

# Two-dimensional acoustic scattering, conformal mapping, and the Rayleigh hypothesis

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Methods for solving two-dimensional scattering problems using conformal mappings are investigated. It is shown that their convergence relies on a mapped form of the Rayleigh hypothesis. It is concluded that methods based on conformal mappings offer no advantages over established methods in which the scattered field is expanded as a series of circular-cylindrical outgoing wavefunctions.

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## I. INTRODUCTION

There are several effective rigorous methods for solving the exterior boundary-value problems of acoustics in two dimensions. For simple geometries (circles and ellipses), we can use separation of variables. More generally, we can use a boundary integral equation or a  $T$ -matrix method.

Our purpose here is to discuss certain methods in which conformal mappings are used. The basic problem is to solve the Helmholtz equation,  $(\nabla^2 + k^2)\phi = 0$ , exterior to a simple closed curve  $\mathcal{C}$ , with boundary and radiation conditions. Application of a conformal mapping can simplify the geometry ( $\mathcal{C}$  can be mapped to a circle) but at the cost of a more complicated differential equation: the constant  $k^2$  is replaced by a function of the independent variables; see Eq. (24) below. This basic approach has a long history, going back to Garabedian.<sup>1</sup> More recent papers include Refs. 2–4.

At the heart of our discussion are singularities of the analytic continuation of the (physical) exterior field inside  $\mathcal{C}$ . The relevance of these singularities to scattering problems is well known, especially in the context of the *Rayleigh hypothesis*. This concerns the use of circular-cylindrical wave functions when the scatterer is not a circle; see Sec. III. In Sec. II C, we give methods for calculating singularity locations inside  $\mathcal{C}$ , with emphasis on the role of the *Schwarz function*.<sup>5</sup> This function has been used in other applied areas, such as vortex dynamics (see, for example, Ref. 6 or Sec. 9.2 of Ref. 7). It can be constructed by conformal mapping (Chap. 8 of Ref. 5) and by several other methods.

In Sec. IV A, we introduce transformations (changes of independent variables), with conformal mappings as a special case. It is observed that solutions of the transformed Helmholtz equation can be obtained simply by introducing the same transformation into known solutions of the Helmholtz equation. Then, in Sec. IV B, we consider published methods in which a conformal mapping is used to map  $\mathcal{C}$  onto a unit circle. It is demonstrated that doing this offers no advantages (and has the additional disadvantage of having to find the appropriate conformal mapping). The new methods

are tantamount to trying to solve the problem of scattering by  $\mathcal{C}$  using circular-cylindrical wave functions (for which several numerical strategies are available), which means they are subject to (a mapped form of) the Rayleigh hypothesis.

## II. SCATTERING PROBLEMS AND SINGULARITIES

We consider acoustic scattering in two dimensions. Thus, we want to solve the Helmholtz equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = \nabla^2 \phi + k^2 \phi = 0, \quad (1)$$

in the unbounded region,  $B$ , exterior to the smooth closed curve,  $\mathcal{C}$ . Here,  $k = \omega/c$ ,  $c$  is the speed of sound and  $\omega$  is the frequency: the suppressed time dependence is  $e^{-i\omega t}$ . We use Cartesian coordinates  $x, y$ , with the origin,  $O$ , inside  $\mathcal{C}$ . The total field is  $\phi + \phi_{\text{inc}}$ , where  $\phi_{\text{inc}}$  is the given incident field. For simplicity, we assume that  $\mathcal{C}$  is sound-soft, which means

$$\phi + \phi_{\text{inc}} = 0 \text{ on } \mathcal{C} \quad (2)$$

In addition,  $\phi$  must satisfy the Sommerfeld radiation condition at infinity. It is well known that the boundary-value problem for  $\phi$  has exactly one solution.

### A. Circular scatterer

When  $\mathcal{C}$  is a circle, centered at  $O$ , we can solve for  $\phi$  by separation of variables in polar coordinates. Thus, as is well known, radiating solutions of Eq. (1) are

$$\psi_n(x, y) = H_n(kr)e^{in\theta}, \quad n = \text{integer}, \quad (3)$$

where  $x + iy = re^{i\theta}$  and  $H_n \equiv H_n^{(1)}$  is a Hankel function. Then, we write

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} c_n \psi_n(x, y), \quad (4)$$

where the coefficients  $c_n$  are to be found. If convergent, this representation for  $\phi$  ensures that Eq. (1) and the radiation condition are satisfied. For the boundary condition, suppose  $\mathcal{C}$  has radius  $a$  and put  $f(\theta) = -\phi_{\text{inc}}$  evaluated on  $r = a$ . Then, Eqs. (2) and (4) give

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$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n H_n(ka) e^{in\theta}, \quad -\pi < \theta \leq \pi. \quad (5)$$

As this is a Fourier expansion, we obtain

$$c_n = \frac{1}{2\pi H_n(ka)} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta.$$

Let  $u_n = \max\{|c_n|, |c_{-n}|\}$  for  $n=0, 1, 2, \dots$ . As

$$H_n(\kappa) \sim -(i/\pi)(2/\kappa)^n (n-1)!, \quad n \rightarrow \infty, \text{ fixed } \kappa, \quad (6)$$

the series in Eq. (4) will be absolutely convergent for

$$r > \lim_{n \rightarrow \infty} \frac{2nu_{n+1}}{ku_n}. \quad (7)$$

For a simple example, consider an incident plane wave

$$\phi_{\text{inc}} = e^{ikx} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta}, \quad (8)$$

where  $J_n$  is a Bessel function. Then

$$c_n = -\frac{i^n J_n(ka)}{H_n(ka)} \quad \text{and} \quad \frac{u_{n+1}}{u_n} \sim \frac{(ka)^2}{4n^2} \quad \text{as } n \rightarrow \infty,$$

using Eq. (6) and a similar estimate for Bessel functions,  $J_n(\kappa) \sim (\kappa/2)^n/n!$  for large  $n$  and fixed  $\kappa$ . Hence, from Eq. (7), the series in Eq. (4) converges absolutely for  $r > 0$ , for an incident plane wave, Eq. (8). In other words,  $\phi$  can be analytically continued inside the circle  $\mathcal{C}$ , using Eq. (4), all the way to  $r=0$ , where there is a singularity.

Indeed, there must be at least one singularity inside  $\mathcal{C}$ , assumed now to be a simple, smooth closed curve, because the only radiating solution of the Helmholtz equation in the whole of space is identically zero. These singularities can be induced by the incident field or by the shape of  $\mathcal{C}$ .

The singularity locations are intrinsic to the boundary-value problem: they do not depend on  $k$  (it is enough to consider  $k=0$ ), the choice of coordinate origin, or the method used for solving the boundary-value problem. They do depend on the shape of  $\mathcal{C}$  and on the incident field (or the boundary condition on  $\mathcal{C}$ ).

## B. Two more examples

Suppose that the circular scatterer of Sec. II A is displaced so that its equation is

$$(x-x_0)^2 + y^2 = a^2. \quad (9)$$

The analytic continuation of  $\phi$  will still have a singularity at the circle's center,  $(x, y) = (x_0, 0)$ . If we use Eq. (4) for  $\phi$ , with wave functions centered at  $O$ , the series will converge for  $r > |x_0|$ ; this region will include all of the exterior,  $B$ , provided that  $|x_0| < (1/2)a$ .

For another example, consider an ellipse,

$$(x/a)^2 + (y/b)^2 - 1 = 0, \quad 0 < b \leq a. \quad (10)$$

The analytic continuation of  $\phi$  will have singularities at the foci,  $(x, y) = (\pm c, 0)$  with  $c = \sqrt{a^2 - b^2}$  (see Sec. II C). If we use Eq. (4) for  $\phi$ , with wave functions centered at  $O$ , the series will converge for  $r > c$ ; this region will include all of  $B$  provided that  $a/b < \sqrt{2}$ .

## C. Methods for finding singularities

One way to find the singularity locations proceeds as follows, adapting a method used by Keller<sup>8</sup> for grating problems. Thus, suppose that the boundary curve  $\mathcal{C}$  is given by  $F(x, y) = 0$  for some function  $F$ . Write  $\mathcal{C}$  as

$$\bar{z} = S(z), \quad z \in \mathcal{C}, \quad (11)$$

where  $z = x + iy$  and  $\bar{z} = x - iy$ . Hence,  $x = (1/2)(z + \bar{z}) = (1/2)(z + S(z))$  and  $y = (1/2)i(S(z) - z)$ , so that  $S$  is given implicitly by

$$F(x, y) = 0, \quad \text{with } x = \frac{1}{2}(z + S(z)), \quad y = \frac{1}{2}i(S(z) - z). \quad (12)$$

Assuming that  $\mathcal{C}$  is an analytic curve,  $S(z)$  will be an analytic function of  $z$  in a region that includes  $\mathcal{C}$ . We are interested in locating the singularities of  $S(z)$  inside  $\mathcal{C}$ . The reason is that a separate argument (given by Keller<sup>8</sup>) shows that, generically, these locations are where the analytic continuation of  $\phi$  inside  $\mathcal{C}$  will have singularities; see also Ref. 9. (In general, there may be singularities outside  $\mathcal{C}$  too, but these are not relevant when our goal is to solve scattering problems.)

The function  $S(z)$  is known as the *Schwarz function* of  $\mathcal{C}$ . Much is known about properties of Schwarz functions.<sup>5,10</sup> Let us give some explicit examples.

- (1) Displaced circle, Eq. (9):  $S(z) = x_0 + a^2/(z - x_0)$ . Note that  $S$  has a simple pole at the center of the circle. It is known that if the Schwarz function of a closed curve  $\mathcal{C}$  is a rational function, then  $\mathcal{C}$  must be a circle (p. 104 of Ref. 5).
- (2) Ellipse, Eq. (10). With  $F(x, y)$  defined by the left-hand side of Eq. (10), Eq. (12) gives

$$(a^2 - b^2)(z^2 + S^2) - 2(a^2 + b^2)zS + 4a^2b^2 = 0, \quad (13)$$

with solution [see Eq. (5.13) of Ref. 5]

$$S(z) = \frac{1}{c^2} \left[ (a^2 + b^2)z - 2ab\sqrt{z^2 - c^2} \right], \quad c = \sqrt{a^2 - b^2}.$$

Note that  $S$  has branch points at the foci of the ellipse,  $z = \pm c$ .

- (3) Carl Neumann's oval (p. 20 of Ref. 10). Barnett and Betcke<sup>11</sup> define an "inverted ellipse" parametrically by

$$x(s) + iy(s) = e^{is}(1 + ae^{2is})^{-1}, \quad 0 \leq s \leq 2\pi,$$

where  $a$  is a constant,  $0 < a < 1$ . Eliminating the parameter  $s$  gives

$$(x^2 + y^2)^2 - [x/(1+a)]^2 - [y/(1-a)]^2 = 0 \quad (14)$$

from which

$$S(z) = \frac{z}{2} \left[ \frac{(1+a^2) + (1-a^2)\sqrt{1-4az^2}}{(1-a^2)^2 z^2 + a} \right]. \quad (15)$$

We note that  $S$  has branch points at  $z = \pm(2\sqrt{a})^{-1}$  (outside  $\mathcal{C}$ ) and simple poles at  $z = \pm i\sqrt{a}/(1-a^2)$  (inside  $\mathcal{C}$ ). For Eq. (14), see Eq. (3.8) in Ref. 10. In plane polar coordinates, Eq. (14) can be written in the form  $r^2 = A + B \cos^2 \theta$ , called “the rose  $R_2$ ” by Davis; he gives Eq. (15) [Eq. (5.16) in Ref. 5] and some generalizations.

Keller<sup>8</sup> did not identify  $S(z)$  as the Schwarz function of  $\mathcal{C}$  but he did suggest a way of finding the singularities of  $S$  without having an explicit formula for  $S$ . We have

$$F(x, y) = F\left(\frac{1}{2}(z + \bar{z}), \frac{1}{2}i(\bar{z} - z)\right) = g(z, \bar{z}).$$

On  $\mathcal{C}$ ,  $F = 0$  and  $\bar{z} = S(z)$  whence  $g(z, S(z)) = 0$ ,  $z \in \mathcal{C}$ . But  $g$  is analytic in  $z$  and  $\bar{z}$  so that  $g(z, S(z)) = 0$  for  $z$  near  $\mathcal{C}$ . Then, differentiating with respect to  $z$  gives

$$0 = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \bar{z}} S'(z);$$

see p. 114 of Ref. 5. Thus,  $S'$  is infinite when

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} = 0 \quad \text{with } x = \frac{z + S(z)}{2}, \quad y = \frac{z - S(z)}{2i}. \quad (16)$$

Combining Eqs. (12) and (16) determines the singularity locations,  $z$ . This combination of conditions on  $F$  [but not the use of  $\bar{z} = S(z)$ ] was also found by Maestre and Cadilhac.<sup>12</sup>

For an example, consider the ellipse, Eq. (10). Equation (16) gives  $(a^2 + b^2)z = (a^2 - b^2)S$ . Using this to eliminate  $S$  from Eq. (13) then gives  $z = \pm\sqrt{a^2 - b^2}$ , as expected.

The methods described above assume that  $\mathcal{C}$  is given in the form  $F(x, y) = 0$ . However, there are situations where  $\mathcal{C}$  is defined parametrically by

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq 2\pi, \quad (17)$$

or, in polar coordinates (when  $\mathcal{C}$  is “star-shaped”), by

$$r = \rho(\theta), \quad 0 \leq \theta \leq 2\pi. \quad (18)$$

From Eqs. (11) and (17), we have  $x(s) - iy(s) = S(x(s) + iy(s))$ . Differentiation then shows that  $S'$  will be infinite when

$$x'(s) + iy'(s) = 0; \quad (19)$$

solving this equation for  $s = s_0$ , say, the singularity of  $S(z)$  is at  $z = x(s_0) + iy(s_0)$ .

Similarly, Eqs. (11) and (18) give

$$\rho(\theta)e^{-i\theta} = S(\rho(\theta)e^{i\theta}),$$

so that differentiation gives

$$\rho'(\theta) + i\rho(\theta) = 0 \quad (20)$$

as the requirement for a singularity; solving Eq. (20) for  $\theta = \theta_0$ , say, the singularity is at  $z = \rho(\theta_0)e^{i\theta_0}$ . The condition

Eq. (20) is also obtained from Eq. (16) using  $F(x, y) = x^2 + y^2 - \rho^2(\theta)$  and  $\theta(x, y) = \arctan(y/x)$ . We note that Eq. (20) is similar to a condition found by van den Berg and Fokkema;<sup>13</sup> see their Eq. (14).

For the ellipse, Eq. (10), we can take  $x = a \cos s$  and  $y = b \sin s$ . Then Eq. (19) gives  $\tan s_0 = ib/a$ . We also have  $\rho(\theta) = ab(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{-1/2}$ ; Eq. (20) gives

$$e^{2i\theta_0} = \frac{a^2 - b^2}{a^2 + b^2} \quad \text{and} \quad \rho(\theta_0) = \sqrt{a^2 + b^2}.$$

Either way, we obtain the expected singularity locations.

For another example, suppose that  $\mathcal{C}$  is the “rounded triangle” of Barnett and Betcke,<sup>11</sup> defined by

$$x(s) + iy(s) = e^{is} + ae^{-2is}, \quad 0 \leq s \leq 2\pi, \quad 0 < a < \frac{1}{2}.$$

Equation (19) reduces to  $e^{3is} = 2a$ , with three distinct solutions. The corresponding singularities are at

$$z = z_n = \frac{3}{2}(2a)^{1/3}e^{2n\pi i/3}, \quad n = 0, 1, 2.$$

These are all inside  $\mathcal{C}$ ; see Fig. (7a) in Ref. 11 for a sketch when  $a = 0.2$ .

In summary, the methods we have described determine the singularity locations based solely on the shape of  $\mathcal{C}$ . Again, we note that there may be other singularities caused by the incident field: these will not occur for incident plane waves, for example.

### III. RAYLEIGH HYPOTHESIS

The discussion in Sec. II is helpful in understanding the Rayleigh hypothesis. Thus, suppose we write the solution of our scattering problem as in Eq. (4). This infinite series converges outside a circle,  $C_0$ , of some radius,  $r_0$ , centered at  $O$ . The circle  $C_0$  is at least as small as the circumscribed circle (the smallest circle, centered at  $O$  and containing  $\mathcal{C}$ ), but it may be smaller. Thus, it is possible that  $\mathcal{C}$  contains  $C_0$ , so that the series in Eq. (4) converges everywhere outside and on  $\mathcal{C}$ : this is an assumption known as the Rayleigh hypothesis. It is known that the validity of the Rayleigh hypothesis depends on the shape of  $\mathcal{C}$  and the location of  $O$ . Specifically,  $C_0$  must contain all the singularities of the analytic continuation of  $\phi$  inside  $\mathcal{C}$ , as discussed in Sec. II C.

A clear overview of this well-known material is given by Millar.<sup>14</sup>

We emphasize that the Rayleigh hypothesis is concerned with the convergence of infinite series. It says nothing about numerical methods which, inevitably, deal with finite series and approximations. For example, we may truncate the series in Eq. (4) and write

$$\phi(x, y) \simeq \phi_N \equiv \sum_{n=-N}^N c_n^N \psi_n(x, y). \quad (21)$$

Evidently,  $\phi_N$  satisfies Eq. (1) and the radiation condition, so it remains to find the coefficients  $c_n^N$  in order that the

boundary condition, Eq. (2), be satisfied. One way to do this is to apply Eq. (2) at discrete points on  $\mathcal{C}$ ; see, for example, Ref. 15. Another popular way is to impose Eq. (2) in a least-squares sense, minimizing  $\int_S |\phi_N + \phi_{\text{inc}}|^2 ds$ ; see, for example, Refs. 14 and 16. For comparisons and connections, see Ref. 17.

It is interesting to note that singularity locations have been found to play a role in the behavior of the method of fundamental solutions (MFS). In the MFS, sources are placed outside the physical domain along a fictitious “charge curve”  $\Gamma$  and then their strengths are determined by fitting the boundary condition on  $\mathcal{C}$ . For interior problems, Barnett and Betcke<sup>11</sup> show numerically “that the success (numerical stability and hence high accuracy) of the MFS relies on a choice of charge curve which does not enclose any singularities of the analytic continuation of the solution,” meaning that there should be no singularities between  $\mathcal{C}$  and  $\Gamma$ .

#### IV. CONFORMAL MAPPING

As noted in Sec. I, there has been interest in the literature, dating back many years, in using conformal mapping in the context of two-dimensional scattering. We shall discuss some of this work in Sec. IV B.

##### A. Mappings and their effects

Let us start by introducing an arbitrary, smooth invertible change of variables, defined by

$$x = x(u, v), \quad y = y(u, v). \quad (22)$$

When this is used in Eq. (1), together with the chain rule, a new equation is obtained for  $\phi(x(u, v), y(u, v)) = \Phi(u, v)$ , say (see the Appendix in Ref. 18):

$$\begin{aligned} |\nabla u|^2 \frac{\partial^2 \Phi}{\partial u^2} + 2(\nabla u \cdot \nabla v) \frac{\partial^2 \Phi}{\partial u \partial v} + |\nabla v|^2 \frac{\partial^2 \Phi}{\partial v^2} \\ + \frac{\partial \Phi}{\partial u} \nabla^2 u + \frac{\partial \Phi}{\partial v} \nabla^2 v + k^2 \Phi = 0. \end{aligned} \quad (23)$$

The mapping in Eq. (22) can be chosen in many ways. For example, we may choose it so that the boundary curve,  $\mathcal{C}$ , in the  $xy$ -plane is mapped into a simpler curve in the  $uv$ -plane.

One option is to choose a *conformal* mapping. For such mappings, Eq. (23) simplifies substantially. The result is

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} + k^2 J \Phi = 0, \quad (24)$$

where  $J(u, v)$  is the Jacobian of the transformation, defined by

$$J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}. \quad (25)$$

(The Cauchy-Riemann equations can be used to write  $J$  in other ways.) Equation (24) is Eq. (20) in Ref. 3, for example. Separated solutions of Eq. (24) are investigated in Ref. 18.

Return to the general equation, Eq. (23). Evidently, if  $\phi(x, y)$  is any solution of the Helmholtz equation, Eq. (1),

then  $\Phi(u, v) = \phi(x(u, v), y(u, v))$  is a solution of Eq. (23). This result follows by substitution, nothing more: the change of variables does not have to come from a conformal mapping.

One application of this result is that

$$\psi_n(x, y) = \psi_n(x(u, v), y(u, v)) = \Psi_n(u, v), \quad (26)$$

say, solves Eq. (23), where  $\psi_n$  is the outgoing cylindrical wave function, Eq. (3). We see solutions of this form as Eq. (4.2) in Ref. 2, Eq. (27) in Ref. 3, and Eq. (24) in Ref. 4, obtained there by much more complicated procedures.

##### B. Comparisons

In order to compare with previous work, we consider using a conformal mapping. The simplest example concerns the displaced circle, Eq. (9). In the physical  $z$ -plane, we have  $z = x + iy = re^{i\theta}$ . In the mapped  $\zeta$ -plane, we have  $\zeta = u + iv = \sigma e^{i\varphi}$ . The mapping  $z = \omega(\zeta) = a\zeta + x_0$  takes the unit circle  $|\zeta| = 1$  to the displaced circle  $|z - x_0| = a$ . It also takes  $\zeta = 0$  to the center of the displaced circle,  $z = x_0$ , and  $\zeta = -x_0/a$  to  $z = 0$ . From Eq. (25),  $J = a^2$ .

Let us assume that  $x_0 > 0$ . In the physical domain, we know that the infinite series of wave functions, Eq. (4), converges for  $r > x_0$ , and this includes the whole of the exterior region if  $x_0 < (1/2)a$ .

In the mapped domain, we follow Eq. (24) in Ref. 4 and write

$$\Phi(u, v) = \sum_{n=-\infty}^{\infty} B_n \Psi_n(u, v) \quad (27)$$

with

$$\begin{aligned} \Psi_n(u, v) &= H_n^{(1)}\left(k|w(\zeta)|\right) \left(\frac{w(\zeta)}{|w(\zeta)|}\right)^n \\ &= H_n^{(1)}(k|a\zeta + x_0|) e^{in\theta(\sigma, \varphi)}. \end{aligned}$$

Each term in Eq. (27) is singular where  $|a\zeta + x_0| = 0$ , that is, at  $\zeta = -x_0/a$ . However, the analytic continuation of  $\phi$  inside the mapped circle,  $|\zeta| = 1$ , will be singular at the center,  $\zeta = 0$ . The convergence of the series in Eq. (27) depends on the size of  $|\zeta + x_0/a|$ , the distance from  $\zeta$  to the expansion center,  $-x_0/a$ . This distance must be greater than the distance to the farthest singularity which, in our case, is the (only) singularity at the origin,  $\zeta = 0$ . Thus (assuming no other singularities have been introduced by the boundary condition), the condition for convergence of Eq. (27) is  $|\zeta + x_0/a| > x_0/a$ . This is exactly the same as we found in the physical domain,  $r = |a\zeta + x_0| > x_0$ .

##### C. Discussion

The simple example just described is typical. Its study leads to the following conclusions. First, the Rayleigh hypothesis is still present in the mapped problem, even though the mapped geometry is a circle. Conformal mapping has merely disguised this fact.

Second, as the functions  $\Psi_n$  are not orthogonal on  $|\zeta|=1$ , some kind of numerical scheme will be needed to find the coefficients,  $B_n$ , in Eq. (27). Exactly the same observation can be made of the series expansion in the physical domain, Eq. (4). Consequently, one may as well work directly with a truncated form of Eq. (4), which is Eq. (21): nothing is gained by introducing a conformal mapping (which would also have to be found).

Third, if  $\mathcal{C}$  has a more complicated shape, one would have to examine the curves  $|\omega(\zeta)| = \text{constant}$  in the  $\zeta$ -plane in order to understand the convergence of Eq. (27). These curves may or may not enclose singularities of the analytic continuation of  $\Phi$  inside  $|\zeta|=1$ . Making this determination is not worthwhile: it is better to remain in the physical domain.

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