SIAM J. APPL. MATH.
Vol. 72, No. 3, pp. 725–739

GENERATION OF INTERNAL GRAVITY WAVES BY AN OSCILLATING HORIZONTAL ELLIPTICAL PLATE*

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Abstract. Time-harmonic oscillations of a horizontal plate generate internal gravity waves in an unbounded stratified fluid. A plane-wave (Fourier) decomposition is used in which waves with outgoing group velocity are selected. The pressure and the velocity in the far field are estimated in terms of the Fourier transform of the pressure jump across the plate. Explicit solutions are obtained for arbitrary prescribed motions of an elliptical plate. Energy is confined to certain wave beams, bounded by conical characteristic surfaces of the underlying hyperbolic partial differential equation; the solution itself is not axisymmetric. The details of the beam structure are complicated (because each characteristic surface has a vertical axis of symmetry, whereas the elliptical plate does not), but they emerge naturally from the asymptotic method, a method that has wider applicability. Results for analogous piston problems are also obtained; these problems arise when a piece of a rigid horizontal plane is forced to oscillate, thus generating waves above the plane.

Key words. internal gravity waves, method of stationary phase, dual integral equations

AMS subject classifications. 76B55, 76B70, 41A60

DOI. 10.1137/110854965

1. Introduction.

Internal gravity waves occur in the atmosphere and in the oceans. They arise in fluids where the density varies continuously as a function of depth, and they are known to be important in the context of oceanic mixing [13], for example.

Internal waves can be generated by scattering, where an ambient oscillatory flow interacts with topography or an immersed object. They can also be generated directly by oscillating an object; we shall consider such *radiation problems* below.

The mathematical framework for small-amplitude internal gravity waves is well established; see, for example, Lighthill [19, Chapter 4], Brekhovskikh and Goncharov [5, sect. 10.4] or Vallis [27, sect. 2.4]. The simplest situation is when the fluid is incompressible, inviscid, and uniformly stratified, and there is no rotation. (These assumptions may be relaxed.) Then, with the Boussinesq approximation and time-harmonic motions, it is found that the pressure $p(x, y, z)$ satisfies a linear hyperbolic partial differential equation (PDE); see (2.1) below. This equation is to be solved subject to boundary and far-field conditions.

We shall suppose that internal waves are generated by oscillating a bounded object in an unbounded three-dimensional fluid; the effects of other boundaries (such as a horizontal plane) are also of interest, but we ignore those here. The prototype problem is when the object is a sphere [1, 10, 16, 18, 30, 31, 32]: for example, the sphere could be pulsating or it could be oscillating as a rigid body. There are also some publications on spheroids with a vertical axis of symmetry and on horizontal circular discs [18, 11, 8, 21].

*Received by the editors November 11, 2011; accepted for publication (in revised form) March 2, 2012; published electronically May 17, 2012.
http://www.siam.org/journals/siap/72-3/85496.html

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A characteristic feature of internal-wave generation is the formation of energy beams. These beams are bounded by characteristic surfaces of the governing hyperbolic PDE. These surfaces are circular cones with vertical axes of symmetry; see (2.2) below. The group velocity is directed along the beams, away from the oscillating body. The phase velocity is perpendicular to the group velocity. These known facts [19, sect. 4.10] indicate how to formulate a far-field condition.

We choose to study the generation of internal waves by an oscillating horizontal elliptical plate: we are not aware of any previous results on internal-wave generation by objects without a vertical axis of symmetry. This lack of axisymmetry is interesting because the beam structure is complicated: the beam thickness varies with azimuthal angle. To solve the problem, we use Fourier transforms in the horizontal directions, giving a plane-wave representation for p. We choose those plane waves that give outgoing group velocity. Application of the boundary condition gives dual integral equations or, alternatively, a hypersingular boundary integral equation. Exact solutions are obtained. We then give an asymptotic method for calculating the pressure and velocity in the far field; the thickness of the beam comes out of the calculations naturally. In fact, the asymptotic method is more general: it can be used for plates of any shape. It can also be used for "piston problems," where fluid occupies the region $z > 0$ above a rigid floor in which is mounted an oscillating vibrator or piston. Such problems have been studied extensively by Chashechkin and his colleagues [2, 6, 7, 28, 29]. They are easier to solve, formally, because the normal velocity is prescribed everywhere over the plane $z = 0$, whereas the plate problem leads to a mixed boundary-value problem. Our asymptotic method enables the far field of the piston to be calculated.

Section 2 begins by recalling the governing equations. The oscillating-plate problem is formulated and then reduced to integral equations. Section 3 gives the basic far-field estimates. A variant of the two-dimensional method of stationary phase is used, in which there is a line of stationary-phase points within the integration domain [34]. Subsequently, the plate is assumed to be elliptical. Exact solutions are obtained, adapting a method used previously for pressurized flat elliptical cracks in an elastic solid [20]. The method makes use of expansions in terms of Gegenbauer polynomials. The far field of the oscillating elliptical plate is calculated in section 6. Two special cases are investigated further in section 7. They are a "heaving plate," where a rigid horizontal plate oscillates vertically, and a "rolling plate," where a rigid horizontal plate makes small oscillations about its minor axis. In the last section (section 8), the problem of an elliptical piston is solved. It is shown that the energy input by the piston is balanced by the energy found in the far field. This provides a good check on the far-field analysis.

2. An oscillating horizontal plate. Consider a variable-density inviscid fluid without rotation. Under the Boussinesq approximation, the (rescaled) pressure has the form $\mathrm{Re}\,\{p(x,y,z)e^{-i\omega t}\}$, where p satisfies

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \gamma^2 \frac{\partial^2 p}{\partial z^2} = 0 \quad \text{with} \quad \gamma^2 = \frac{\omega^2}{N^2 - \omega^2}. \tag{2.1}$$

As noted in section 1, the theory behind (2.1) is given in textbooks; see [19, Chapter 4], [5, sect. 10.4] or [27, sect. 2.4]. See also [21, sect. 2] or [22, sect. 2].

We use Cartesian coordinates $Oxyz$, with z pointing upwards; ω is the oscillation frequency, and N is the (constant) Brunt–Väisälä frequency. For internal waves, we suppose that $0 < \omega < N$, and we write $\omega = N \cos\theta_{\mathrm{c}}$ with $0 < \theta_{\mathrm{c}} < \pi/2$. Then

$\gamma = \cot\theta_c$ and the differential equation (2.1) is hyperbolic; it is to be solved subject to boundary and far-field conditions (which we shall specify later).

The characteristic surfaces for (2.1) are given by (see, for example, [12, sect. 6.1])

$$(z - z_0)^2 = \{(x - x_0)^2 + (y - y_0)^2\} \cot^2\theta_c, \tag{2.2}$$

where x_0, y_0, and z_0 are constants. The surfaces defined by (2.2) are circular cones with vertical axes and generators inclined at angle θ_c to the vertical. We are interested in the generation of internal waves by objects that do not have a vertical axis of symmetry: later, we shall choose a thin horizontal elliptical plate.

The actual excess pressure in the fluid is $\rho_0 p$, where ρ_0 is the constant background density. Thus, p has dimensions of velocity squared. The velocity field, $\boldsymbol{v} = (u_x, u_y, w)$, is given in terms of p by

$$u_x = \frac{1}{i\omega}\frac{\partial p}{\partial x}, \quad u_y = \frac{1}{i\omega}\frac{\partial p}{\partial y}, \quad w = \frac{i\gamma^2}{\omega}\frac{\partial p}{\partial z}. \tag{2.3}$$

Consider a thin flat plate, Ω, in the xy-plane. Denote the rest of the xy-plane by Ω'. The plate oscillates with prescribed normal velocity on Ω, w_p. Thus

$$w = \frac{i\gamma^2}{\omega}\frac{\partial p}{\partial z} = w_p(x, y), \quad (x, y) \in \Omega. \tag{2.4}$$

As this holds on both sides of Ω, it follows that p must be an odd function of z. Hence, we can reduce the problem to one in the half-space $z > 0$. This splitting of the problem into two half-space problems means that we must also impose continuity of p across Ω'.

To solve the problem, we introduce two-dimensional (horizontal) Fourier transforms, defined by

$$F(\boldsymbol{\xi}) = \mathcal{F}[f(\boldsymbol{x}); \boldsymbol{\xi}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x}) \exp(-i\,\boldsymbol{\xi}\cdot\boldsymbol{x})\, d\boldsymbol{x}, \tag{2.5}$$

where $\boldsymbol{x} = (x, y)$ and $\boldsymbol{\xi} = (\xi, \eta)$. The corresponding inverse is

$$f(\boldsymbol{x}) = \mathcal{F}^{-1}[F(\boldsymbol{\xi}); \boldsymbol{x}] = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{\xi}) \exp(i\,\boldsymbol{\xi}\cdot\boldsymbol{x})\, d\boldsymbol{\xi}. \tag{2.6}$$

Henceforth, we write $\iint$ when the integration limits are as in (2.5) or (2.6).

Taking the Fourier transform of (2.1) with respect to $\boldsymbol{x}$ gives $P'' = -(\kappa/\gamma)^2 P$, where $P = \mathcal{F}p$, $P'' = \partial^2 P/\partial z^2$, and $\kappa = |\boldsymbol{\xi}|$. Hence,

$$P(\boldsymbol{\xi}, z) = C(\boldsymbol{\xi})\, e^{i(\kappa/\gamma)z} + D(\boldsymbol{\xi})\, e^{-i(\kappa/\gamma)z} \quad \text{for } z > 0. \tag{2.7}$$

The C term corresponds to waves propagating upwards: their phase velocity has a positive z-component. Similarly, the D term corresponds to waves propagating downwards.

If we invert, $p = \mathcal{F}^{-1}P$ with (2.7), we obtain a plane-wave decomposition of p. Explicitly, (2.1) has solutions of the form

$$p(x, y, z) = \exp\{i(k_1 x + k_2 y + k_3 z)\} \quad \text{if} \quad \omega^2 = N^2(k_1^2 + k_2^2)/K^2,$$

where $K^2 = k_1^2 + k_2^2 + k_3^2$. The corresponding group velocity, $\boldsymbol{c}_g$, is defined by

$$\boldsymbol{c}_g = (\partial\omega/\partial k_1,\ \partial\omega/\partial k_2,\ \partial\omega/\partial k_3).$$

It is given by

$$\omega \boldsymbol{c}_{\mathrm{g}} = (N/K)^2 (k_1 \sin^2\theta_{\mathrm{c}},\ k_2 \sin^2\theta_{\mathrm{c}},\ -k_3\cos^2\theta_{\mathrm{c}}).$$

Then, in the half-space $z > 0$, we insist that the group velocity be upwards, away from the plate, implying that we should take $k_3 < 0$, that is, we should take $C = 0$ in (2.7). This is our radiation condition. It has been used previously [3, 24], and it has been shown to be consistent with causality in the time-domain [21]. As a result,

$$p(\boldsymbol{x}, z) = \mathcal{F}^{-1}[D(\boldsymbol{\xi})\, e^{-i(\kappa/\gamma)z}; \boldsymbol{x}]. \tag{2.8}$$

This representation ensures that (2.1) and the radiation condition are satisfied. The boundary conditions are (2.4) and $p = 0$ on Ω'; they give

$$\mathcal{F}^{-1}[\kappa D(\boldsymbol{\xi}); \boldsymbol{x}] = (\omega/\gamma) w_{\mathrm{p}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{2.9}$$

$$\mathcal{F}^{-1}[D(\boldsymbol{\xi}); \boldsymbol{x}] = 0, \qquad \boldsymbol{x} \in \Omega'. \tag{2.10}$$

This is a pair of dual integral equations for $D(\boldsymbol{\xi})$. To interpret this quantity, define the discontinuity in p across $z = 0$ by

$$2\delta(\boldsymbol{x}) \equiv p(x, y, 0+) - p(x, y, 0-) = 2p(x, y, 0+). \tag{2.11}$$

As $\delta(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega'$,

$$\mathcal{F}[\delta(\boldsymbol{x}); \boldsymbol{\xi}] = \int_\Omega \delta(\boldsymbol{x}) \exp(-i\,\boldsymbol{\xi}\cdot\boldsymbol{x})\, d\boldsymbol{x} = \mathcal{F}[p(\boldsymbol{x}, 0+); \boldsymbol{\xi}] = D(\boldsymbol{\xi}). \tag{2.12}$$

Hence, substituting for D in (2.9) gives

$$\mathcal{F}^{-1}[|\boldsymbol{\xi}|\, \mathcal{F}\{\delta; \boldsymbol{\xi}\}; \boldsymbol{x}] = (\omega/\gamma) w_{\mathrm{p}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{2.13}$$

which is a hypersingular boundary integral equation for the pressure discontinuity, δ. Equations of this type arise in many branches of mechanics where there are thin objects, such as cracks or screens; see [4] for a review.

3. The far-field pressure. Once $D = \mathcal{F}\delta$ has been found, either by solving the dual integral equations (2.9) and (2.10) or the integral equation (2.13), the pressure is given by (2.8). Using cylindrical polar coordinates for $\boldsymbol{x}$ and $\boldsymbol{\xi}$, defined by

$$x = r\cos\phi, \quad y = r\sin\phi, \quad \xi = \kappa\cos\beta, \quad \text{and} \quad \eta = \kappa\sin\beta,$$

(2.8) becomes

$$p(\boldsymbol{x}, z) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} D(\kappa, \beta) \exp\left\{ i\kappa\left(r\cos[\beta - \phi] - \frac{z}{\gamma} \right) \right\} \kappa\, d\beta\, d\kappa. \tag{3.1}$$

In the far field, $R \equiv \sqrt{r^2 + z^2}$ is large. As we are expecting wave beams bounded by characteristic cones, (2.2), it is useful to introduce conical polar coordinates, σ and ζ, defined by (see, e.g., [31, p. 250])

$$\sigma = r\cos\theta_{\mathrm{c}} - z\sin\theta_{\mathrm{c}} \quad \text{and} \quad \zeta = r\sin\theta_{\mathrm{c}} + z\cos\theta_{\mathrm{c}}; \tag{3.2}$$

equivalently, $r = \sigma \cos\theta_c + \zeta \sin\theta_c$ and $z = -\sigma \sin\theta_c + \zeta \cos\theta_c$. The quantity σ is a lateral coordinate, across the beam. In the far field, $\zeta \to \infty$, but σ is finite; note that $\zeta \sim R$ in this limit. Then (3.1) becomes

$$p(\boldsymbol{x}, z) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} g(\kappa, \beta)\, e^{i\zeta f(\kappa,\beta)}\, d\beta\, d\kappa, \tag{3.3}$$

where

$$g(\kappa, \beta) = \kappa D(\kappa, \beta)\, e^{i\kappa\sigma\{\tan\theta_c \sin\theta_c + \cos(\beta-\phi)\cos\theta_c\}},$$
$$f(\kappa, \beta) = \{\cos(\beta - \phi) - 1\}\kappa \sin\theta_c.$$

To estimate (3.3), we use the two-dimensional method of stationary phase. We have

$$\frac{\partial f}{\partial \kappa} = \{\cos(\beta - \phi) - 1\} \sin\theta_c \quad \text{and} \quad \frac{\partial f}{\partial \beta} = -\kappa \sin(\beta - \phi) \sin\theta_c.$$

These both vanish when $\beta = \phi$, giving a line of stationary points. On this line, $f = 0$, $f_{\kappa\kappa} = 0$, $f_{\beta\beta} = -\kappa \sin\theta_c$, and $g(\kappa, \phi) = \kappa D(\kappa, \phi)\, e^{i\kappa\sigma \sec\theta_c}$. Then, from [34, Theorem 1, p. 454], we obtain the estimate

$$p \sim b_0 \zeta^{-1/2} \quad \text{with} \quad b_0 = \frac{1}{4\pi^2} \sqrt{\frac{2\pi}{\sin\theta_c}}\, e^{-i\pi/4} \int_0^\infty D(\kappa, \phi)\, e^{i\kappa\sigma \sec\theta_c} \sqrt{\kappa}\, d\kappa. \tag{3.4}$$

Within the wave beams, we can also estimate the velocity. Thus [21, sect. 7b],

$$\boldsymbol{v} \sim v\, \hat{\boldsymbol{\zeta}} \quad \text{with} \quad v = (iN \sin\theta_c)^{-1} \partial p/\partial\sigma, \tag{3.5}$$

where $\hat{\boldsymbol{\zeta}}$ is a unit vector in the ζ direction, pointing away from the plate.

At this stage, our far-field estimates for p and v are quite general: they involve D, the Fourier transform of the pressure jump across the plate Ω (see (2.12)), but the shape of Ω has not been used.

In order to obtain more explicit results, we now specialize to elliptical plates.

4. Elliptical plate.

For an elliptical plate, let

$$\Omega = \{(x, y, z) : 0 \le \rho < 1,\, 0 \le \chi < 2\pi,\, z = 0\}, \tag{4.1}$$

where

$$x = a\rho \cos\chi, \quad y = b\rho \sin\chi, \quad 0 < b \le a.$$

Thus, $(x/a)^2 + (y/b)^2 \le 1$. Note that the coordinates ρ and χ are not orthogonal.

For the Fourier transform variable $\boldsymbol{\xi}$, write

$$\xi = (\lambda/a) \cos\psi \quad \text{and} \quad \eta = (\lambda/b) \sin\psi$$

so that $\boldsymbol{\xi} \cdot \boldsymbol{x} = \lambda\rho \cos(\chi - \psi)$.

Suppose, for simplicity, that $f(\boldsymbol{x})$ is an even function of y. Then it has a Fourier expansion, $f(\boldsymbol{x}) = \sum_{m=0}^\infty f_m(\rho) \cos m\chi$. It follows that when computing $\mathcal{F}f$, we can integrate over χ, giving

$$\mathcal{F}[f(\boldsymbol{x}); \boldsymbol{\xi}] = 2\pi ab \sum_{m=0}^\infty (-i)^m \cos m\psi\, \mathcal{H}_m[f_m(\rho); \lambda], \tag{4.2}$$

where

$$\mathcal{H}_\nu[f(\rho);\lambda] = \int_0^\infty f(\rho) J_\nu(\lambda\rho)\rho\, d\rho$$

is a Hankel transform and J_ν is a Bessel function.

Now, considering (2.13), suppose that the unknown pressure jump is written as

$$\delta(\boldsymbol{x}) = a\omega\mathcal{U} \sum_{m=0}^\infty d_m(\rho)\cos m\chi, \tag{4.3}$$

where $\mathcal{U}$ is a velocity scale and the functions $d_m(\rho)$ are dimensionless. From (4.2),

$$D(\boldsymbol{\xi}) = \mathcal{F}[\delta(\boldsymbol{x});\, \boldsymbol{\xi}] = 2\pi\omega\mathcal{U}a^2 b \sum_{m=0}^\infty (-i)^m \cos m\psi\, \mathcal{H}_m[d_m(\rho);\lambda]. \tag{4.4}$$

Then, as $|\boldsymbol{\xi}| = (\lambda/b)(1-k^2\cos^2\psi)^{1/2}$, where

$$k^2 = 1-(b/a)^2,$$

(2.13) reduces to

$$w_{\mathrm{p}}(\boldsymbol{x}) = \mathcal{U}\gamma \sum_{n=0}^\infty \epsilon_n w_n(\rho)\cos n\chi, \quad 0\le\rho<1, \quad 0\le\chi<2\pi, \tag{4.5}$$

where

$$w_n(\rho) = \frac{a}{\pi b}\sum_{m=0}^\infty I_{mn}(k)\mathcal{H}_n[\lambda\mathcal{H}_m\{d_m(\rho);\lambda\};\rho], \tag{4.6}$$
$$I_{mn}(k) = \frac{1}{2}(-i)^m\, i^n \int_0^{2\pi} (1-k^2\cos^2\psi)^{1/2}\cos m\psi\cos n\psi\, d\psi,$$

$\epsilon_0 = 1$ and $\epsilon_n = 2$ for $n\ge 1$.

The integrals I_{mn} can be expressed in terms of the elliptic integrals,

$$E_m(k) = \int_0^{\pi/2} (1-k^2\sin^2 x)^{1/2}\cos 2mx\, dx.$$

Thus $I_{2m,2n} = E_{m-n}+E_{m+n}$, $I_{2m+1,2n+1} = E_{m-n}-E_{m+n+1}$ and $I_{2m,2n+1}=0$. In particular,

$$I_{00} = 2E(k) \quad \text{and} \quad I_{11} = \tfrac{2}{3}k^{-2}\{(2k^2-1)E(k)+(1-k^2)K(k)\}, \tag{4.7}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively.

5. Elliptical plate: Polynomial solutions. Our task is to solve (4.6) for d_m, given w_n. We expand $\delta(\boldsymbol{x})$ as (4.3) with

$$d_m(\rho) = H(1-\rho)\sum_{j=0}^\infty \frac{j!\,\Gamma(m+1/2)}{\Gamma(j+m+3/2)}\, D_j^m \Phi_j^{(m)}(\rho), \tag{5.1}$$

where $H(t)$ is the Heaviside unit function,

$$\Phi_m^{(n)}(\rho) = \rho^n C_{2m+1}^{n+1/2}(\sqrt{1-\rho^2}\,), \tag{5.2}$$

and C_n^μ is a Gegenbauer polynomial. (The gamma functions in (5.1) were inserted so that a later formula, (5.4), takes a simple form.) As $\Phi_m^{(n)}(\rho)$ is of the form $\sqrt{1-\rho^2}$ multiplied by a polynomial in ρ, the square-root zero at the plate edge is incorporated automatically. The Heaviside function ensures that $\delta(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega'$. The coefficients D_j^m are to be found.

Substitution in (4.4) leads to Tranter's integral,

$$\int_0^1 \Phi_m^{(n)}(x) J_n(\xi x)\, x\, dx = \frac{2\Gamma(m+n+3/2)}{m!\,\Gamma(n+1/2)} \frac{j_{n+2m+1}(\xi)}{\xi}, \tag{5.3}$$

where $j_n(x) = \sqrt{\pi/(2x)} J_{n+1/2}(x)$ is a spherical Bessel function; see [25, eq. (5)], [26, eq. (8.6)] and [17, eq. (72)]. Hence

$$\mathcal{H}_m[d_m(\rho);\lambda] = \frac{2}{\lambda} \sum_{j=0}^{\infty} D_j^m j_{2j+m+1}(\lambda). \tag{5.4}$$

Substitution of (5.1) in (4.6) requires

$$\mathcal{H}_n[\lambda \mathcal{H}_m\{d_m(\rho);\lambda\};\rho] = 2 \sum_{j=0}^{\infty} D_j^m L_{m,n}^{2j}(\rho), \tag{5.5}$$

where

$$L_{m,n}^{2j}(\rho) = \int_0^\infty \lambda J_n(\lambda\rho) j_{2j+m+1}(\lambda)\, d\lambda.$$

In (5.5), we can assume that m and n are both even or both odd (because $I_{2m,2n+1} = 0$), so we can define integers p and q by $m+n = 2p$ and $m-n = 2q$. The integral $L_{m,n}^{2j}(\rho)$ can be evaluated for $0 \le \rho < 1$; its value is zero when $j+q < 0$, whereas

$$L_{m,n}^{2j}(\rho) = \frac{\Gamma(n+1/2)\Gamma(j+q+3/2)}{(j+p)!\,\sqrt{1-\rho^2}}\, \Phi_{j+q}^{(n)}(\rho), \quad j+q \ge 0,$$

so that $L_{m,n}^{2j}$ is a polynomial when $j+q \ge 0$. Using this result in (5.5), (4.6) gives

$$\begin{aligned} w_{2n}(\rho) &= \frac{2a}{\pi b} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} I_{2m,2n} D_j^{2m} \frac{\Gamma(2n+1/2)\Gamma(m-n+j+3/2)}{(m+n+j)!\,\sqrt{1-\rho^2}}\, \Phi_{j+m-n}^{(2n)}(\rho) \\ &= \frac{2a}{\pi b} \sum_{j=0}^{\infty} \frac{\Gamma(2n+1/2)\Gamma(j+3/2)}{(2n+j)!\,\sqrt{1-\rho^2}}\, \Phi_j^{(2n)}(\rho) \sum_{m=0}^{n+j} I_{2m,2n} D_{n+j-m}^{2m}, \end{aligned}$$

where $0 \le \rho < 1$, and we have used the fact that, in the first line, the summation is over only those j and m satisfying $j+m \ge n$. There is a similar equation for w_{2n+1}. Therefore, if we write

$$w_n(\rho) = \sum_{j=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(j+3/2)}{(n+j)!\,\sqrt{1-\rho^2}} W_j^n \Phi_j^{(n)}(\rho), \tag{5.6}$$

we obtain

$$W_j^{2n} = \frac{2a}{\pi b} \sum_{m=0}^{n+j} I_{2m,2n} D_{n+j-m}^{2m} \tag{5.7}$$

for $n \geq 0$ and $j \geq 0$. Similarly,

$$W_j^{2n+1} = \frac{2a}{\pi b} \sum_{m=0}^{n+j} I_{2m+1,2n+1} D_{n+j-m}^{2m+1}. \tag{5.8}$$

These are linear systems for D_i^m in terms of W_j^n, and these coefficients are known in terms of $w_n(\rho)$. It turns out that the linear systems (5.7) and (5.8) can be truncated properly [20, sect. 6].

Once the coefficients D_i^m have been found, we can compute the pressure and velocity anywhere in the fluid.

6. Elliptical plate: Far-field pressure. For the far-field pressure, we use the results in section 3. Specifically, we use the estimate (3.4), giving p in terms of $D(\kappa, \phi)$.

Equations (4.4) and (5.4) give

$$D(\kappa, \beta) = \frac{4\pi}{\lambda} \omega \mathcal{U} a^2 b \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-i)^m D_j^m j_{2j+m+1}(\lambda) \cos m\psi,$$

where κ and β are related to λ and ψ by $\kappa \cos\beta = (\lambda/a)\cos\psi$ and $\kappa \sin\beta = (\lambda/b)\sin\psi$. Let $\psi = \psi_0$ correspond to $\beta = \phi$. Thus, $\tan\psi_0 = (b/a)\tan\phi$ with ψ_0 and ϕ being in the same quadrant. Then, (3.4) gives

$$p \sim \frac{2\omega \mathcal{U} a^2 b}{\sqrt{2\pi R \sin\theta_c}}\, e^{-i\pi/4} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-i)^m D_j^m \cos m\psi_0 \int_0^{\infty} j_{2j+m+1}(\lambda)\, e^{i\kappa\sigma \sec\theta_c} \frac{\sqrt{\kappa}}{\lambda}\, d\kappa.$$

This estimate can be written as

$$p \sim \sqrt{\frac{a}{R}}\, \frac{\omega \mathcal{U} b}{\Delta^{3/2}\sqrt{\sin\theta_c}}\, e^{-i\pi/4} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-i)^m D_j^m\, h_{2j+m}(X) \cos m\psi_0, \tag{6.1}$$

where $\Delta(\phi) = (1 - k^2 \sin^2\phi)^{1/2}$,

$$h_n(X) = \int_0^{\infty} J_{n+3/2}(\lambda)\, e^{i\lambda X}\, \frac{d\lambda}{\lambda}, \quad \text{and} \quad X(\sigma, \phi) = \frac{\sigma \sec\theta_c}{a\Delta(\phi)}.$$

This integral can be evaluated. From [33, eqs. (2) and (3), p. 405], we have

$$\int_0^{\infty} J_\mu(\lambda)\, e^{\pm i\lambda c}\, \frac{d\lambda}{\lambda} = \begin{cases} \mu^{-1} \exp(\pm i\mu \arcsin c), & 0 \leq c \leq 1, \\ \mu^{-1}[c + \sqrt{c^2 - 1}]^{-\mu}\, e^{\pm i\pi\mu/2}, & c \geq 1. \end{cases} \tag{6.2}$$

The first of these gives

$$h_n(X) = (n + 3/2)^{-1} \exp\{i(n + 3/2) \arcsin X\}, \quad |X(\sigma, \phi)| \leq 1,$$

which shows the phase variation across the wave beam. The inequality $|X| \leq 1$ gives the thickness of the beam at azimuth ϕ, namely,

$$|\sigma| \leq a\Delta(\phi)\cos\theta_c = \cos\theta_c \left\{a^2 \cos^2\phi + b^2 \sin^2\phi\right\}^{1/2} = B(\phi), \tag{6.3}$$

say. The far-field velocity then follows from (3.5), using

$$\frac{\partial}{\partial\sigma}h_n(X) = \frac{i}{\sqrt{B^2-\sigma^2}}\exp\left\{i\left(n+\frac{3}{2}\right)\arcsin X\right\}, \quad |\sigma| < B(\phi).$$

Outside the beam, we can use the second part of (6.2): there is no phase variation, which implies that there is no energy transport outside the wave beams. Within this region, the pressure decays as R^{-1} as $R \to \infty$.

7. Two examples. The method described above can be used for arbitrary prescribed normal velocity of the plate, w_{p}. In this section, we investigate two simple but realistic forcings, namely heaving and rolling motions.

7.1. The heaving plate. This is the simpler problem. We suppose that $w_{\mathrm{p}} = \mathcal{U}_0$, a given constant, so that we are concerned only with $n = 0$ in (4.5). Put $\mathcal{U}\gamma = \mathcal{U}_0$. As $C_1^{\mu}(x) = 2\mu x$, $\Phi_0^{(0)}(\rho) = \sqrt{1-\rho^2}$ and so (5.6) gives $W_0^0 = 2/\pi$, all other W_m^n being zero. Then, (5.7) gives

$$D_0^0 = \frac{b/a}{2E(k)},$$

using $I_{00} = 2E(k)$, where $E(k)$ is a complete elliptic integral and $k = (1 - b^2/a^2)^{1/2}$; see (4.7). The pressure discontinuity across the plate is $2\rho_0\delta$, where

$$\delta = a\omega\mathcal{U}d_0(\rho) = 2a\omega\mathcal{U}D_0^0\sqrt{1-\rho^2}, \quad 0 \le \rho < 1.$$

The force on the plate is in the vertical direction with magnitude

$$\int_\Omega 2\rho_0\delta\,d\boldsymbol{x} = 4\pi\rho_0 ab\int_0^1 \delta\,\rho\,d\rho = \frac{4\pi}{3E(k)}\omega\mathcal{U}_0\rho_0 ab^2\tan\theta_{\mathrm{c}}.$$

For a circular plate, $a = b$, $k = 0$, $E(0) = \pi/2$, and we recover a known result [18, eq. (47)].

The far-field pressure within the wave beams is given by (6.1) as

$$p \sim \sqrt{\frac{a}{R}}\,\frac{b^2N\mathcal{U}_0\sqrt{\sin\theta_{\mathrm{c}}}}{3a(1-k^2\sin^2\phi)^{3/4}E(k)}\,e^{-i\pi/4}\exp\{(3i/2)\arcsin X(\sigma,\phi)\} \tag{7.1}$$

for $|X(\sigma,\phi)| \le 1$. Thus, although the prescribed motion is very simple, the dependence of p on the azimuthal angle ϕ is complicated.

Let us interpret the formula (7.1) for the far-field pressure within the beam. (Similar discussions can be given for related formulas.) First, there is the slow algebraic decay with distance from the plate, $R^{-1/2}$. Then, we note that the amplitude contains the factor $(1 - k^2\sin^2\phi)^{-3/4}$, which increases from 1 in the xz-plane ($\phi = 0$; see Figure 7.1) to $(a/b)^{3/2}$ in the yz-plane ($\phi = \frac{1}{2}\pi$). (By symmetry, it is enough to consider $0 \le \phi \le \frac{1}{2}\pi$, $z \ge 0$.) The beam has width $2B(\phi)$, with B decreasing from $a\cos\theta_{\mathrm{c}}$ at $\phi = 0$ to $b\cos\theta_{\mathrm{c}}$ at $\phi = \frac{1}{2}\pi$; see (6.3). The beam itself is bounded by the characteristic circular cones (2.2), with apexes located at all points around the edge of the plate. Thus, in a plane $\phi =$ constant, we would have a sketch similar to Figure 7.1 but with a narrower beam.

In the xz-plane, we can define a quantity ϑ by $\sigma = a\cos\theta_{\mathrm{c}}\sin\vartheta$, so that $|\vartheta| < \frac{1}{2}\pi$ within the beam; see Figure 7.1. Then, the last exponential in (7.1) simplifies to

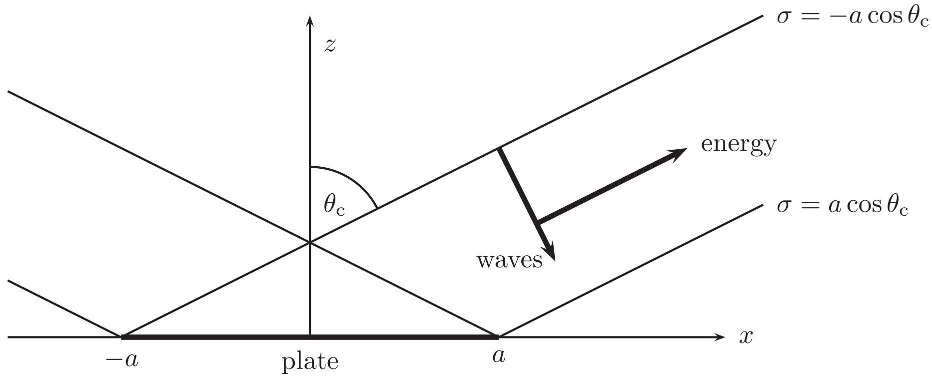

FIG. 7.1. *The group velocity is away from the plate, along the wave beams. The phase velocity is perpendicular to the group velocity; waves propagate downwards, across the beam.*

$\exp\{\frac{3}{2}i\vartheta\}$. In terms of ϑ, the beam has width π, and the wavelength is $\frac{4}{3}\pi$. The wave propagates downwards, with crests parallel to the beam boundaries. Similar interpretations can be given in other planes, $\phi =$ constant. Note that the numbers appearing in this description, $\frac{3}{2}$ and $\frac{4}{3}$, are a consequence of the simple forcing of the plate; more generally, see section 6.

7.2. The rolling plate. Consider a horizontal elliptical plate making small oscillations about the y-axis. For such rolling motions, we have $w_{\rm p} = \mathcal{V}_0 x/a = \mathcal{V}_0\rho\cos\chi$, where $\mathcal{V}_0$ is a given constant. Thus, we consider only $n = 1$ in (4.5). Put $\mathcal{U}\gamma = \mathcal{V}_0$. Then, as $\Phi_0^{(1)}(\rho) = 3\rho\sqrt{1-\rho^2}$, (5.6) gives $W_0^1 = 2/(3\pi)$, all other W_m^n being zero. From (5.8),

$$D_0^1 = \frac{b/a}{3I_{11}},$$

where $I_{11}(k)$ is given by (4.7). The pressure discontinuity is

$$\delta = a\omega\mathcal{U}d_1(\rho)\cos\chi = 2a\omega\mathcal{U}D_0^1\rho\sqrt{1-\rho^2}\cos\chi, \quad 0 \le \rho < 1.$$

Thus, there is no net force on the plate, but there is a moment about the y-axis with magnitude

$$\int_\Omega 2x\rho_0\delta\,d\boldsymbol{x} = 4\pi\rho_0\omega\mathcal{U}a^3bD_0^1\int_0^1\rho^3\sqrt{1-\rho^2}\,d\rho = \frac{8\pi\rho_0}{45I_{11}(k)}a^2b^2\mathcal{V}_0N\sin\theta_{\rm c}.$$

8. Piston problems. Suppose that the plane $z = 0$ is rigid apart from a piston, Ω, on which $w = w_{\rm p}$, a given function. On the rigid part of the plane, Ω', $w = 0$. The piston generates waves in the fluid above the plane, $z > 0$. The pressure is given by (2.8) in terms of $D(\boldsymbol{\xi})$. From (2.3) and (2.8), $\mathcal{F}\{w\} = \kappa(\gamma/\omega)D$ on $z = 0$, and this determines D.

Suppose now that Ω is an ellipse (4.1). Suppose for simplicity that $w_{\rm p} = \mathcal{U}_0$, a constant. Then,

$$\begin{aligned} D(\boldsymbol{\xi}) &= \frac{\omega}{\kappa\gamma}\mathcal{F}\{w\} = \frac{\omega\mathcal{U}_0}{\kappa\gamma}\int_\Omega \exp(-i\,\boldsymbol{\xi}\cdot\boldsymbol{x})\,d\boldsymbol{x} \\ &= 2\pi ab\frac{\omega\mathcal{U}_0}{\kappa\gamma}\int_0^1 J_0(\lambda\rho)\rho\,d\rho = 2\pi ab\frac{\omega\mathcal{U}_0}{\kappa\lambda\gamma}J_1(\lambda), \end{aligned}$$

using (4.2). As $\int_\Omega \exp(i\boldsymbol{\xi}\cdot\boldsymbol{x})\,d\boldsymbol{x} = 2\pi ab\lambda^{-1}J_1(\lambda)$, the net force on the piston is

$$\int_\Omega \rho_0 p\,d\boldsymbol{x} = \frac{\rho_0}{2\pi}\int_0^\infty\int_0^{2\pi} D(\boldsymbol{\xi})J_1(\lambda)\,d\psi\,d\lambda = \frac{16\rho_0}{3\pi}ab^2\omega\mathcal{U}_0K(k)\tan\theta_{\mathrm{c}}, \tag{8.1}$$

using $\kappa = (\lambda/b)\sqrt{1-k^2\cos^2\psi}$ and $\int_0^\infty \lambda^{-2}J_1^2(\lambda)\,d\lambda = 4/(3\pi)$ [15, eq. 6.575 (2)].

The time-averaged energy input to the fluid by the piston oscillations is

$$\mathcal{E}_{\mathrm{in}} = \frac{\rho_0}{2}\int_\Omega \mathrm{Re}\,\{p\,\overline{w_{\mathrm{p}}}\}\,d\boldsymbol{x} = \frac{8\rho_0}{3\pi}ab^2\omega|\mathcal{U}_0|^2K(k)\tan\theta_{\mathrm{c}}, \tag{8.2}$$

using (8.1), where the overline denotes complex conjugation. The quantity $\mathcal{E}_{\mathrm{in}}$ will be shown to match the energy transport in the far field, thus providing a check on the calculations.

In the far field, $p \sim b_0\zeta^{-1/2}$ with b_0 given by (3.4):

$$b_0 = \frac{ab\omega\mathcal{U}_0}{\gamma\sqrt{2\pi}\sin\theta_{\mathrm{c}}}\,e^{-i\pi/4}\int_0^\infty \frac{J_1(\lambda)}{\kappa\lambda}\,e^{i\kappa\sigma\sec\theta_{\mathrm{c}}}\sqrt{\kappa}\,d\kappa.$$

In this formula, $\lambda = \{(a\xi)^2+(b\eta)^2\}^{1/2} = \kappa\{a^2\cos^2\beta + b^2\sin^2\beta\}^{1/2}$ evaluated at $\beta=\phi$. Thus, $\lambda = \kappa B\sec\theta_{\mathrm{c}}$, where $B(\phi)$, defined by (6.3), gives the (expected) beam thickness at azimuth ϕ. Then, the substitution $s = \kappa\sec\theta_{\mathrm{c}}$ shows that

$$b_0 = \frac{ab\omega\mathcal{U}_0}{\pi B(\phi)}\,\sqrt{\tan\theta_{\mathrm{c}}}\,\mathcal{L}(B,\sigma), \tag{8.3}$$

where

$$\mathcal{L}(B,\sigma) = e^{-i\pi/4}\sqrt{\frac{\pi}{2}}\int_0^\infty s^{-3/2}J_1(sB)\,e^{is\sigma}\,ds. \tag{8.4}$$

Here, $B>0$, but σ can take any value. However, when evaluating $\mathcal{L}$, we can assume that $\sigma \ge 0$ because of the relation

$$\mathcal{L}(B,\sigma) = -i\,\overline{\mathcal{L}(B,-\sigma)}. \tag{8.5}$$

The integral $\mathcal{L}(B,\sigma)$ can be written in terms of Weber–Schafheitlin integrals; its value depends on whether $0<\sigma<B$ (within the wave beam) or $0<B<\sigma$ (outside the beam). The relevant formulas are [15, eqs. 6.699 (1) and (2)].

For $0<B<\sigma$, we obtain

$$\mathcal{L}(B,\sigma) = \frac{B\pi}{\sqrt{8\sigma}}\,F\left(\frac{1}{4},\frac{3}{4};2;\left(\frac{B}{\sigma}\right)^2\right),$$

where F is the hypergeometric function. Putting $\mu=-1$ and $\nu=\frac{1}{2}$ in [9, eq. 14.3.7] gives

$$Q_{1/2}^{-1}(X) = -\frac{\pi}{2^{3/2}\sqrt{X}\sqrt{X^2-1}}F\left(\frac{1}{4},\frac{3}{4};2;X^{-2}\right)$$

for $X>1$, where Q_ν^μ is an associated Legendre function of the second kind. Hence

$$\mathcal{L}(B,\sigma) = -B^{-1/2}\sqrt{\sigma^2-B^2}\,Q_{1/2}^{-1}(\sigma/B) \quad \text{for } \sigma > B > 0.$$

As expected, there is no phase variation with σ, implying no energy transport.

For $0 < \sigma < B$, [15, eqs. 6.699 (1) and (2)] gives

$$\mathcal{L}_c \equiv \int_0^\infty s^{-3/2} J_1(sB)\cos(s\sigma)\,ds = \frac{\sqrt{B}\,\Gamma(\frac{1}{4})}{2\sqrt{2}\,\Gamma(\frac{7}{4})} F\left(\frac{1}{4}, -\frac{3}{4}; \frac{1}{2}; \left(\frac{\sigma}{B}\right)^2\right),$$
$$\mathcal{L}_s \equiv \int_0^\infty s^{-3/2} J_1(sB)\sin(s\sigma)\,ds = \frac{\sigma\Gamma(\frac{3}{4})}{\sqrt{2B}\,\Gamma(\frac{5}{4})} F\left(\frac{3}{4}, -\frac{1}{4}; \frac{3}{2}; \left(\frac{\sigma}{B}\right)^2\right).$$

The hypergeometric functions appearing here also appear in expressions for certain Ferrers functions. Thus, putting $\mu = 1$ and $\nu = \frac{1}{2}$ in [9, eqs. 14.3.13 and 14.3.14] (noting [9, eq. 14.3.3]) gives

$$w_1\left(\frac{1}{2}, 1, X\right) = \frac{\Gamma(\frac{5}{4})}{\sqrt{\pi}\,\Gamma(\frac{3}{4})} \frac{2}{\sqrt{1-X^2}} F\left(\frac{1}{4}, -\frac{3}{4}; \frac{1}{2}; X^2\right),$$
$$w_2\left(\frac{1}{2}, 1, X\right) = \frac{\Gamma(\frac{7}{4})}{\sqrt{\pi}\,\Gamma(\frac{1}{4})} \frac{4X}{\sqrt{1-X^2}} F\left(\frac{3}{4}, -\frac{1}{4}; \frac{3}{2}; X^2\right).$$

So, with $0 < X = \sigma/B < 1$,

$$\mathcal{L}_c = \frac{2}{3}\sqrt{2\pi B}\sqrt{1-X^2}\, w_1\left(\frac{1}{2}, 1, X\right),$$
$$\mathcal{L}_s = \frac{2}{3}\sqrt{2\pi B}\sqrt{1-X^2}\, w_2\left(\frac{1}{2}, 1, X\right).$$

Then, from [9, eqs. 14.3.11 and 14.3.12],

$$w_1\left(\frac{1}{2}, 1, X\right) - w_2\left(\frac{1}{2}, 1, X\right) = -\sqrt{2}\,\mathsf{P}^1_{1/2}(X),$$
$$w_1\left(\frac{1}{2}, 1, X\right) + w_2\left(\frac{1}{2}, 1, X\right) = -\sqrt{2}\left(\frac{2}{\pi}\right)\mathsf{Q}^1_{1/2}(X),$$

where P^μ_ν and Q^μ_ν are Ferrers functions. Hence,

$$\sqrt{2}\,w_1\left(\frac{1}{2}, 1, X\right) = -\mathsf{P}^1_{1/2}(X) - \frac{2}{\pi}\mathsf{Q}^1_{1/2}(X),$$
$$\sqrt{2}\,w_2\left(\frac{1}{2}, 1, X\right) = \mathsf{P}^1_{1/2}(X) - \frac{2}{\pi}\mathsf{Q}^1_{1/2}(X).$$

From [9, eqs. 14.6.1 and 14.6.2],

$$\mathsf{P}^1_\nu(X) = -\sqrt{1-X^2}\,\mathsf{P}'_\nu(X) \quad \text{and} \quad \mathsf{Q}^1_\nu(X) = -\sqrt{1-X^2}\,\mathsf{Q}'_\nu(X),$$

whereas [9, eqs. 14.5.20 and 14.5.22] gives

$$\mathsf{P}_{1/2}(X) = (2/\pi)\{2E(X_-) - K(X_-)\} \quad \text{and} \quad \mathsf{Q}_{1/2}(X) = K(X_+) - 2E(X_+),$$

where E and K are complete elliptic integrals and $X_\pm = \sqrt{(1 \pm X)/2}$. As $dX_\pm/dX = \pm\frac{1}{4}/X_\pm$, $\mathsf{P}'_{1/2}(X) = (2/\pi)G(X_-)$ and $\mathsf{Q}'_{1/2}(X) = G(X_+)$, where

$$G(k) = \frac{1}{4k}\frac{d}{dk}\{K(k) - 2E(k)\} = \frac{1}{4k^2k'^2}\left\{(1-2k'^2)E(k) + k'^2K(k)\right\},$$

$k'^2 = 1 - k^2$, and we have used [15, eqs. 8.123 (2) and (4)]. Thus, when $k = X_\pm$, $k' = X_\mp$, $4k^2k'^2 = 1 - X^2$ and $1 - 2k'^2 = \pm X$. Remove the factor of $(1 - X^2)^{-1}$ by writing

$$2G(X_\pm) = (1 - X^2)^{-1}\mathcal{G}_\pm \quad \text{with} \quad \mathcal{G}_\pm = K(X_\pm) \pm X\{2E(X_\pm) - K(X_\pm)\}. \tag{8.6}$$

Hence

$$\sqrt{2}\, w_1\left(\frac{1}{2}, 1, X\right) = \frac{1}{\pi}(1 - X^2)^{-1/2}\left\{\mathcal{G}_+ + \mathcal{G}_-\right\},$$
$$\sqrt{2}\, w_2\left(\frac{1}{2}, 1, X\right) = \frac{1}{\pi}(1 - X^2)^{-1/2}\left\{\mathcal{G}_+ - \mathcal{G}_-\right\}.$$

From (8.4), we want

$$\begin{aligned} \mathcal{L}(B,\sigma) &= e^{-i\pi/4}\sqrt{\frac{\pi}{2}}\,(\mathcal{L}_c + i\,\mathcal{L}_s) \\ &= \frac{2}{3}e^{-i\pi/4}\pi\sqrt{B}\sqrt{1 - X^2}\left\{w_1\left(\frac{1}{2}, 1, X\right) + i\,w_2\left(\frac{1}{2}, 1, X\right)\right\} \\ &= \frac{2}{3}e^{-i\pi/4}\sqrt{\frac{B}{2}}\,\left\{(\mathcal{G}_+ + \mathcal{G}_-) + i\,(\mathcal{G}_+ - \mathcal{G}_-)\right\} \\ &= \frac{2}{3}\sqrt{B}\,(\mathcal{G}_+ - i\,\mathcal{G}_-). \end{aligned} \tag{8.7}$$

This formula was derived assuming that $0 \le \sigma < B$. However, use of (8.5) shows that (8.7) is valid across the beam, $-B < \sigma < B$.

Having found $\mathcal{L}$, b_0 is given by (8.3), and then the far-field pressure within the beam is given by

$$p \sim \frac{2}{3\pi}ab\omega\mathcal{U}_0\,\sqrt{\tan\theta_c}\,[B(\phi)]^{-1/2}\,(\mathcal{G}_+ - i\,\mathcal{G}_-)\,\zeta^{-1/2}.$$

The far-field velocity, v, is given by (3.5) in terms of $\partial p/\partial\sigma$; as $X = \sigma/B$,

$$v \sim \frac{2ab\mathcal{U}_0}{3\pi i\zeta^{1/2}\sqrt{\tan\theta_c}\,[B(\phi)]^{3/2}}\frac{\partial}{\partial X}\,(\mathcal{G}_+ - i\,\mathcal{G}_-).$$

Now, if we define $\mathcal{J}_\pm = 2E(X_\pm) - K(X_\pm)$, (8.6) gives $\mathcal{G}_\pm = K(X_\pm) \pm X\mathcal{J}_\pm$ and then direct calculation gives $\partial\mathcal{G}_\pm/\partial X = \pm(3/2)\mathcal{J}_\pm$.

In the far field, the time-averaged energy transport vector is $I\hat{\boldsymbol{\zeta}}$, with $I = (\rho_0/2)\mathrm{Re}\,\{p\overline{v}\}$ [21, 31]. Substituting for p and v gives

$$I = \frac{2}{3}\omega\rho_0\left(\frac{ab}{\pi B(\phi)}\right)^2\frac{|\mathcal{U}_0|^2}{\zeta}\left(\frac{\mathcal{G}_-}{3}\frac{\partial\mathcal{G}_+}{\partial X} - \frac{\mathcal{G}_+}{3}\frac{\partial\mathcal{G}_-}{\partial X}\right).$$

The quantity inside the last pair of parentheses reduces to

$$\frac{1}{2}\{K(X_-)\mathcal{J}_+ + K(X_+)\mathcal{J}_-\} = E(X_+)K(X_-) + E(X_-)K(X_+) - K(X_+)K(X_-) = \frac{\pi}{2},$$

using Legendre's relation [15, eq. 8.122]. Hence,

$$I = \frac{\omega\rho_0}{3\pi\zeta}\left(\frac{ab}{B(\phi)}\right)^2|\mathcal{U}_0|^2,$$

which is positive: the energy transport vector $I\hat{\boldsymbol{\zeta}}$ points away from the piston.

The total energy transported away, $\mathcal{E}_{\text{out}}$, is found by integrating over the beam cross-section [21, 31]:

$$\mathcal{E}_{\text{out}} = \zeta \sin\theta_{\text{c}} \int_0^{2\pi} \int_{-B(\phi)}^{B(\phi)} I \, d\sigma \, d\phi = \frac{2}{3\pi}(ab)^2 \omega \rho_0 |\mathcal{U}_0|^2 \sin\theta_{\text{c}} \int_0^{2\pi} \frac{d\phi}{B(\phi)}.$$

The remaining integral is $(4/a)K(k)\sec\theta_{\text{c}}$ using (6.3), whence $\mathcal{E}_{\text{out}} = \mathcal{E}_{\text{in}}$ (see (8.2)), as expected.

We conclude by noting that results for a circular piston are readily obtained by setting $a = b$ ($k = 0$). In this case, the beam width does not depend on ϕ, $B = a\cos\theta_{\text{c}}$.

It is possible that some of the techniques developed in this paper (and also in [22]) could be adapted to analyze the motions generated by other kinds of wavemakers [14, 23], but this remains for future work.

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