Comment on "Elastic wave propagation in a solid layer with laser-induced point defects" [J. Appl. Phys. 110, 064906 (2011)]

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Mirzade [J. Appl. Phys. **110**, 064906 (2011)] developed a linear theory for the propagation of waves in an elastic solid with atomic point defects, and then sought time-harmonic solutions. It is shown that Mirzade's analysis is incomplete: substantial corrections are required. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4747830]

The title problem concerns waves in an isotropic solid in which there are atomic point defects. The density of the defects is $n(\mathbf{r}, t)$, where $\mathbf{r} = (x_1, x_2, x_3)$ is a point in the solid. (Although Ref. 1 starts with two kinds of defects, most of the analysis is restricted to one type.) The constitutive relation between the stresses σ_{ij}, n, and the displacement components, $u_i(\mathbf{r}, t)$ $(i = 1,2,3)$, is

$$\sigma_{ij} = \lambda\delta_{ij}\Delta + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \vartheta_{\mathrm{d}} n\delta_{ij}, \tag{1}$$

where λ and μ are Lamé moduli, $\Delta = \partial u_i/\partial x_i$ is the dilatation (with the usual summation convention), and the constant ϑ_{d} controls the strain-defect interaction. We note, in passing, that Eq. (1) has the same structure as the constitutive relation for thermoelasticity, with n playing the role of temperature.

The governing equations of motion are

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1, 2, 3, \tag{2}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial Q_i}{\partial x_i} + g - \gamma n. \tag{3}$$

Here, ρ is the mass density, and g and γ are nonlinear functions of the dilatation,

$$g = \mathcal{G}\exp(\vartheta_{\mathrm{g}}\Delta/k_T), \quad \gamma = \tau^{-1}\exp(\vartheta_{\mathrm{m}}\Delta/k_T), \tag{4}$$

where $\mathcal{G}$, ϑ_{g}, $k_T = k_B T$, τ, and ϑ_{m} are constants. (In another paper, Mirzade[2] has considered a simpler problem, with $g = \mathcal{G}$ and $\gamma = \tau^{-1}$.) The defect flux has components Q_i given by[3]

$$Q_i = -D\frac{\partial n}{\partial x_i} + v_i n, \tag{5}$$

where D is a diffusion constant and the components of the defect-drift velocity are (see above Eq. (3) in Ref. 1 or above Eq. (4) in Ref. 2)

$$v_i = \frac{D}{k_T}F_i = -\frac{D}{k_T}\frac{\partial U_{\mathrm{int}}}{\partial x_i} = \frac{D\vartheta_{\mathrm{d}}}{k_T}\frac{\partial \Delta}{\partial x_i}.$$

Thus,

$$\frac{\partial Q_i}{\partial x_i} = -D\nabla^2 n + \frac{D\vartheta_{\mathrm{d}}}{k_T}\frac{\partial}{\partial x_i}\left(n\frac{\partial \Delta}{\partial x_i}\right). \tag{6}$$

To make progress, Mirzade linearizes Eq. (3). Thus, assume small strains and put $n = n_0(x, y, z) + n_1(x, y, z, t)$ with $|n_1/n_0| \ll 1$. For Eq. (2) to be satisfied at leading order, we must have $n_0 =$ constant. Then, from Eq. (3) at leading order, we obtain $n_0 = \mathcal{G}\tau$.

At next order, Eqs. (1) and (2) give

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)\frac{\partial \Delta}{\partial x_i} + \mu\nabla^2 u_i - \vartheta_{\mathrm{d}}\frac{\partial n_1}{\partial x_i}, \quad i = 1, 2, 3. \tag{7}$$

From Eqs. (3), (4), and (6), we obtain

$$\frac{\partial n_1}{\partial t} = g_{\mathrm{e}}\Delta - \tilde{g_{\mathrm{d}}}\nabla^2\Delta + D\nabla^2 n_1 - \tau^{-1}n_1, \tag{8}$$

where $\tilde{g_{\mathrm{d}}} = Dn_0\vartheta_{\mathrm{d}}/k_T$ and $g_{\mathrm{e}} = \mathcal{G}(\vartheta_{\mathrm{g}} - \vartheta_{\mathrm{m}})/k_T$. Equation (8) should be compared with Eq. (9) in Ref. 1. Notationally, our g_{e} and $\tilde{g_{\mathrm{d}}}$ are Mirzade's g and g_d, respectively. As g_{e} and $\tilde{g_{\mathrm{d}}}$ have different dimensions, for later use, we define

$$g_{\mathrm{d}} = \mathcal{G}\vartheta_{\mathrm{d}}/k_T \quad \text{giving} \quad \tilde{g_{\mathrm{d}}} = D\tau g_{\mathrm{d}}. \tag{9}$$

Using our notation, Mirzade's equation (9) has γ instead of the constant τ^{-1}; see Eq. (4). For a consistent linearization, the approximation $\gamma \simeq \tau^{-1}$ should be used.

PLANE WAVES

Mirzade[1] considers waves in a layer, in a half-space, and in an unbounded space. Here, we focus on the simplest problem of determining plane waves in an unbounded space. Mirzade introduces various potentials; we bypass this step. Thus, we try $\boldsymbol{u} = \mathrm{Re}\,\{\boldsymbol{A}\mathcal{E}\}$ and $n_1 = \mathrm{Re}\,\{\mathcal{N}\mathcal{E}\}$ with $\mathcal{E} = \exp\{i(\boldsymbol{K}\cdot\mathbf{r} - \omega t)\}$. The constant vectors $\boldsymbol{A}$ and $\boldsymbol{K}$ are allowed to be complex: they are *bivectors*.[4] Also, $\mathcal{N}$ is a complex constant. We have $\Delta = \mathrm{Re}\,\{i(\boldsymbol{A}\cdot\boldsymbol{K})\mathcal{E}\}$ and $\nabla^2\mathcal{E} = -q^2\mathcal{E}$, where

$$\begin{aligned} q^2 &= \boldsymbol{K}\cdot\boldsymbol{K} = K_1^2 + K_2^2 + K_3^2 \\ &= \boldsymbol{K}^+\cdot\boldsymbol{K}^+ - \boldsymbol{K}^-\cdot\boldsymbol{K}^- + 2i\boldsymbol{K}^+\cdot\boldsymbol{K}^- \end{aligned}$$

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and we have written $\boldsymbol{K} = (K_1, K_2, K_3) = \boldsymbol{K}^+ + i\boldsymbol{K}^-$ (see p. 16 of Ref. 4). We have used the notation q^2 so that we can compare with Ref. 1, but we emphasise that q^2 is complex unless $\boldsymbol{K}$ is real ($\boldsymbol{K} = \boldsymbol{K}^+$, $\boldsymbol{K}^- = \boldsymbol{0}$). Substitution in Eqs. (7) and (8) gives

$$-\rho\omega^2 \boldsymbol{A} = -(\lambda+\mu)(\boldsymbol{A}\cdot\boldsymbol{K})\boldsymbol{K} - \mu q^2 \boldsymbol{A} - i\vartheta_{\mathrm{d}}\mathcal{N}\boldsymbol{K},$$
$$-i\omega\mathcal{N} = -Dq^2\mathcal{N} + i(g_{\mathrm{e}} + \tilde{g}_{\mathrm{d}}q^2)(\boldsymbol{A}\cdot\boldsymbol{K}) - \tau^{-1}\mathcal{N}.$$

We seek non-trivial solutions of this system. Simplify the notation by putting $X = \mu q^2 - \rho\omega^2$, $L = \lambda + \mu$, $G = g_{\mathrm{e}} + \tilde{g}_{\mathrm{d}}q^2$, and $H = i\omega - \tau^{-1} - Dq^2$. Then, the system becomes

$$L(\boldsymbol{A}\cdot\boldsymbol{K})\boldsymbol{K} + X\boldsymbol{A} + i\vartheta_{\mathrm{d}}\mathcal{N}\boldsymbol{K} = \boldsymbol{0},$$
$$iG(\boldsymbol{A}\cdot\boldsymbol{K}) + H\mathcal{N} = 0.$$

Write this system in matrix form as $C\mathbf{x} = \mathbf{0}$ with $\mathbf{x}^T = (\mathcal{N}, A_1, A_2, A_3)$ and

$$C = \begin{pmatrix} H & iGK_1 & iGK_2 & iGK_3 \\ i\vartheta_{\mathrm{d}}K_1 & X + LK_1^2 & LK_1K_2 & LK_1K_3 \\ i\vartheta_{\mathrm{d}}K_2 & LK_1K_2 & X + LK_2^2 & LK_2K_3 \\ i\vartheta_{\mathrm{d}}K_3 & LK_1K_3 & LK_2K_3 & X + LK_3^2 \end{pmatrix}.$$

Direct calculation gives

$$\det C = X^2\Lambda \quad \text{with} \quad \Lambda = H(X + Lq^2) + \vartheta_{\mathrm{d}}Gq^2.$$

Allowable solutions follow by setting $\det C = 0$. Thus, $X^2 = 0$ or $\Lambda = 0$. The first of these gives $\omega^2 = c_{\mathrm{T}}^2 q^2$, where $c_{\mathrm{T}}^2 = \mu/\rho$ and c_{T} is the speed of transverse (shear) waves in an isotropic elastic solid: such waves propagate independently of any atomic point defects. This result was found by Mirzade; see Eq. (25) in Ref. 1.

The second option, $\Lambda = 0$, gives

$$[(\lambda + 2\mu)q^2 - \rho\omega^2](Dq^2 + \tau^{-1} - i\omega) - \vartheta_{\mathrm{d}}q^2(g_{\mathrm{e}} + \tilde{g}_{\mathrm{d}}q^2) = 0. \tag{10}$$

We compare this with Mirzade's equation (25b). Thus, introduce a length ℓ defined by $D\tau = \ell^2$ and let $c_{\mathrm{L}}^2 = (\lambda + 2\mu)/\rho$ so that c_{L} is the speed of longitudinal (compressional) waves in an isotropic elastic solid. In addition, introduce two independent dimensionless parameters, δ_{e} and δ_{d}, defined by (recall Eq. (9))

$$\delta_{\mathrm{e}} = \frac{\vartheta_{\mathrm{d}}g_{\mathrm{e}}\tau}{\lambda + 2\mu} \quad \text{and} \quad \delta_{\mathrm{d}} = \frac{\vartheta_{\mathrm{d}}g_{\mathrm{d}}\tau}{\lambda + 2\mu}. \tag{11}$$

Mirzade's δ is our δ_{e}; see below Eq. (19) in Ref. 1. Then, Eq. (10) becomes

$$(q^2 - \omega^2 c_{\mathrm{L}}^{-2})[q^2 + (1 - i\omega\tau)\ell^{-2}] - \delta_{\mathrm{e}}\ell^{-2}q^2 - \delta_{\mathrm{d}}q^4 = 0. \tag{12}$$

This should be compared with Eq. (25b) in Ref. 1, namely,

$$(q^2 - \omega^2 c_{\mathrm{L}}^{-2})[q^2 + (1 + i\omega\tau)\ell^{-2}] - \delta_{\mathrm{e}}\ell^{-2}q^2 = 0. \tag{13}$$

The difference between $(1 - i\omega\tau)$ in Eq. (12) and $(1 + i\omega\tau)$ in Eq. (13) is simply due to us assuming $e^{-i\omega t}$ and Mirzade taking $e^{+i\omega t}$. However, the most striking difference is the absence of the last term in Eq. (12). This error can be traced to Eq. (13) in Ref. 1, where a term proportional to $\nabla^4\varphi$ has been omitted. This omission implies that much of the analysis and computation in Ref. 1 for layers and half-spaces will require correction.

One could regard Eq. (13) as a special case of Eq. (12), obtained by putting $\delta_{\mathrm{d}} = 0$. However, this case is not very interesting because it implies that $\vartheta_{\mathrm{d}} = 0$, which means that there is no strain-defect interaction; see Eq. (7). In addition, $\vartheta_{\mathrm{d}} = 0$ implies that $\delta_{\mathrm{e}} = 0$ (see Eq. (11)), in which case Eq. (12) factors.

Mirzade also gives a perturbation analysis of Eq. (13) in which it is assumed that $\delta_{\mathrm{e}} \ll 1$. One could presumably give a similar analysis of Eq. (12), but this would require both $\delta_{\mathrm{e}} \ll 1$ and $\delta_{\mathrm{d}} \ll 1$.

Further analysis of the dispersion relation Eq. (12) could be interesting. It can be regarded as a cubic equation for ω, given q, or as a quadratic equation for q^2, given the frequency ω. (Recall that q^2 need not be real.) It is noted that the case $\delta_{\mathrm{d}} = 1$ is special because Eq. (12) contains a term $(1 - \delta_{\mathrm{d}})q^4$.

[1]F. Mirzade, J. Appl. Phys. **110**, 064906 (2011).
[2]F. K. Mirzade, Physica B **406**, 119 (2011).
[3]Equation (4) in Ref. 1 and Eq. (3b) in Ref. 2 give $-v_i n$ in Eq. (5), but these are typographical errors, F. Mirzade, private communication (2011).
[4]P. Boulanger and M. Hayes, *Bivectors and Waves in Mechanics and Optics* (Chapman and Hall, 1993).