begin by examining two special cases. First, suppose we let $h_2 \to \infty$, giving a defect buried in a single inhomogeneous half-space with a free surface at $x = -h_1$. This problem was considered by Fang et al. (2010a). However, their use of images is erroneous; in particular, they take Eq. (6) for their incident field.

For a second special case, suppose instead that we let $h_1 \to \infty$, giving two half-spaces with an interface at $x = h_2$: the defect is within an inhomogeneous half-space. This problem has been considered in many earlier papers by Fang and his colleagues: seven of these papers are cited by Martin (2009). The emphasis is on using images to account for the presence of the interface at $x = h_2$. It was pointed out by Martin (2009) that simple images are inadequate (the interface does not behave like a mirror), and it was shown how to construct a proper image system: this system is complicated, involving various contour integrals, but it could be used to actually construct solutions, if desired.

2.6. The four cited papers

Finally, let us return to the problem of an inhomogeneous layer bonded to a homogeneous (right) half-space. Surprisingly, the four cited papers hardly mention any effects of the interface. There is one explicit statement on p. 235 of Yang et al. (2010): "At $x = h_2$, the structure is continuous, no boundary condition exists." Thus, although the earlier papers (discussed by Martin (2009)) tried to account for the interface, the later papers simply ignore it. We note that, although the material properties are continuous across the interface, their derivatives are not, and this discontinuity cannot be ignored.

3. Discussion

One could develop the approach described in Martin (2009), so as to construct special solutions that are singular at a point in the layer and that satisfy the boundary condition at $x = -h_1$ and the interface conditions at $x = h_2$. These multipole solutions would be more complicated than the multipole solutions given in Martin (2009) (because of the additional boundary condition to be satisfied). It is unclear if the effort would be worthwhile, mainly because the underlying physical model is rather simplified.

While this paper was being reviewed, another erroneous paper (Fang et al., 2011) appeared.

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